Solitons and Vortices in BECs: Some Recent Developments

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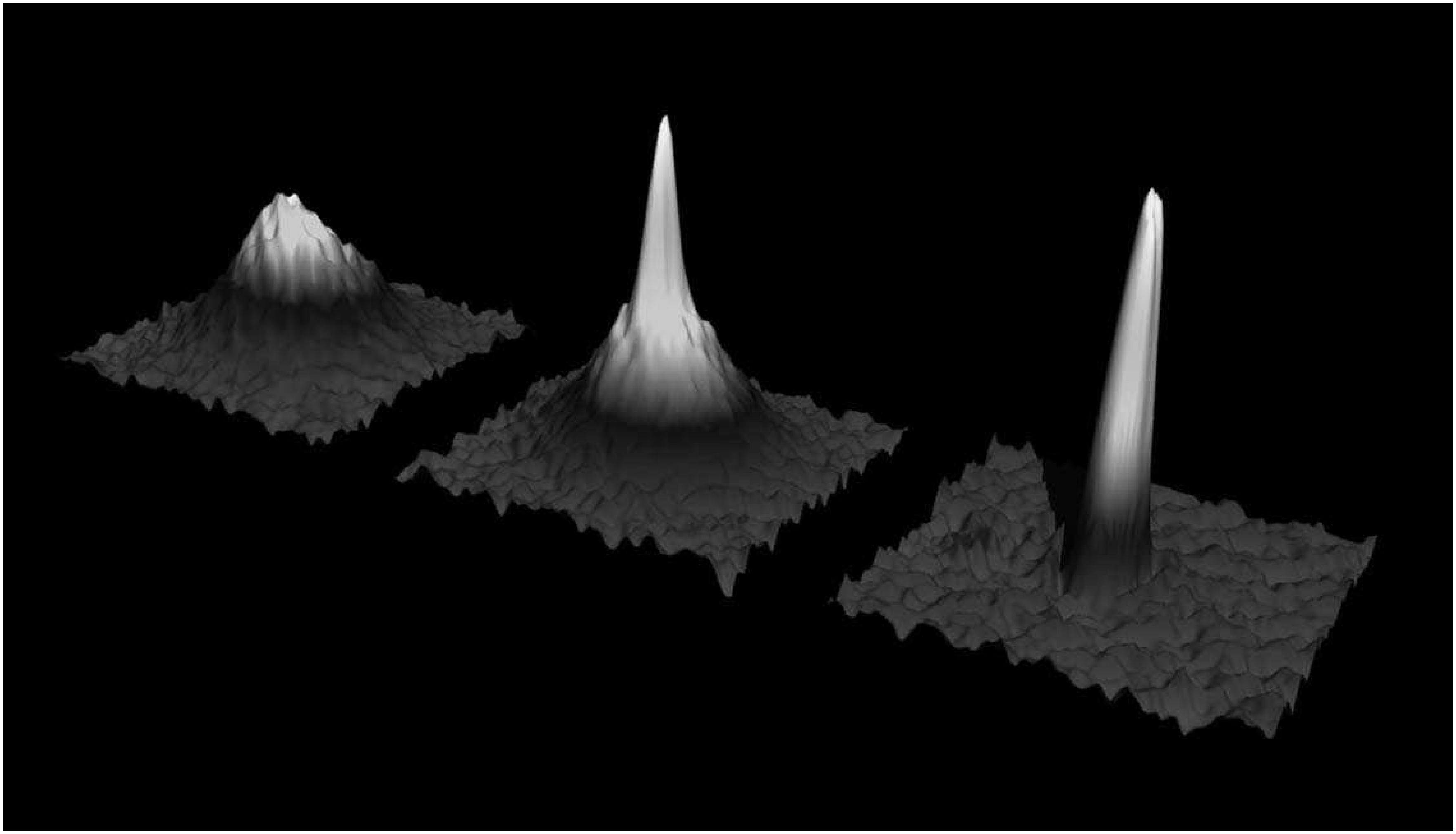
References

- 1d GPE:
  - Contemp. Math. **473**, 159 (2008); ZAMP **59**, 559 (2008)
  - Phys. Rev. Lett. **104**, 244302 (2010); PRA **81**, 053618 (2010)
  - Nonlinearity **23**, 1753 (2010); SIAP **73**, 1368 (2013).
  - arXiv:1403.1038, arXiv:1402.1895, arXiv:1407.8049, arXiv:1407.7965,
- Experimental Results:
  - Phys. Rev. Lett. **101**, 130401 (2008)
  - Phys. Rev. A **81**, 063604 (2010)
  - Phys. Rev. A **84**, 011605 (2011), NJP **15**, 113028 (2013)
- Generalizations to Vortices:
  - AMRX 1-38 (2012); Nonlinearity **24**, 1271 (2011).
  - J. Phys. B **43**, 155303 (2010); Phys. Rev. A **82**, 013646 (2010).
  - EPL **93**, 20008 (2011); EPL **103**, 20002 (2013)
  - PRA **86**, 053601 (2012); Phys. Rev. Lett. **110**, 225301 (2013).
  - Phys. Rev. Lett. **110**, 264101 (2013); PRL **111**, 235301(2013).

Brief Introduction to BECs

- 1924: S. Bose and A. Einstein realize that Bose statistics predicts a Maximum Atom Number in the Excited States: a Quantum Phase Transition.
- 1995: E. Cornell, C. Wieman and W. Ketterle realize BEC in a dilute gas of ^{87}Rb and ^{23}Na: 2001 Nobel Prize.
- Today:
  - $\sim$ 35 Experimental Groups have achieved BEC (in 100-10^8 atoms of Rb, Li, Na, H).
  - O(10^3) Theoretical and O(10^2) Experimental papers ! Check out: http://amo.phy.gasou.edu/bec.html/bibliography.html



Mean-Field Models of BEC: why do we care ?

BEC

- Many Body Hamiltonian

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^\dagger \left[-\frac{\hbar^2}{2m}\Delta + V_{\text{ext}}(\mathbf{r}) \right] \hat{\Psi} + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\Psi}^\dagger(\mathbf{r}) \hat{\Psi}^\dagger(\mathbf{r}') V(\mathbf{r}-\mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}) \tag{1}$$

- Bogoliubov Decomposition:

$$\hat{\Psi} = \Phi(\mathbf{r}, t) + \hat{\Psi}'(\mathbf{r}, t) \tag{2}$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Phi + V_{\text{ext}}(\mathbf{r})\Phi + g|\Phi|^2\Phi \tag{3}$$

- for dilute, cold, binary collision gas.

- But: This is 3D NLS with a Potential: GP !

