Parallel spectral-element direction splitting method for incompressible Navier-Stokes equations

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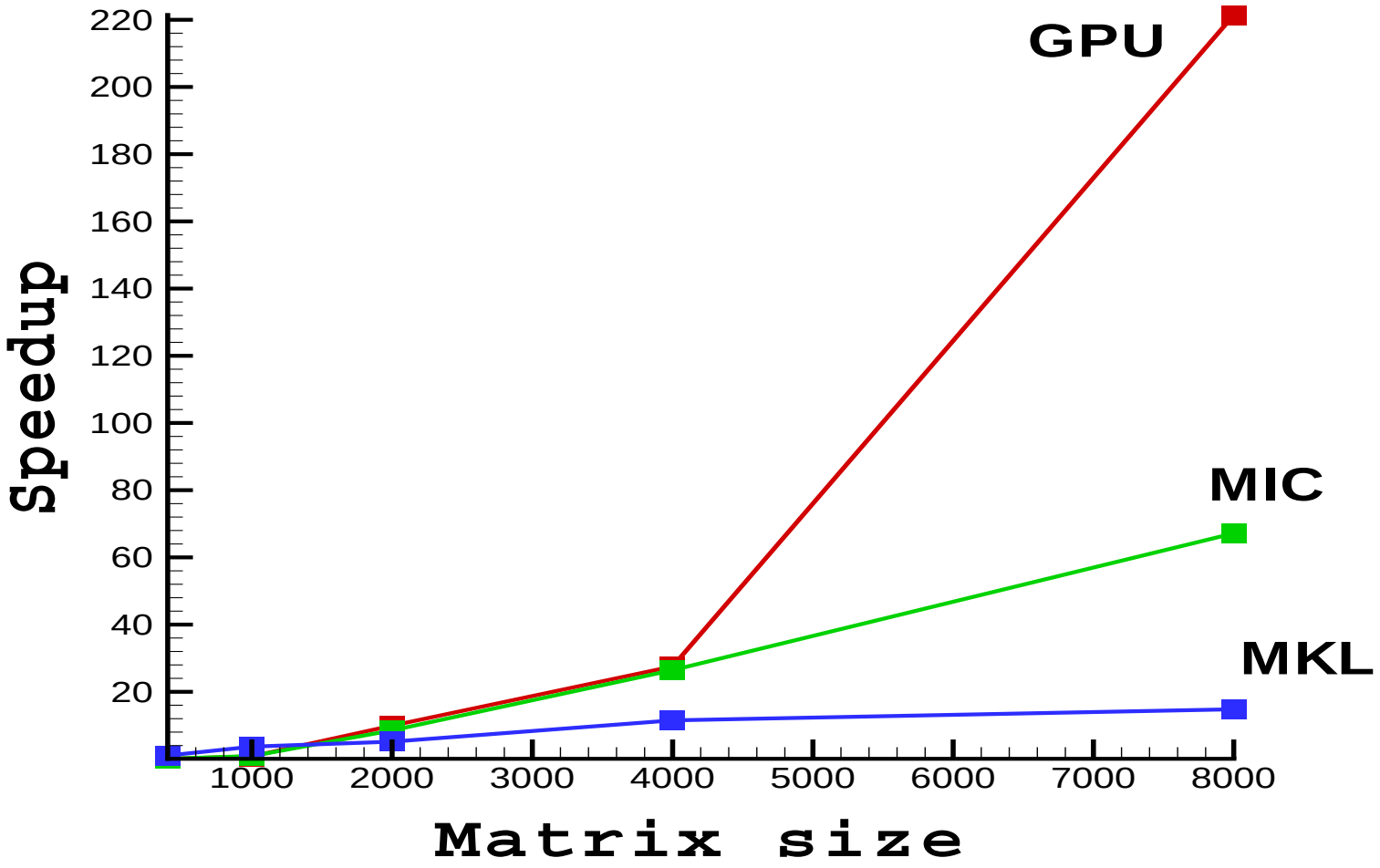
Matrix multiplication speedup based on GPU, MIC and MKL

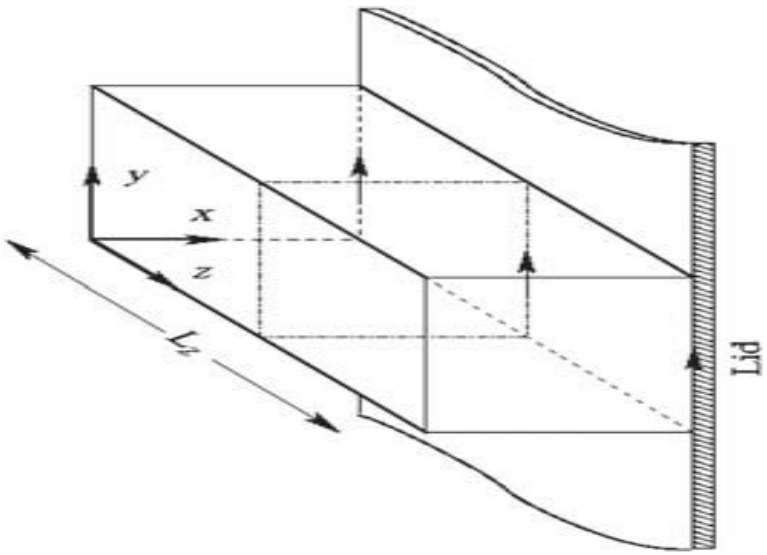


Outline

Backgrounds of PSEM

J.L. Guermond, P.D. Minev and A.J. Salgado.

- A new class of fractional step techniques for the incompressible Navier-Stokes equations using direction splitting
- Convergence analysis of a class of massively parallel direction splitting algorithms for the Navier-Stokes equations.
- Start-up flow in a three-dimensional lid-driven cavity by means of a massively parallel direction splitting algorithm.



The continuous problem

We shall restrict our attention to $\Omega = (-1,1)^d, d = 2,3$, and consider the time-dependent Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= \mathbf{f}\ , \quad \text{in } \Omega \times (0, T], \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega \times [0, T], \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}, \quad \text{in } [0, T], \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0, \quad \text{in } \Omega, \end{cases} \tag{1}$$

where ν is the viscosity coefficient, $\mathbf{u}$ and p stand for the velocity vector and the pressure respectively.