Low Dimensional Reductions

- 1d Magnetic Trap and/or Optical Lattice

$$V(x) = \frac{1}{2}\Omega^2 x^2 + V_0 \sin^2(kx + \theta) \tag{4}$$

- 2d Magnetic Trap and/or Optical Lattice

$$V(x,y) = \frac{1}{2}\left(\Omega_x^2 x^2 + \Omega_y^2 y^2\right) + V_0\left(\sin^2(kx+\theta) + \sin^2(ky+\theta)\right) \tag{5}$$

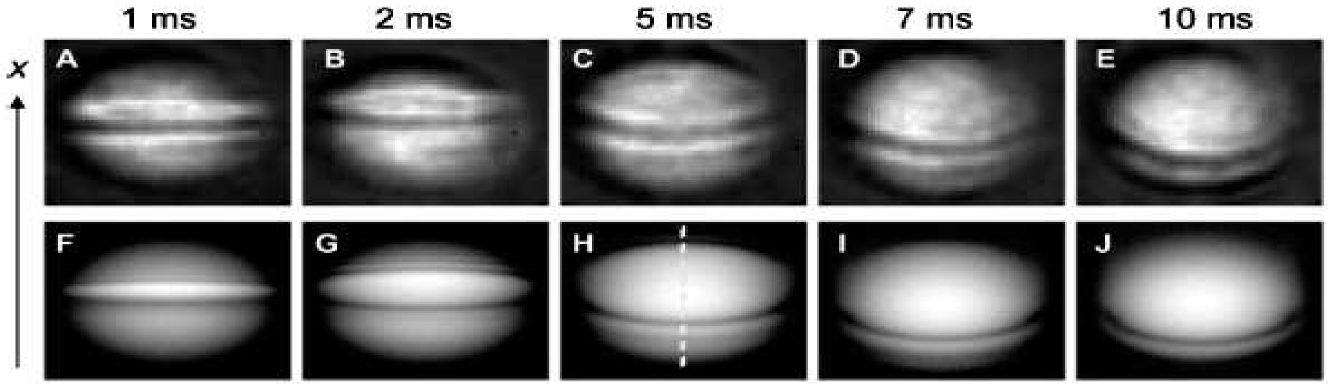
- Discrete Models: Reduction through Wannier functions (WF)

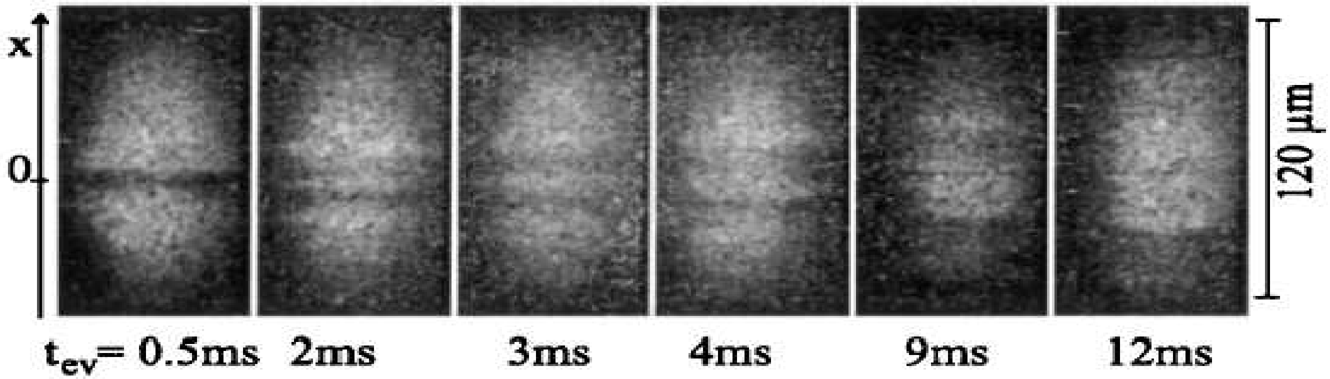
$$\psi(x,t) = \sum_{n\alpha} c_{n,\alpha}(t) w_{n,\alpha}(x) \tag{6}$$

where the WF w_n of band α are expressed in terms of the Bloch functions $\phi_{k,\alpha}$ as:

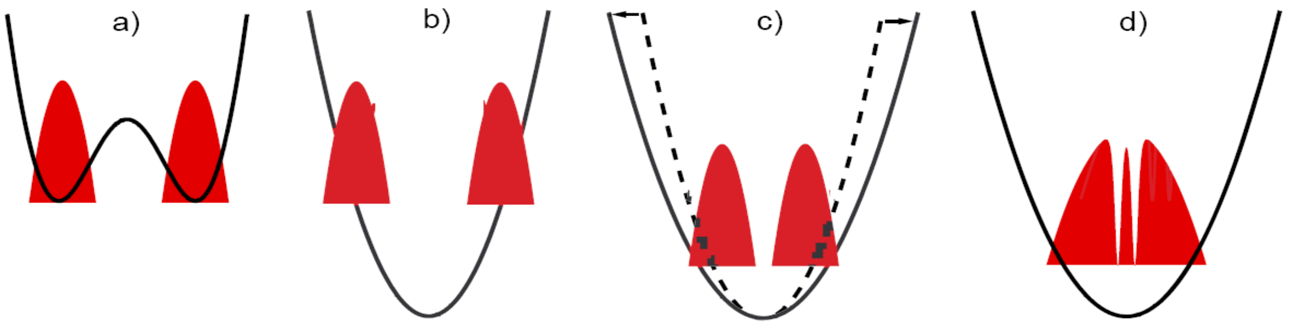
$$w_\alpha(x - nL) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \varphi_{k,\alpha}(x) e^{-inkL} dk. \tag{7}$$

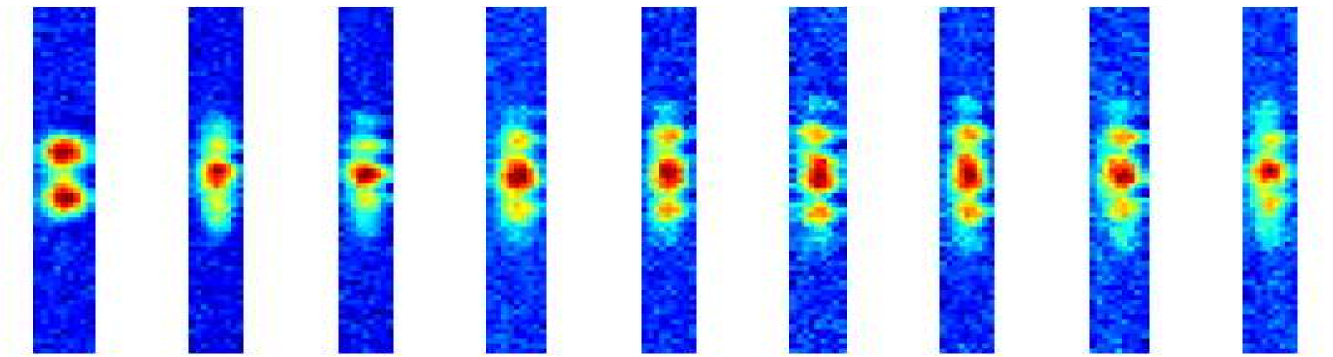
Early Experiments



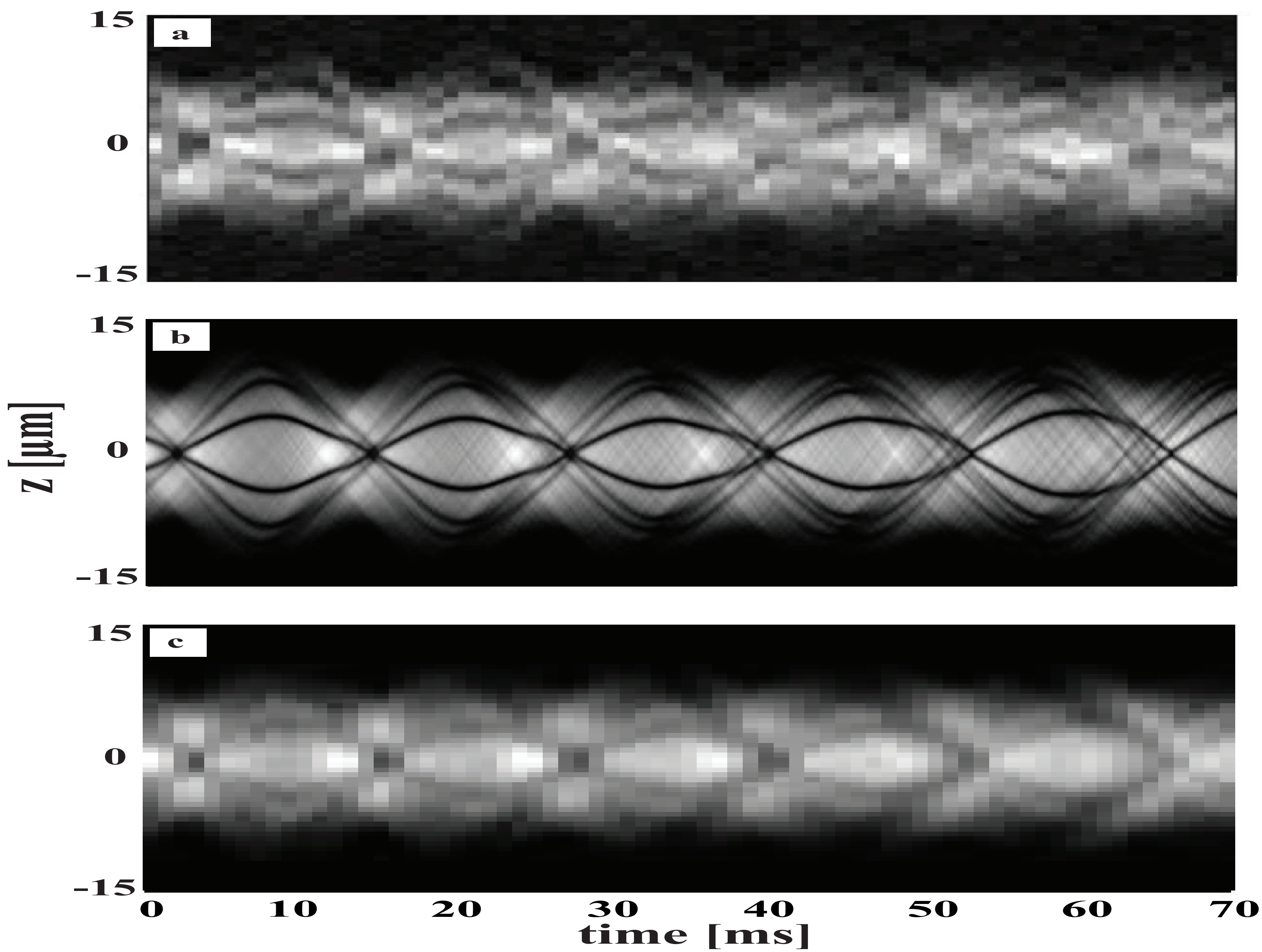


A Specific Motivation: Dark Soliton Experiments in Heidelberg

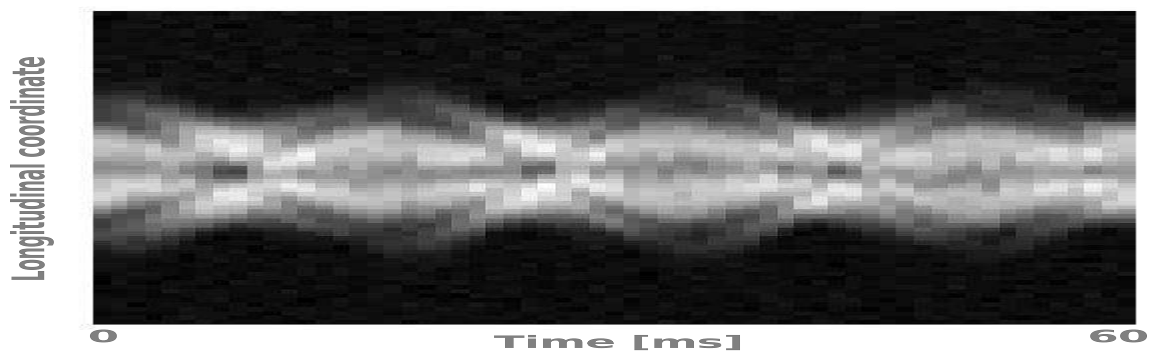


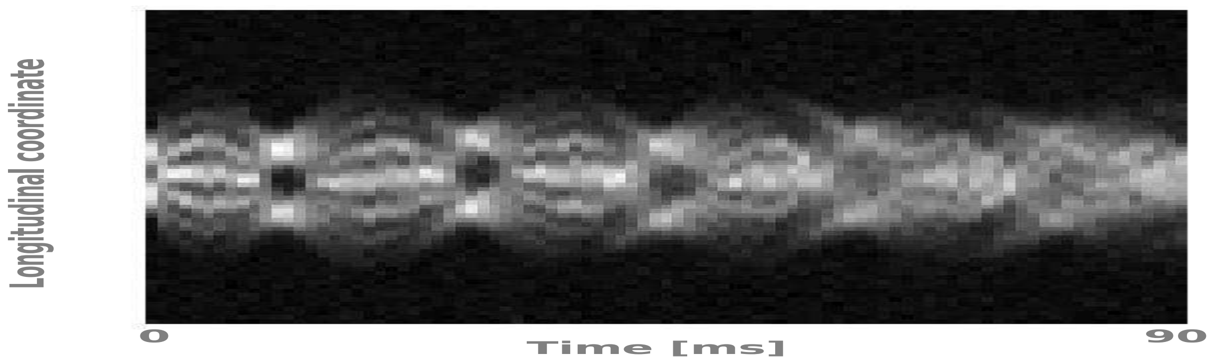


Detailed Dynamics and Computational Comparison



3-, 4-, N-soliton States





A Line of Attack: A Particle (Variational) Approach for Dark Solitons

- For simplicity, consider the GPE model:

$$iv_\tau = -\frac{1}{2}v_{\xi\xi} + \frac{1}{2}\xi^2 v + |v|^2 v - \mu v, \tag{8}$$