A second-order pressure-stabilization scheme

$$\begin{cases} \frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^n) - \nu\triangle\frac{\mathbf{u}^{n+1} + u^n}{2} + \nabla p^n = \mathbf{f}^{n+\frac{1}{2}}, \text{ in } \Omega, \\ \mathbf{u}^{n+1}|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

$$\begin{cases} -\triangle\phi^{n+1} = -\frac{1}{\Delta t}\nabla\cdot\mathbf{u}^{n+1}, \text{ in } \Omega, \\ \frac{\partial\phi^{n+1}}{\partial\mathbf{n}}|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

$$p^{n+1} = \phi^{n+1} + p^n - \chi\nu\nabla\cdot(\frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)), 0 \le \chi \le 1. \tag{4}$$

Error estimate:

$$\begin{aligned} &\|\mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}\|_{H^1} \le O(\Delta t^2), \\ &\|p(t^{n+1}) - p^{n+1}\|_{L^2} \le O(\Delta t), \quad \chi = 0, \\ &\|p(t^{n+1}) - p^{n+1}\|_{L^2} \le O(\Delta t^{\frac{3}{2}}), \quad 0 < \chi \le 1, \end{aligned}$$

$$\begin{cases} \frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^n) - \nu \triangle \frac{\mathbf{u}^{n+1} + u^n}{2} + \nabla p^n = \mathbf{f}^{n+\frac{1}{2}}, \text{ in } \Omega, \\ \mathbf{u}^{n+1}|_{\partial\Omega} = 0, \end{cases} \tag{5}$$

$$\begin{cases} (1 - \partial_{xx})(1 - \partial_{yy})\phi^{n+1} = -\frac{1}{\Delta t}\nabla \cdot \mathbf{u}^{n+1}, \text{ in } \Omega, \\ \frac{\partial \phi^{n+1}}{\partial \mathbf{n}}|_{\partial\Omega} = 0, \end{cases} \tag{6}$$

$$p^{n+1} = \phi^{n+1} + p^n - \chi\nu\nabla \cdot (\frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)), 0 \le \chi \le 1. \tag{7}$$

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Direction splitting scheme

- Velocity splitting:

$$\frac{\boldsymbol{\xi}^{n+1}-\mathbf{u}^n}{\Delta t}-\nu\Delta\mathbf{u}^n+\nabla p^{*,n+\frac{1}{2}}+\mathbf{NL}^{n+1}(\mathbf{u}^n,\mathbf{u}^{n-1})=\mathbf{f}^{n+\frac{1}{2}},\quad \boldsymbol{\xi}^{n+1}|_{\partial\Omega=\mathbf{0}},$$

$$\begin{aligned}
\frac{\boldsymbol{\eta}^{n+1}-\boldsymbol{\xi}^{n+1}}{\Delta t}-\frac{1}{2}\nu\partial_{xx}\left(\boldsymbol{\eta}^{n+1}-\mathbf{u}^n\right) &= \mathbf{0},\quad \boldsymbol{\eta}^{n+1}|_{x=\pm1}=\mathbf{0},\\
\frac{\boldsymbol{\zeta}^{n+1}-\boldsymbol{\eta}^{n+1}}{\Delta t}-\frac{1}{2}\nu\partial_{yy}\left(\boldsymbol{\zeta}^{n+1}-\mathbf{u}^n\right) &= \mathbf{0},\quad \boldsymbol{\zeta}^{n+1}|_{y=\pm1}=\mathbf{0}.\\
\frac{\mathbf{u}^{n+1}-\boldsymbol{\zeta}^{n+1}}{\Delta t}-\frac{1}{2}\nu\partial_{zz}\left(\mathbf{u}^{n+1}-\mathbf{u}^n\right) &= \mathbf{0},\quad \mathbf{u}^{n+1}|_{z=\pm1}=\mathbf{0}.
\end{aligned}$$

- Pressure splitting:

$$
\begin{aligned}
&\psi^{n+\frac{1}{2}} - \partial_{xx}\psi^{n+\frac{1}{2}} = -\frac{\nabla\cdot\mathbf{u}^{n+1}}{\Delta t}, && \partial_x\psi^{n+\frac{1}{2}}|_{x=\pm1} = 0,\\
&\varphi^{n+\frac{1}{2}} - \partial_{yy}\varphi^{n+\frac{1}{2}} = \psi^{n+\frac{1}{2}}, && \partial_y\varphi^{n+\frac{1}{2}}|_{y=\pm1} = 0;\\
&\phi^{n+\frac{1}{2}} - \partial_{zz}\phi^{n+\frac{1}{2}} = \varphi^{n+\frac{1}{2}}, && \partial_z\phi^{n+\frac{1}{2}}|_{z=\pm1} = 0;
\end{aligned}
\tag{9}
$$

and
$p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \phi^{n+\frac{1}{2}} - \chi\nu\nabla\cdot\left(\frac{1}{2}(\mathbf{u}^{n+1}+\mathbf{u}^{n})\right).$

Stability analysis

The solution with $\mathbf{f} = \mathbf{0}$ and $\chi = 0$ satisfies the following inequality:

$$\|\mathbf{u}_{\mathcal{N}}\|^2_{l^\infty(0,T;L^2)} + \frac{\nu}{2}\|\nabla \mathbf{u}_{\mathcal{N}}\|^2_{l^2(0,T;L^2)} + \Delta t\|p_{\mathcal{N}}\|^2_{l^2(-\frac{\Delta t}{2},T-\frac{\Delta t}{2};A)} + \frac{\Delta t\nu^2}{4}\|\partial_{xy}\mathbf{u}^{n+1}_{\mathcal{N}}\|^2_{l^2(0,T;L^2}$$
$$\leq c\left(\|\mathbf{u}_0\|^2_{L^2} + \frac{\nu}{2}\|\nabla \mathbf{u}_0\|^2_{L^2} + \Delta t\|p_0\|^2_A + \frac{\Delta t\nu^2}{4}\|\partial_{xy}\mathbf{u}_0\|^2_{L^2}\right),$$

The solution with $\mathbf{f} = \mathbf{0}$ and $0 < \chi \leq 1$ satisfies the following inequality:

$$\|\delta\mathbf{u}^{n+1}_{\mathcal{N}}\|^2_{l^\infty(0,T;L^2)} + \frac{\nu}{2}\|\nabla \times \delta\mathbf{u}^{n+1}_{\mathcal{N}}\|^2_{l^2(0,T;L^2)} + \frac{\nu}{2}(1-\chi)\|\nabla \cdot \delta\mathbf{u}^{n+1}_{\mathcal{N}}\|^2_{l^2(0,T;L^2)}$$
$$+\frac{\Delta t\nu^2}{4}\|\partial_{xy}\delta\mathbf{u}^{n+1}_{\mathcal{N}}\|^2_{l^2(0,T;L^2)} + \Delta t\|p^{n+\frac{1}{2}}_{\mathcal{N}} - p^{n-\frac{1}{2}}_{\mathcal{N}}\|^2_{l^2(-\frac{\Delta t}{2},T-\frac{\Delta t}{2};A)} + \chi\nu\|\nabla \cdot \mathbf{u}^{n+1}_{\mathcal{N}}\|^2_{l^2(0,T}$$
$$\leq c\big(\|\delta\mathbf{u}^1_{\mathcal{N}}\|^2_{L^2} + \frac{\nu}{2}\|\nabla \times \delta\mathbf{u}^1_{\mathcal{N}}\|^2_{L^2} + \frac{\nu}{2}(1-\chi)\|\nabla \cdot \delta\mathbf{u}^1_{\mathcal{N}}\|^2_{L^2}$$
$$+\frac{\Delta t\nu^2}{4}\|\partial_{xy}\delta\mathbf{u}^1_{\mathcal{N}}\|^2_{L^2} + \Delta t\|p^{\frac{1}{2}}_{\mathcal{N}}\|^2_A + \chi\nu\|\nabla \cdot \mathbf{u}^1_{\mathcal{N}}\|^2_{L^2}\big),$$

where c is a constant independent of discretization parameters.