- under the Change of Variables:

$$v(\xi, t) = \mu^{1/2} u(x, t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t, \tag{9}$$

leads to (with $\epsilon = (2\mu)^{-1}$ used as a Small Parameter)

$$i\epsilon u_t + \epsilon^2 u_{xx} + (1 - x^2 - |u|^2)u = 0, \tag{10}$$

- Then using information established about the ground state:

$$\eta_0(x) := \lim_{\epsilon \to 0} \eta_\epsilon(x) = \begin{cases} (1 - x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1. \end{cases} \tag{11}$$

we can use the ansatz $u(x, t) = \eta_\epsilon(x) v(x, t)$ within the Lagrangian $L(v) = K(v) + \Lambda(v)$, where

$$\Lambda(v) = \epsilon^2 \int_{\mathbb{R}} \eta_\epsilon^2(x) |v_x|^2 dx + \frac{1}{2} \int_{\mathbb{R}} \eta_\epsilon^4(x) (1 - |v|^2)^2 dx. \tag{12}$$

$$K(v) = \frac{i}{2}\epsilon \int_{\mathbb{R}} \eta_\epsilon^2(x) (v\bar{v}_t - \bar{v}v_t) dx, \tag{13}$$

$1, 2, 3, \ldots \infty$: Towards a Lattice of Dark Solitons

Single Trapped Dark Soliton

- For a Single Dark Soliton (choosing $A = \sqrt{1-b^2}$)

$$v_1(x,t) = A(t)\ \tanh\left(\epsilon^{-1} B(t)(x - a(t))\right) + ib(t), \tag{14}$$

the effective Lagrangian becomes

$$L_1 := \lim_{\epsilon \to 0} \frac{L(v_1)}{2\epsilon} = -\frac{\dot{b}}{\sqrt{1-b^2}}(a - \frac{1}{3}a^3) + b\sqrt{1-b^2}(1-a^2)\dot{a} \\ + \frac{2}{3}(1-a^2)(1-b^2)B + \frac{1}{3B}(1-a^2)^2(1-b^2)^2.$$

This leads to:

$$B = \frac{\sqrt{1-a^2}\sqrt{1-b^2}}{\sqrt{2}}.$$

$$\dot{a} = \sqrt{2}\sqrt{1-a^2}b, \quad \dot{b} = -\frac{\sqrt{2}a(1-b^2)}{\sqrt{1-a^2}},$$

which, in turn, leads to:

$$\ddot{a} + 2a = 0.$$

2 Trapped Dark Solitons

- Now: 2-Soliton State (use $a_1 = -a_2 = -a$ and $b_1 = -b_2 = -b$):

$$\begin{aligned} v_2(x,t) = &\left[A_1(t)\ \tanh\left(\epsilon^{-1}B_1(t)(x-a_1(t))\right) + ib_1(t)\right] \\ &\times \left[A_2(t)\ \tanh\left(\epsilon^{-1}B_2(t)(x-a_2(t))\right) + ib_2(t)\right], \end{aligned} \tag{15}$$

yields the corresponding Lagrangian. After simplifications, one can write

$$\Lambda_2 := \frac{\Lambda(v_2)}{2\epsilon} = \Lambda_+ + \Lambda_- + \Lambda_{\text{overlap}}, \quad \Lambda_\pm = \frac{4(1-a^2)^{3/2}(1-b^2)^{3/2}}{3\sqrt{2}} + \mathcal{O}(\epsilon^{1/3}).$$

$$\Lambda_{\text{overlap}} = -8\sqrt{2}(1-a^2)^{3/2}(1-b^2)^{5/2}\, e^{-4Ba\epsilon^{-1}} \left(1 + \mathcal{O}(\epsilon^{1/3})\right).$$

- This, in turn, yields the nonlinear oscillator:

$$\ddot{a} + 2a = 8\sqrt{2}\epsilon^{-1}e^{-\frac{2\sqrt{2}a}{\epsilon}},$$

with equilibrium position:

$$a = \frac{\epsilon}{\sqrt{2}}\left(-\log(\epsilon) - \frac{1}{2}\log|\log(\epsilon)| + \frac{3}{2}\log(2) + o(1)\right) \quad \text{as} \quad \epsilon \to 0 \tag{16}$$

and oscillation frequency around it:

$$\begin{aligned} \omega_0^2(\epsilon) &= 2 + \frac{32}{\epsilon^2}e^{-2\sqrt{2}a_0(\epsilon)\epsilon^{-1}} = 2 + \frac{4\sqrt{2}a_0(\epsilon)}{\epsilon} \\ &= -4\log(\epsilon) - 2\log|\log(\epsilon)| + 2 + 6\log(2) + o(1), \quad \text{as} \quad \epsilon \to 0. \end{aligned} \tag{17}$$

m Trapped Dark Solitons

- Use the Ansatz:

$$v_m(x,t) = \prod_{j=1}^{m} \left(A_j(t)\ \tanh\left(\epsilon^{-1}B_j(t)(x-a_j(t))\right) + ib_j(t)\right). \qquad (18)$$

- Obtain the Lagrangian

$$Ł_m \sim -\sqrt{2}\sum_{j=1}^{m}\left(a_j^2+b_j^2\right) - 2\sum_{j=1}^{m} a_j\dot{b}_j - 8\sqrt{2}\sum_{j=1}^{m-1} e^{-\sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}}.$$

- Derive the Equations of Motion

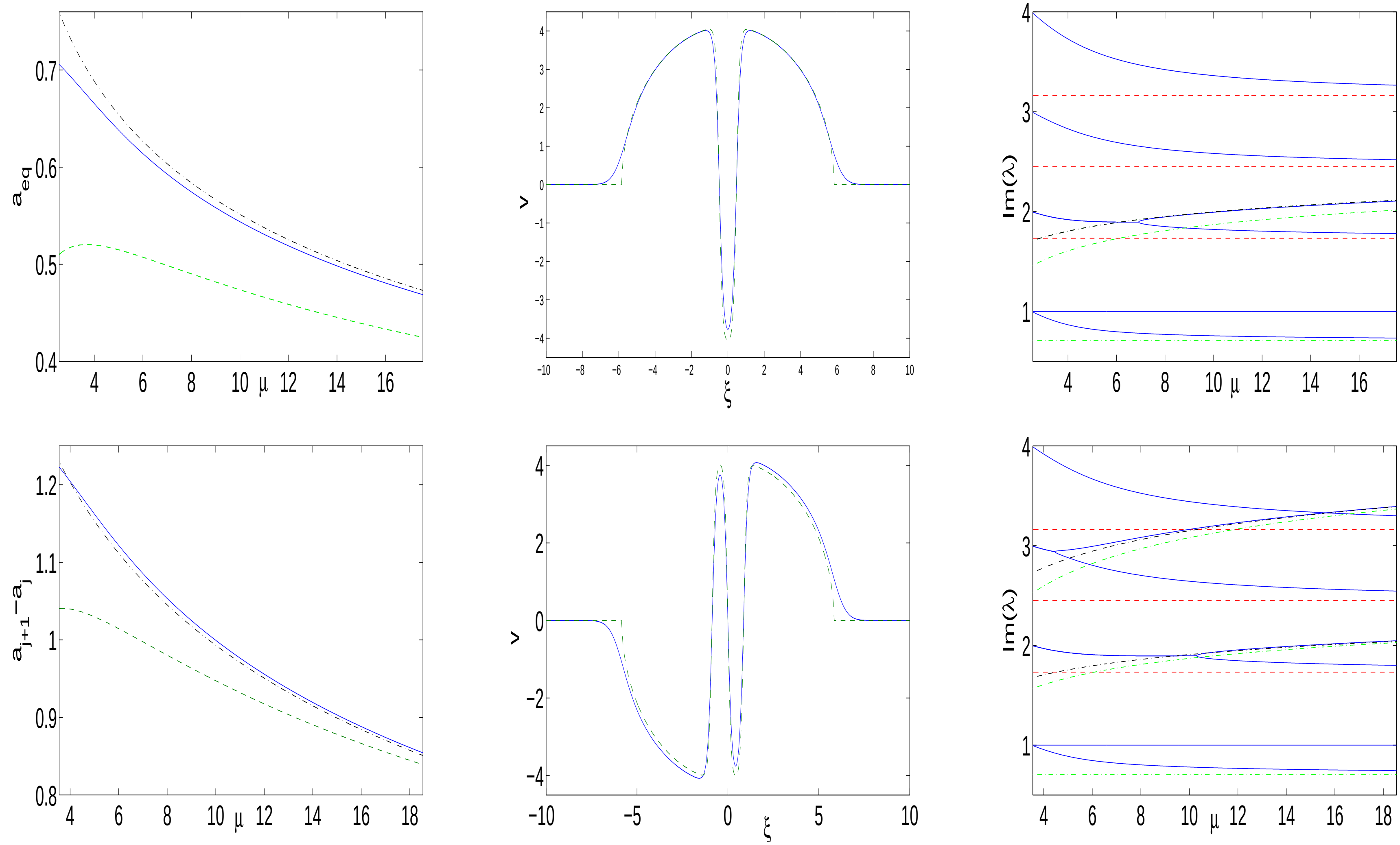
$$\dot{a}_j = \sqrt{2}b_j, \quad \dot{b}_j = -\sqrt{2}a_j - 8\epsilon^{-1}\left(e^{-\sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}} - e^{-\sqrt{2}(a_j-a_{j-1})\epsilon^{-1}}\right). \qquad (19)$$

- Define $x_j = \sqrt{2}(a_{j+1}-a_j)\epsilon^{-1}$, to find Equilibrium Distances and Oscillation Frequencies [With $\Omega^2 \in \left\{1,3,6,...,\frac{m(m-1)}{2}\right\}$].

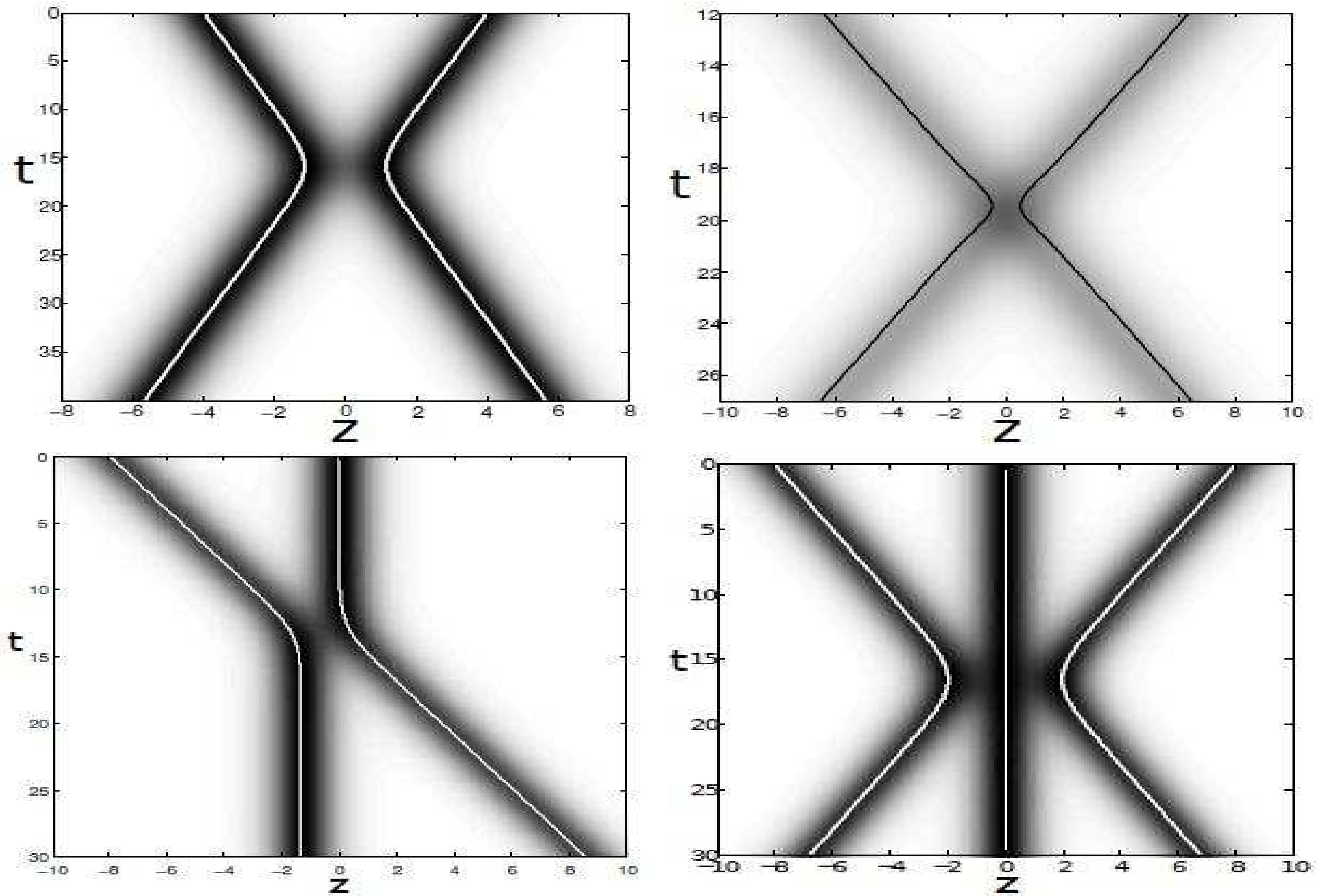
$$\mathbf{x} = -2\log(\epsilon)\mathbf{1} - \log|\log(\epsilon)|\mathbf{1} + 2\log(2)\mathbf{1} - \log(\mathbf{A}^{-1}\mathbf{1}) + o(1), \quad \text{as} \quad \epsilon \to 0, \qquad (20)$$