1D poisson equation

$$\begin{aligned} &\alpha u - \beta \partial_x^2 u = f, x \in \Lambda = (-1, 1) \\ &u(\pm 1) = 0. \end{aligned} \tag{10}$$

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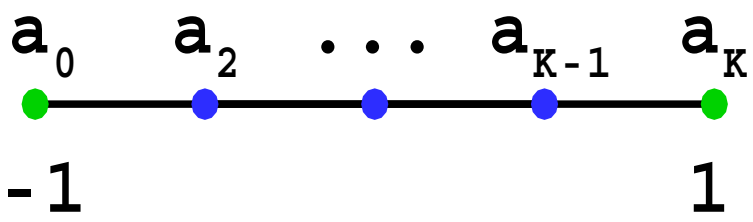
- Domain decomposition

$\Lambda = \bigcup_k \Lambda^k$,

$\Lambda^k = (a_{k-1}, a_k), k = 1, 2, \cdots, K$,

$-1 = a_0 < a_1 < \cdots < a_K = 1.$

$h_k = a_k - a_{k-1}.$



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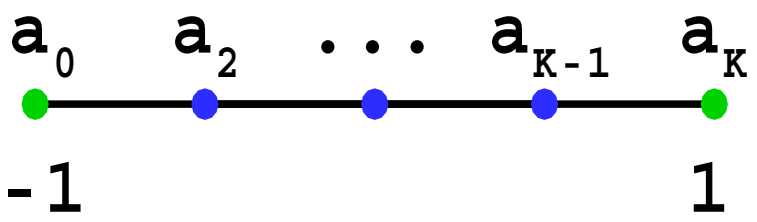

- Spectral element space

$$X_{\mathcal{N}}^0 = \left\{ u_{\mathcal{N}} \mid_{\Lambda^i} \in \mathbb{P}_N, u_{\mathcal{N}} \in C(\cup_{i=1}^K \Lambda^i), u_{\mathcal{N}}(\pm 1) = 0 \right\}, \tag{11}$$

where the $\mathcal{N}$ denotes the integer pair (N, K).

- Construct the basis function:

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$$\phi_j^k(x) = \begin{cases} \dfrac{1}{\sqrt{4j+6}}(L_j(\hat{x}_k) - L_{j+2}(\hat{x}_k)), & x \in \Lambda^k, \\ 0, & \text{others}, \end{cases} \quad j = 0, 1, \cdots, N-2, \quad k = 1, 2, \cdots$$

$$\mathring{X}_{\mathcal{N}} = \{v; v_{|\Lambda^k} \in \mathring{V}_N^k, k = 1, 2, \cdots, K\},$$

$$eex \quad bex \qquad \mathring{V}_N^k = \text{span}\{\phi_0^k(x), \phi_1^k(x), \cdots, \phi_{N-2}^k(x)\}, \quad k = 1, 2, \cdots, K.$$

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★

$$\varphi_k(x) = \begin{cases} \dfrac{1}{2}(L_0(\hat{x}_k) + L_1(\hat{x}_k)), & x \in \Lambda^k, \\ \dfrac{1}{2}(L_0(\hat{x}_{k+1}) - L_1(\hat{x}_{k+1})), & x \in \Lambda^{k+1}, \\ 0, & \text{others}, \end{cases} \tag{12}$$

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$$X_{\mathcal{N}}^0 = \mathring{X}_{\mathcal{N}} \oplus \text{span}\{\varphi_1, \varphi_2, \cdots, \varphi_{K-1}\}.$$

With the Neumann boundary condition,

$$X_{\mathcal{N}} = \mathring{X}_{\mathcal{N}} \oplus \text{span}\{\varphi_0, \varphi_1, \cdots, \varphi_K\}.$$

where the φ_0 and φ_K is defined as follows:

$$\varphi_0(x) = \begin{cases} \frac{1}{2}(L_0(\hat{x}_1) - L_1(\hat{x}_1)), & x \in \Lambda^1, \\ 0, & \text{others}, \end{cases}$$

$$\varphi_K(x) = \begin{cases} \frac{1}{2}(L_0(\hat{x}_K) + L_1(\hat{x}_K)), & x \in \Lambda^K, \\ 0, & \text{others}, \end{cases}$$

and $\varphi_k, k = 1, 2, \cdots, K-1$ is defined in (12).

Matrix statements

Expanding $u_{\mathcal{N}}$ as

$$u_{\mathcal{N}}(x) = \sum_{k=1}^{K} \sum_{i=0}^{N-2} \hat{u}_i^k \phi_i^k(x) + \sum_{k=1}^{K-1} \bar{u}^k \varphi^k(x),$$

and denoting

$$U_i = (\hat{u}_0^1, \cdots, \hat{u}_{N-2}^1, \cdots, \hat{u}_0^K, \cdots, \hat{u}_{N-2}^K)^T, \quad U_e = (\bar{u}^1, \cdots, \bar{u}^{K-1})^T$$

$$F_i = (\hat{f}_0^1, \cdots, \hat{f}_{N-2}^1, \cdots, \hat{f}_0^K, \cdots, \hat{f}_{N-2}^K)^T, \quad F_e = (\bar{f}^1, \cdots, \bar{f}^{K-1})^T$$

$$(A_{ii})^k_{mn} = b(\phi^k_n, \phi^k_m), \quad n, m = 0, 1, \cdots, N-2, \quad k = 1, 2, \cdots, K;$$

$$A_{ii} = diag(A^1_{ii}, A^2_{ii}, \cdots, A^K_{ii}),$$

$$(A_{ie})^k_{ln} = b(\varphi^l, \phi^k_n), n = 0, 1, \cdots, N-2, k = 1, 2, \cdots, K, l = 1, 2, \cdots, K$$

$$A_{ie} = diag(A^1_{ie}, A^2_{ie}, \cdots, A^K_{ie}),$$

$$(A_{ee})_{lk} = b(\varphi^k, \varphi^l), \quad k, l = 1, 2, \cdots, K-1;$$

$$b(u, v) = \alpha(u, v) + \beta(\partial_x u, \partial_x v).$$

Parallel implementation

In each direction d, $d \in \{1, 2, 3\}$:

$$\begin{pmatrix} A_{ii} & A_{ie} \\ A_{ei} & A_{ee} \end{pmatrix} \begin{pmatrix} U_i \\ U_e \end{pmatrix} = \begin{pmatrix} F_i \\ F_e \end{pmatrix}.$$

$$\Downarrow$$

$$\begin{pmatrix} A_{ii} & A_{ie} \\ \mathbf{0} & A_{ee} - A_{ei}A_{ii}^{-1}A_{ie} \end{pmatrix} \begin{pmatrix} U_i \\ U_e \end{pmatrix} = \begin{pmatrix} F_i \\ F_e - A_{ei}A_{ii}^{-1}F_i \end{pmatrix},$$

$$\Downarrow$$

$$(A_{ee} - A_{ei}A_{ii}^{-1}A_{ie})U_e = F_e - A_{ei}A_{ii}^{-1}F_i$$
$$A_{ii}U_i = F_i - A_{ie}U_e.$$

- Each 1D problem gives a block tridiagonal matrix A_{ii}.

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- Direct solution with Schur complement (direct solution on proc 0) $(A_{ee} - A_{ei}A_{ii}^{-1}A_{ie})U_e = F_e - A_{ei}A_{ii}^{-1}F_i$.

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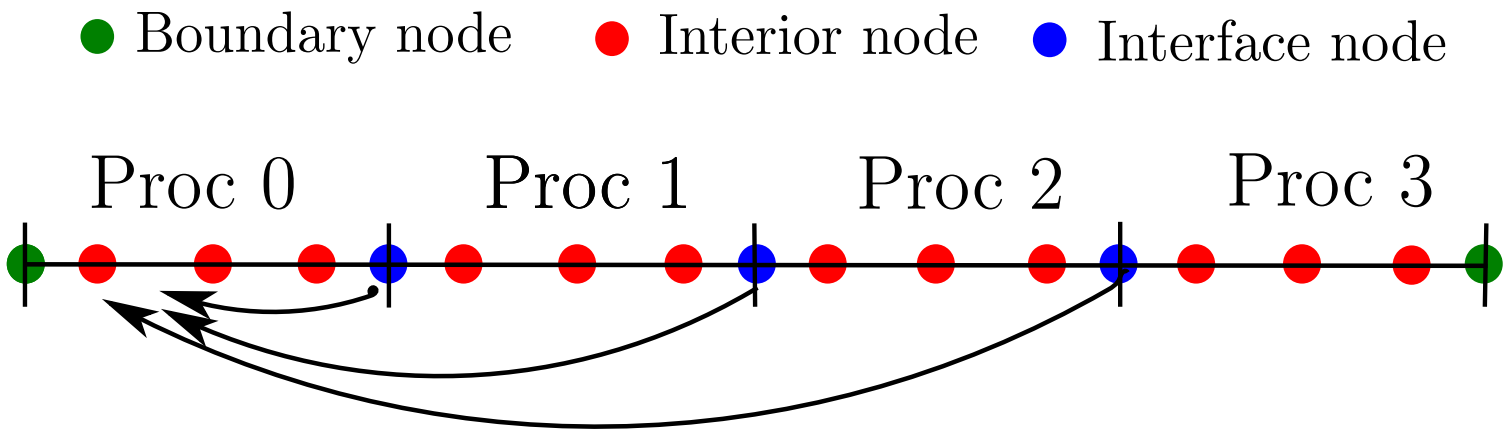
- Number of communications =number of interfaces $A_{ii}^k U_i^k = F_i^k - A_{ie}^k U_e$.



Expression of one order derivative on numerical function

$$\partial_x u_{\mathcal{N}}(x_N^k) = \Big(\sum_{p=0}^{N} D_{Np} u_p^k + \sum_{p=0}^{N} D_{Np} u_p^{k+1}\Big) / \Big(\frac{1}{2h_k} + \frac{1}{2h_{k+1}}\Big)$$

$$\begin{aligned}
\partial_x u_{\mathcal{N}}(x_N^k)(\omega_N^k + \omega_0^{k+1}) &= \partial_x u_{\mathcal{N}}(x_N^k)\omega_N^k + \partial_x u_{\mathcal{N}}(x_N^k)\omega_N^{k+1} \\
&= (\partial_x u_{\mathcal{N}}, \phi_k)_{\mathcal{N},\Omega} \\
&= (\partial_x u_{\mathcal{N}}, \psi_N^k)_{N,\Lambda^k} + (\partial_x u_{\mathcal{N}}, \psi_0^{k+1})_{N,\Lambda^{k+1}} \\
&= -(u_{\mathcal{N}}, \partial_x \psi_N^k)_{N,\Lambda^k} + u_{\mathcal{N}} \psi_N^k|_{\partial\Omega_k} \\
&\quad -(u_{\mathcal{N}}, \partial_x \psi_0^{k+1})_{N,\Lambda^{k+1}} + u_{\mathcal{N}} \psi_0^{k+1}|_{\partial\Omega_{k+1}}
\end{aligned}$$

Expression of second order derivative on numerical function

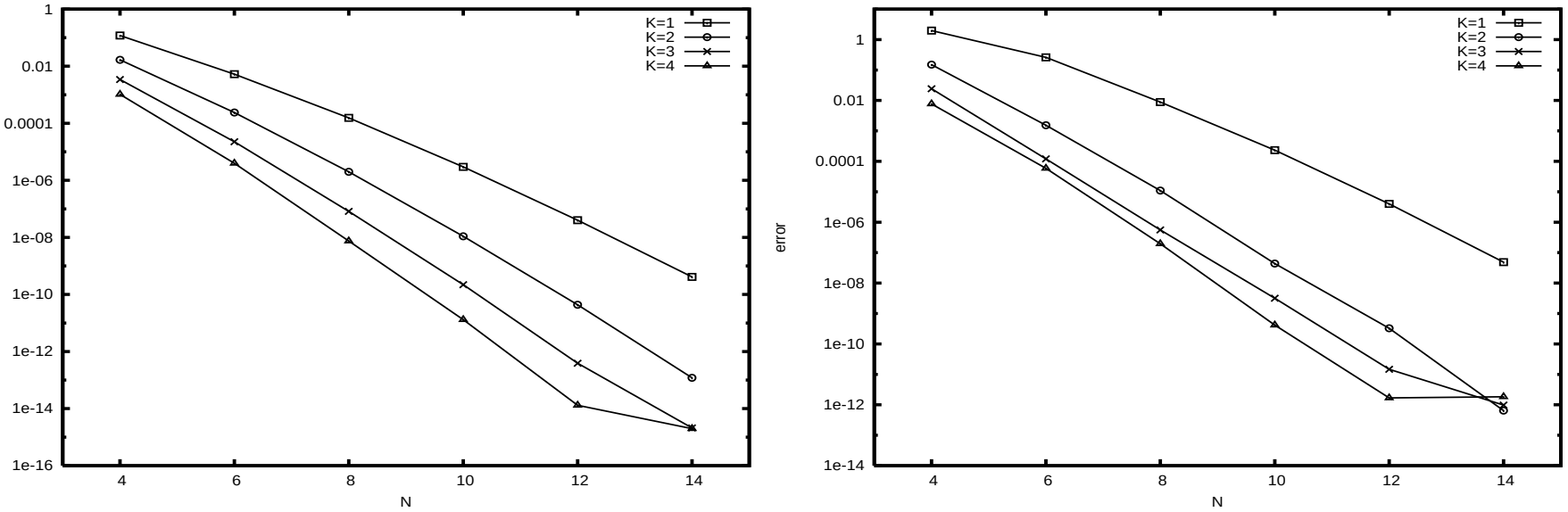
$$\partial_{xx} u_{\mathcal{N}}(x_N^k) = \Big(\sum_{p=0}^{N} D_{Np}\partial_x u_{\mathcal{N}}(x_p^k) + \sum_{p=0}^{N} D_{Np}\partial_x u_{\mathcal{N}}(x_p^{k+1})\Big)/\Big(\frac{1}{2h_k} + \frac{1}{2h_{k+1}}\Big)$$