$$\omega^2 = 2 + \left(-4\log(\epsilon) - 2\log|\log(\epsilon)| + 4\log(2)\right)\Omega^2 + \mathcal{O}(1). \qquad (21)$$

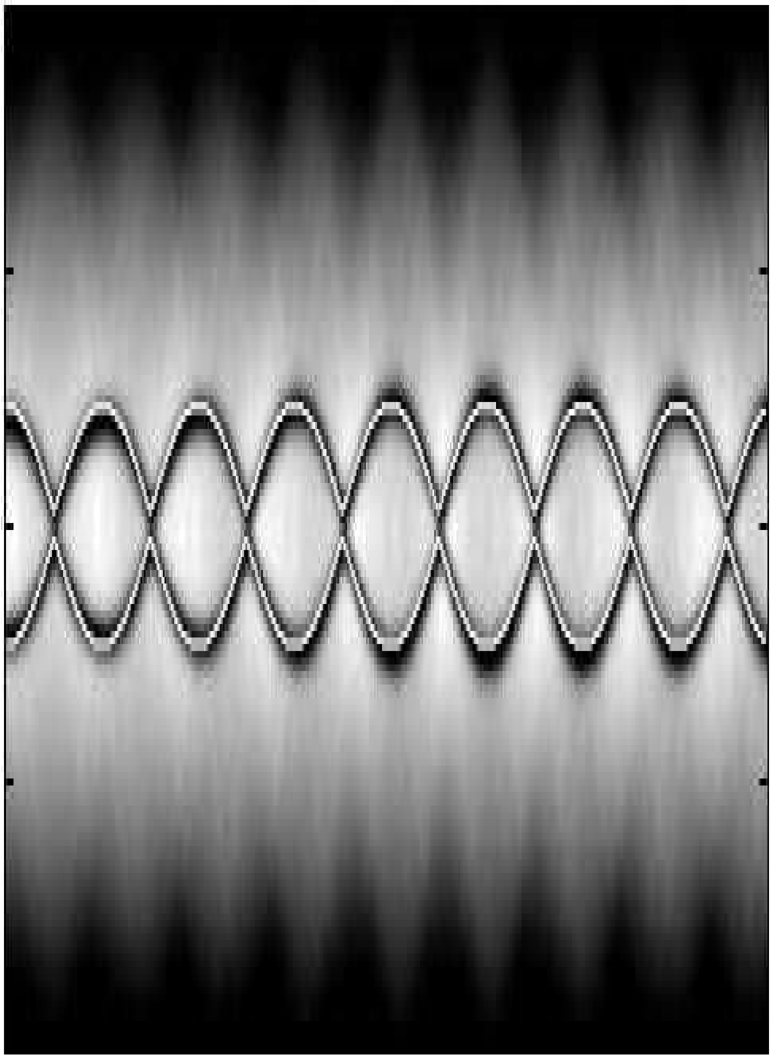
Testing the Prediction: Statics & Oscillation Frequencies

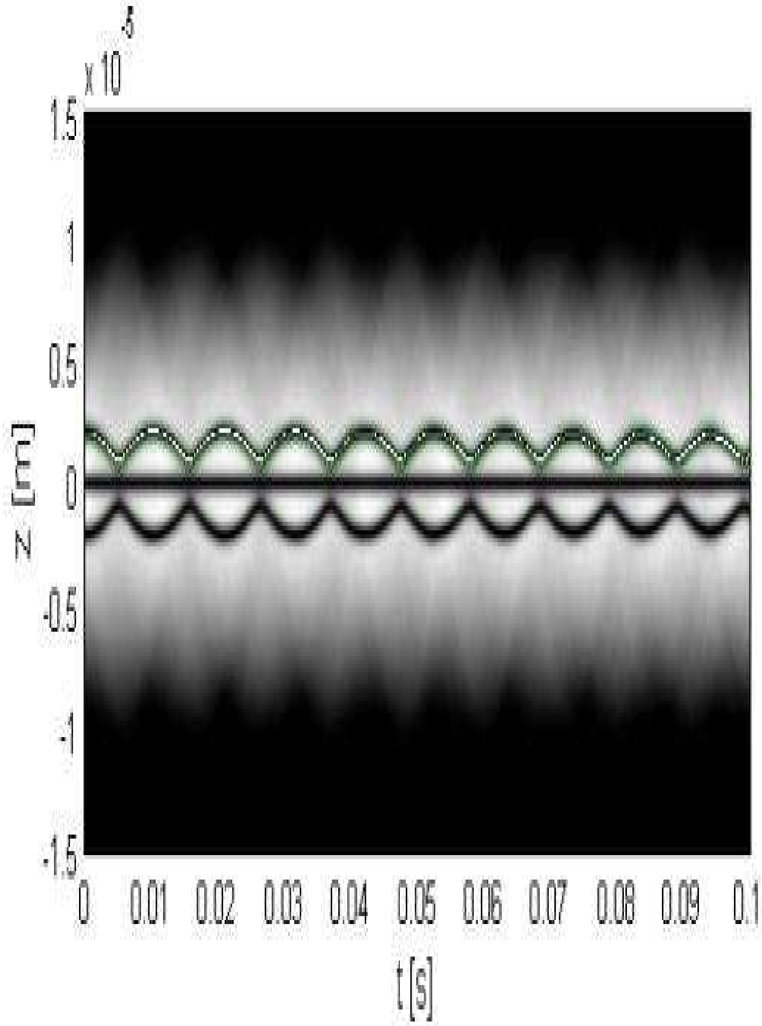


Testing the Prediction: Dynamics (Without Trapping)



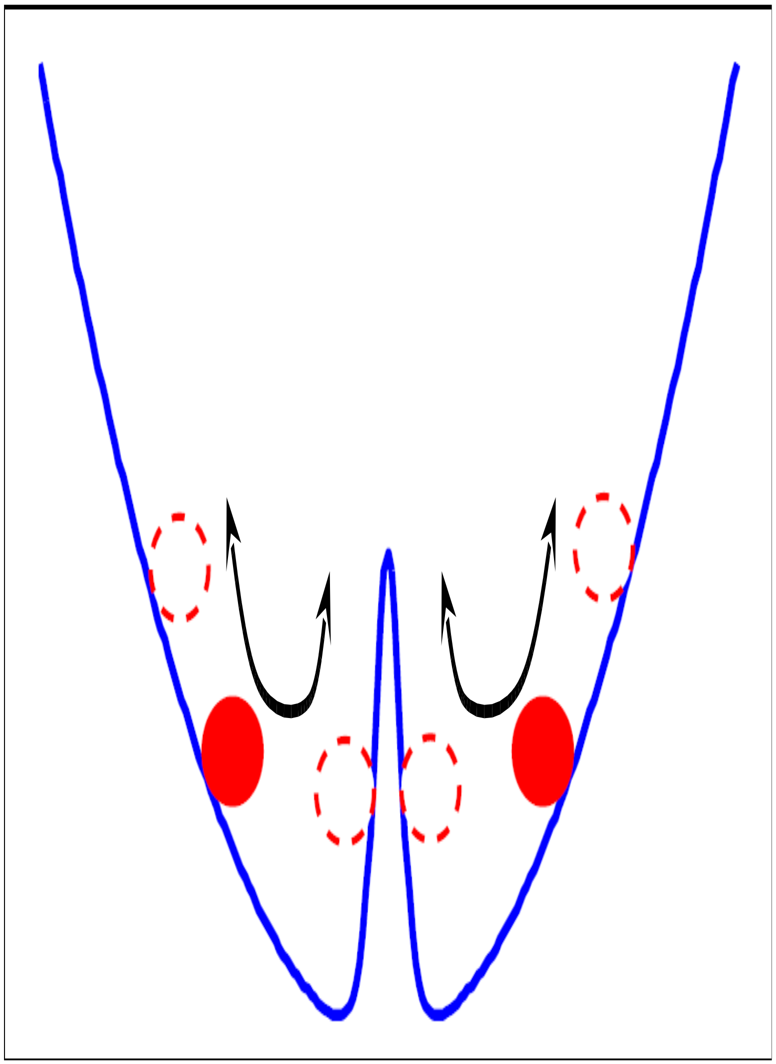
Testing the Prediction: Dynamics (With Trapping)

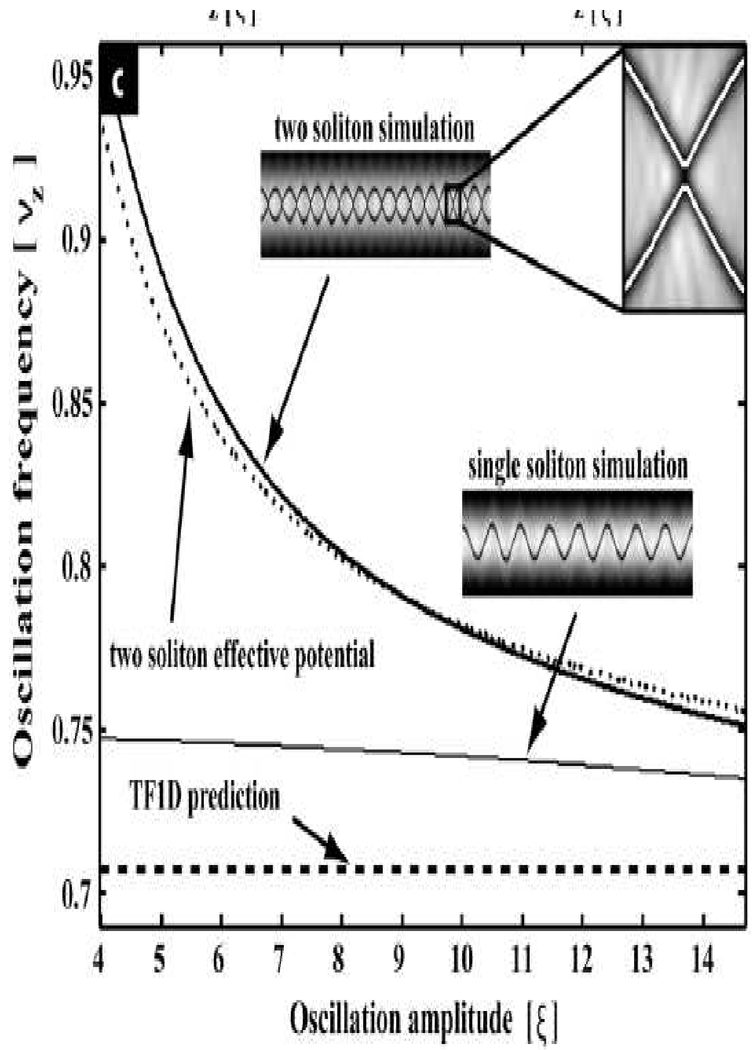




Now Collecting All the Chips !

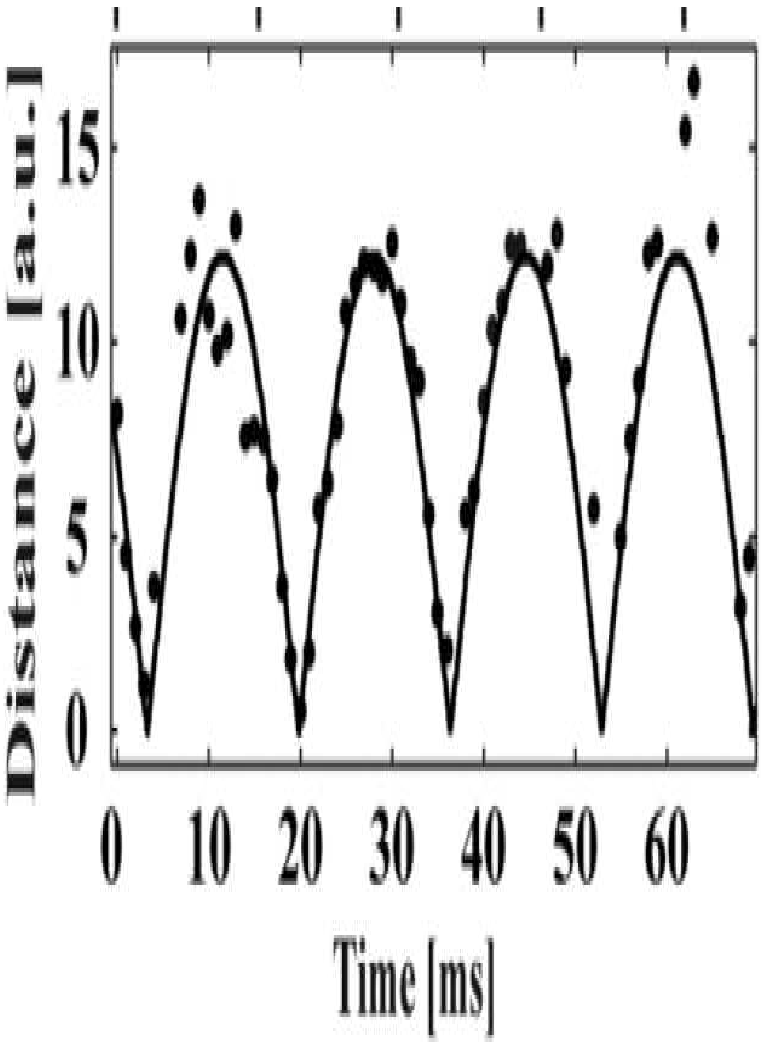
Effective Potential and its Oscillation Frequency Prediction