$$\begin{aligned}
\partial_{xx} u_{\mathcal{N}}(x_N^k)(\omega_N^k + \omega_0^{k+1}) &= (\partial_x u_{\mathcal{N}}, \psi_N^k)_{N,\Lambda^k} + (\partial_x u_{\mathcal{N}}, \psi_N^{k+1})_{N,\Lambda^{k+1}} \\
&= -(\partial_x u_{\mathcal{N}}, \partial_x \psi_N^k)_{N,\Lambda^k} + \partial_x u_{\mathcal{N}} \psi_N^k|_{\partial\Omega_k} \\
&\quad -(u_{\mathcal{N}}, \partial_x \psi_0^{k+1})_{N,\Lambda^{k+1}} + \partial_x u_{\mathcal{N}} \psi_0^{k+1}|_{\partial\Omega_{k+1}}
\end{aligned}$$

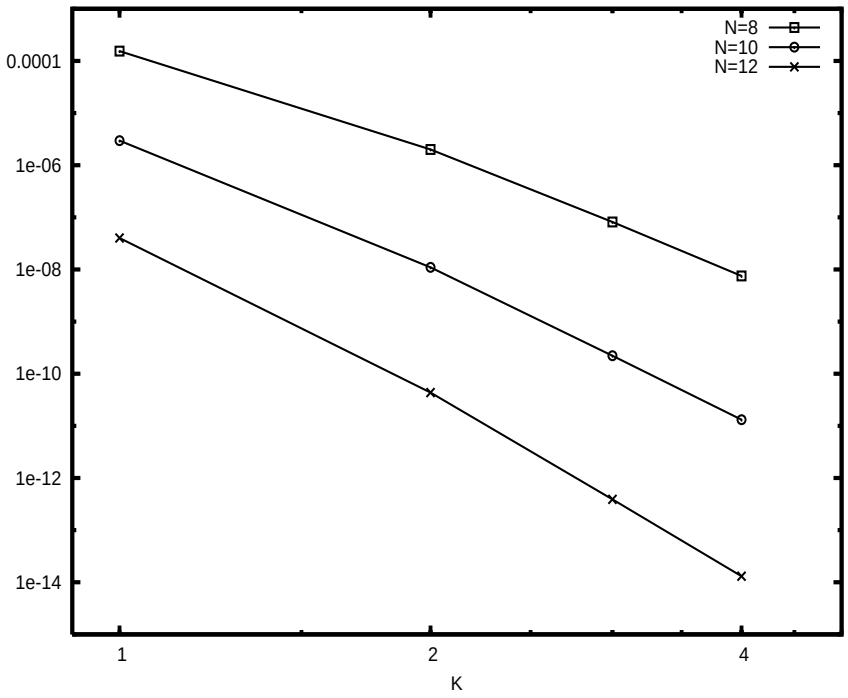
Three dimensional example

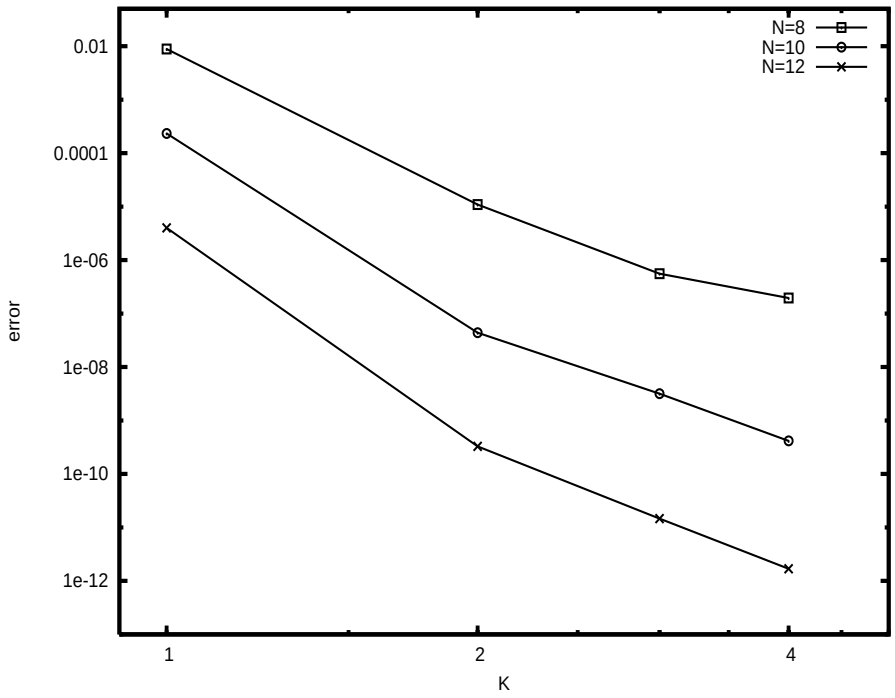
$$\mathbf{u}(x,y,t) = \pi \sin t \begin{pmatrix} \sin^2 \pi x \sin 2\pi y \sin 2\pi z, \\ \sin 2\pi x \sin^2 \pi y \sin 2\pi z, \\ -2 \sin 2\pi x \sin 2\pi y \sin^2 \pi z \end{pmatrix}. \tag{13}$$

$$p(x,y,t) = \sin t \cos \pi x \cos \pi y \sin \pi y.$$

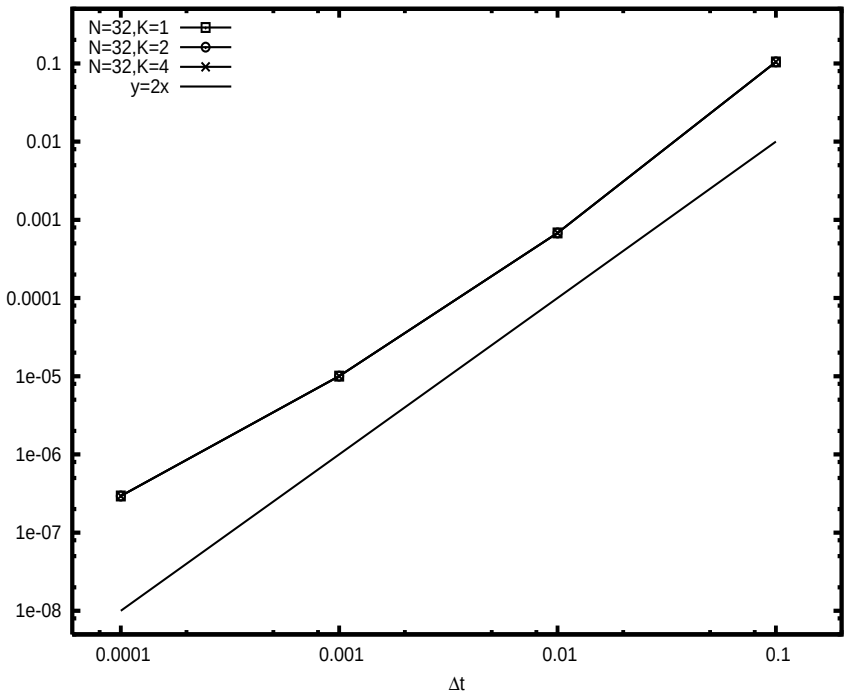


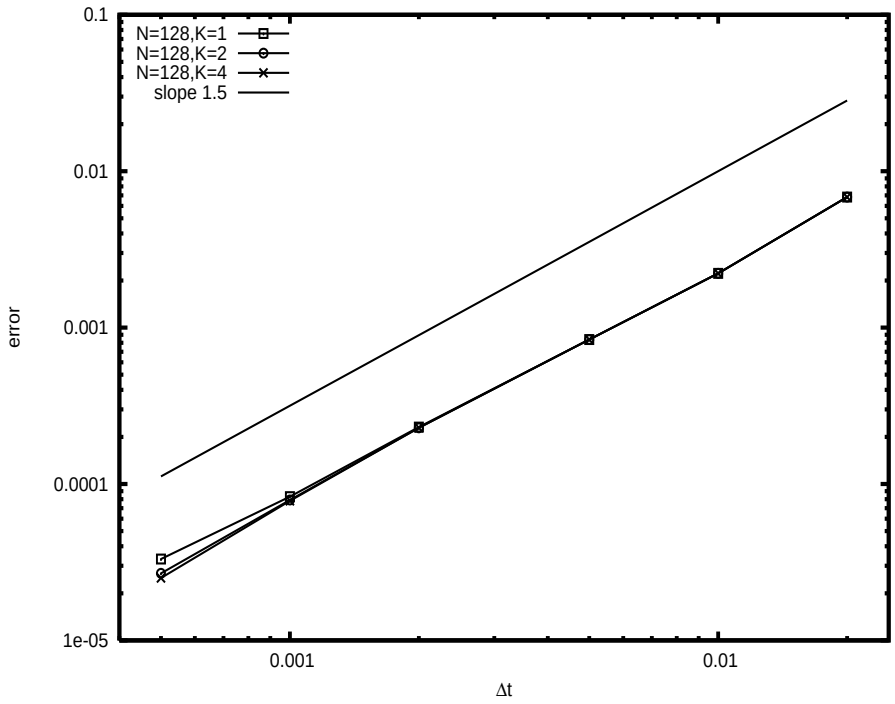
Velocity (left) and pressure (right) error in L^2-norm as a function of N.





Velocity (left) and pressure (right) error in L^2-norm as a function of K.



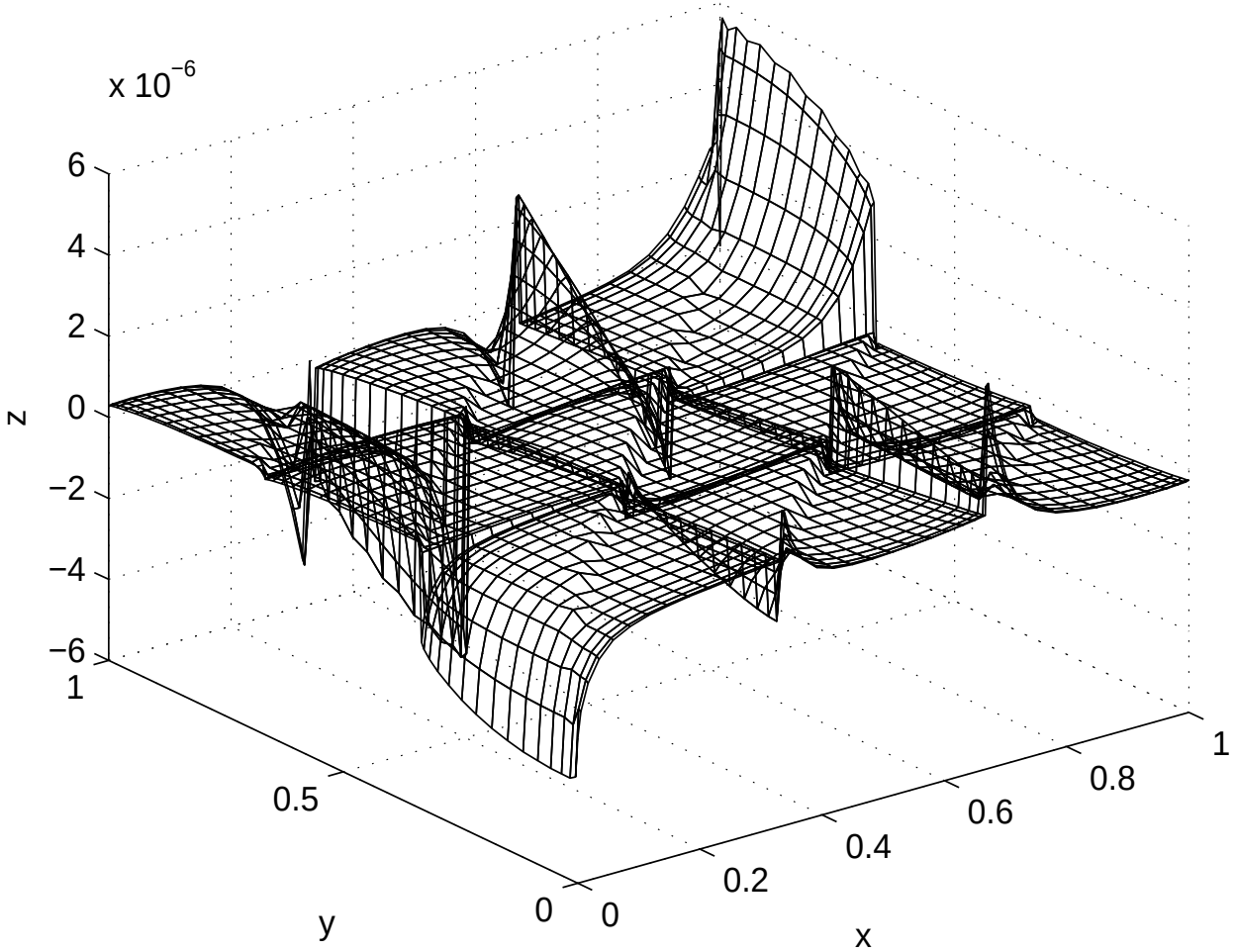


Velocity (left) and pressure (right) error in L^2-norm as a function of Δt.

Parallel efficiencies with weak scaling

N	16	32	64	128
# procs	parallel efficiency	parallel efficiency	parallel efficiency	parallel efficiency
1	–	–	–	–
64	0.8101	0.8608	0.7498	0.8152
216	0.6729	0.8057	0.7307	0.7902
512	0.5819	0.7247	0.7037	0.7467
1000	0.5396	0.6929	0.6992	0.6749

Divergence of the solution with $N = 16, K = 3, \Delta t = 0.0001$ at $T = 1$.



Navier-Stokes equations with variable viscosity

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (\nu(x,y,t)\nabla \mathbf{u}) + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = & \mathbf{f}\ , \quad \text{in } \Omega \times (0,T], \\ \nabla \cdot \mathbf{u} = & 0, \quad \text{in } \Omega \times [0,T], \\ \mathbf{u}|_{\partial\Omega} = & \mathbf{0}, \quad \text{in } [0,T], \\ \mathbf{u}|_{t=0} = & \mathbf{u}_0, \quad \text{in } \Omega, \end{cases} \tag{14}$$

where $\nu(x,y,t) > 0$ is the viscosity which can vary in time and in space.

- Velocity splitting:

$$
\begin{aligned}
&\frac{\mathbf{u}^{n+\frac{1}{2}} - \mathbf{u}^{n}}{\frac{1}{2}\Delta t} - \partial_x\left(\nu^{n+\frac{1}{2}}\partial_x \mathbf{u}^{n+\frac{1}{2}}\right) - \partial_y\left(\nu^{n+\frac{1}{2}}\partial_y \mathbf{u}^{n}\right) + \nabla p^{*,n+\frac{1}{2}} \\
&= \mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+\frac{1}{2}}|_{x=\pm 1} = \mathbf{0}, \\
&\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} - \partial_x\left(\nu^{n+\frac{1}{2}}\partial_x \mathbf{u}^{n+\frac{1}{2}}\right) - \partial_y\left(\nu^{n+\frac{1}{2}}\partial_y \mathbf{u}^{n+1}\right) + \nabla p^{*,n+\frac{1}{2}} \\
&= \mathbf{f}^{n+\frac{1}{2}}, \mathbf{u}^{n+1}|_{y=\pm 1} = \mathbf{0},
\end{aligned}
\tag{15}
$$

where $\nu^{n+\frac{1}{2}} := \nu(x, y, t^{n+\frac{1}{2}})$.

- Pressure splitting:

$$
\begin{aligned}
\psi^{n+\frac{1}{2}} - \partial_{xx}\psi^{n+\frac{1}{2}} &= -\frac{\nabla\cdot\mathbf{u}^{n+1}}{\Delta t}; \quad \partial_x\psi^{n+\frac{1}{2}}|_{x=\pm 1} = 0, \\
\phi^{n+\frac{1}{2}} - \partial_{yy}\phi^{n+\frac{1}{2}} &= \psi^{n+\frac{1}{2}}; \qquad \partial_y\phi^{n+\frac{1}{2}}|_{y=\pm 1} = 0;
\end{aligned}
\tag{16}
$$

$$
p^{n+\frac{1}{2}} = p^{n-\frac{1}{2}} + \phi^{n+\frac{1}{2}} - \chi\nu\nabla\cdot\left(\frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^{n})\right). \tag{17}
$$

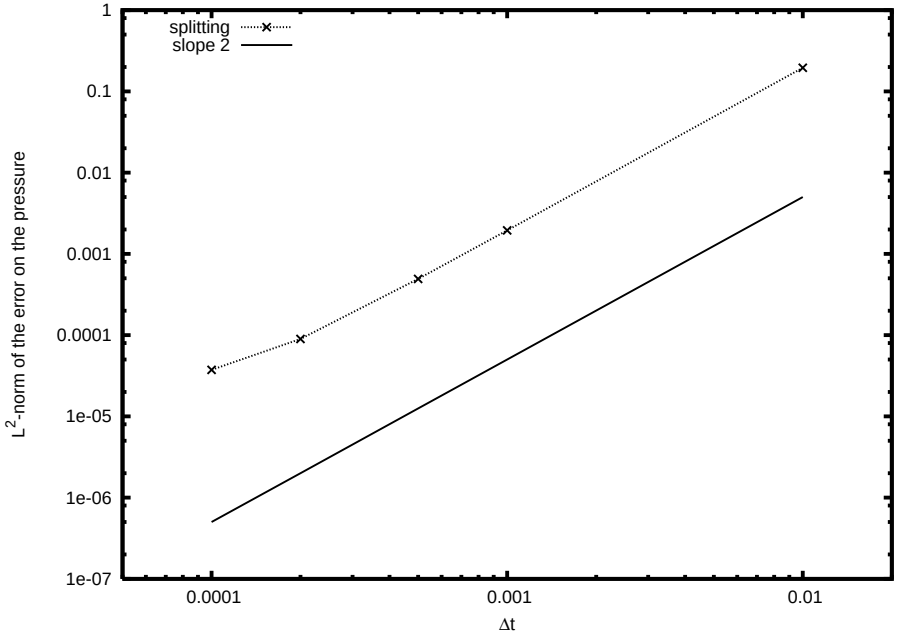
The solution with $\chi = 0$ satisfies the following estimate for all $T \geq 0$:

$$\begin{aligned} &\|\mathbf{u}\|^2_{l^\infty(0,T;L^2)} + \frac{1}{2}\|\sqrt{\nu}\nabla\mathbf{u}\|^2_{l^2(0,T;L^2)} + \Delta t\|p\|_{l^2(-\frac{\Delta t}{2},T-\frac{\Delta t}{2},A)} \\ &\leq \|\mathbf{u}_0\|^2_{L^2} + \Delta t^2\|p_0\|^2_A + \frac{1}{2}\Delta t\|\sqrt{\nu}\nabla\mathbf{u}_0\|^2_{L^2}. \end{aligned} \tag{18}$$

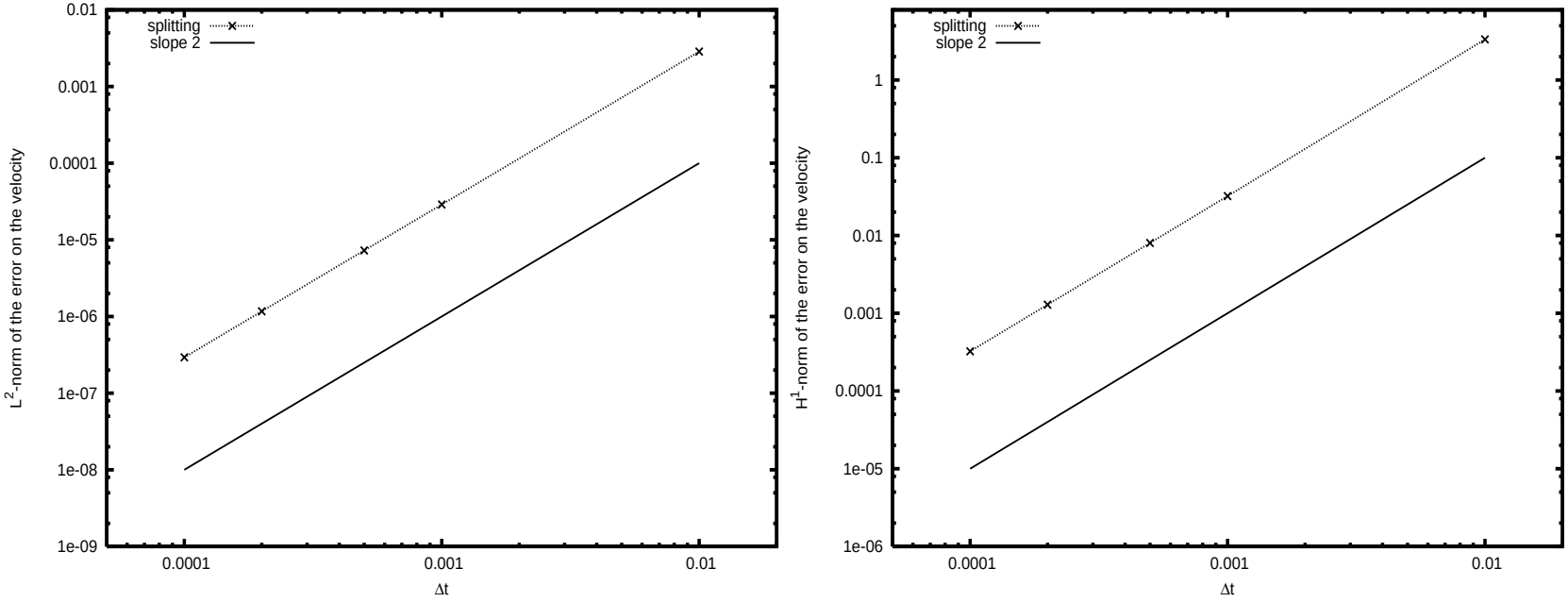
$$\mathbf{u}(x, y, t) = \pi \sin t(\sin 2\pi y \sin^2 \pi x, -\sin 2\pi x \sin^2 \pi y),$$

$$p(x, y, t) = \sin t \cos \pi x \sin \pi y,$$

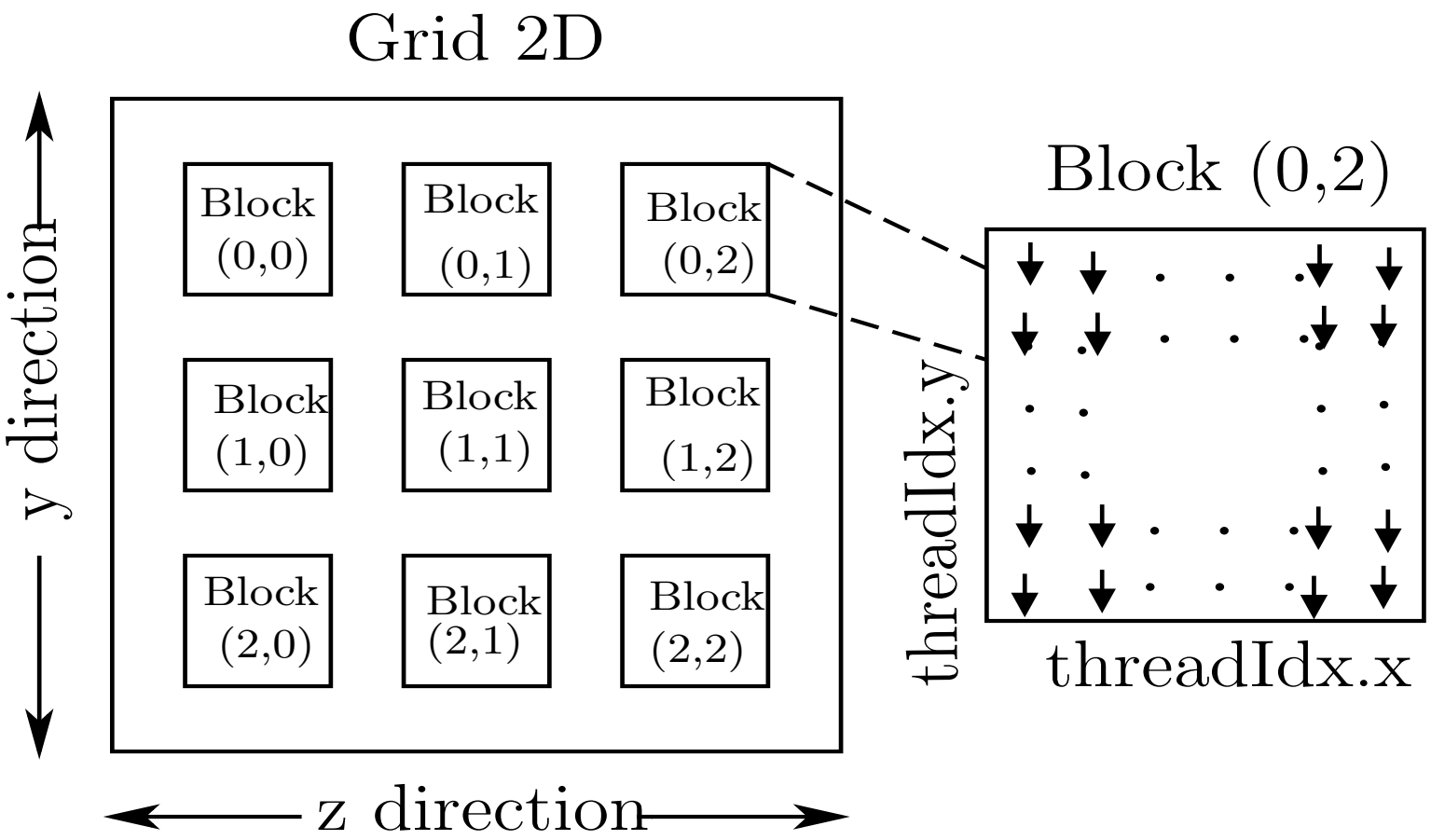
$$\nu(x, y, t) = (\sin^2 t + 1)(x + 2)(y + 2).$$



L^2-norm error on the pressure at $T = 1$.



L^2-norm (left) and H^1-norm (right) error on the velocity at $T = 1$.



$int \quad nx = blockDim.x * blockIdx.x + threadIdx.x;$

$int \quad ny = blockDim.y * blockIdx.y + threadIdx.y;$

CUDA threads organization for each direction solver.

Speedup

N	8	16	32	64	128	256	512	1024	2048	4096
2D PE	–	–	–	3	5	10	20	29	32	31
3D PE	–	4	15	47	56	59				
2D NSE	–	–	–	3	6	9	14	21	25	

without using CUBLAS and LAPACK

N	8	16	32	64	128	256	512	1024	2048	4096
2D PE	–	–	–	–	4	7	15	24	71	98
3D PE	–	–	13	43	60	120				

- Summary
  - An efficient parallel algorithm for the time dependent incompressible Navier-Stokes equations is developed.
  - We achieve about 30 times speedup for the Navier-Stokes Equation by spectral method based on GPU.

- Summary
  - An efficient parallel algorithm for the time dependent incompressible Navier-Stokes equations is developed.
  - We achieve about 30 times speedup for the Navier-Stokes Equation by spectral method based on GPU.
- Future work
  - Treat multiphase flows and/or complex fluids for which the Navier-Stokes solver plays an important role.
  - Treat complex domain and turbulence flow.

Thank You