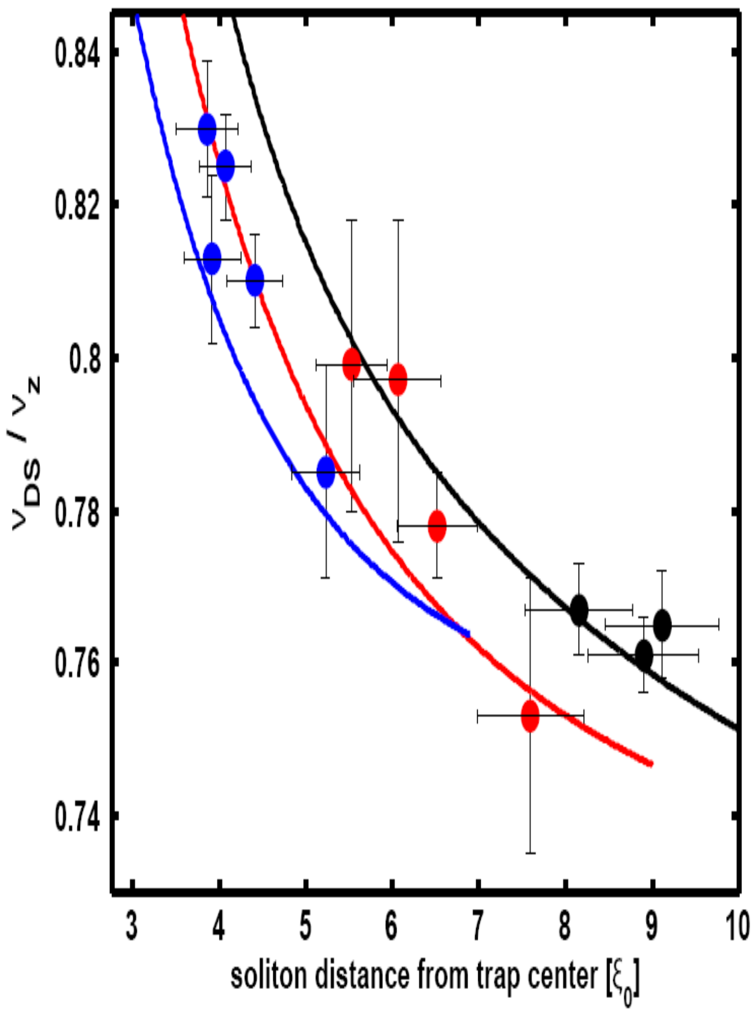




Collecting the Chips (Continued)

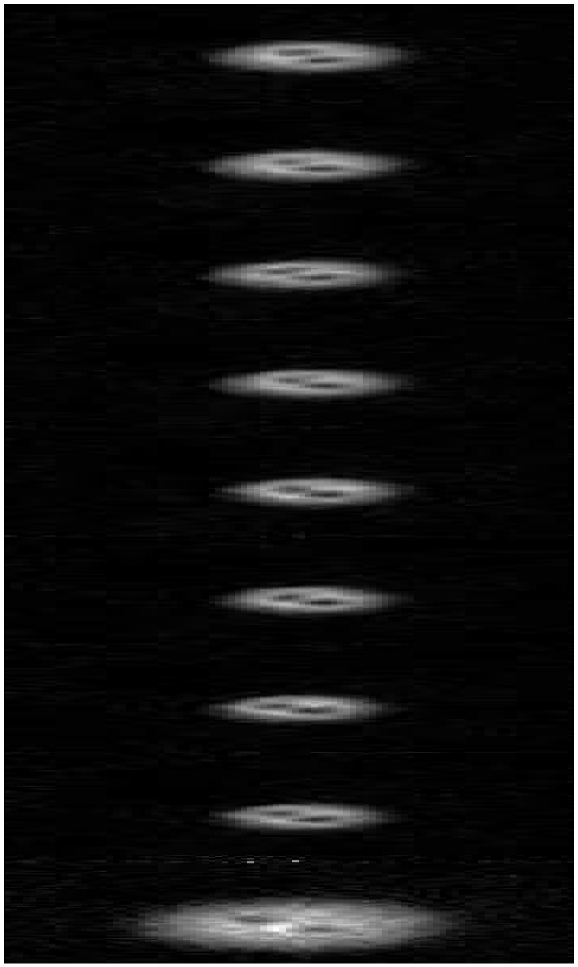
Comparison with Experiments

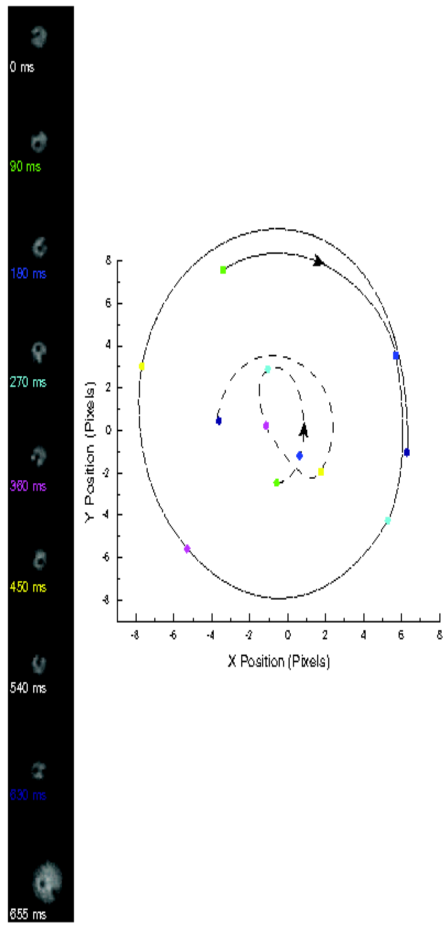


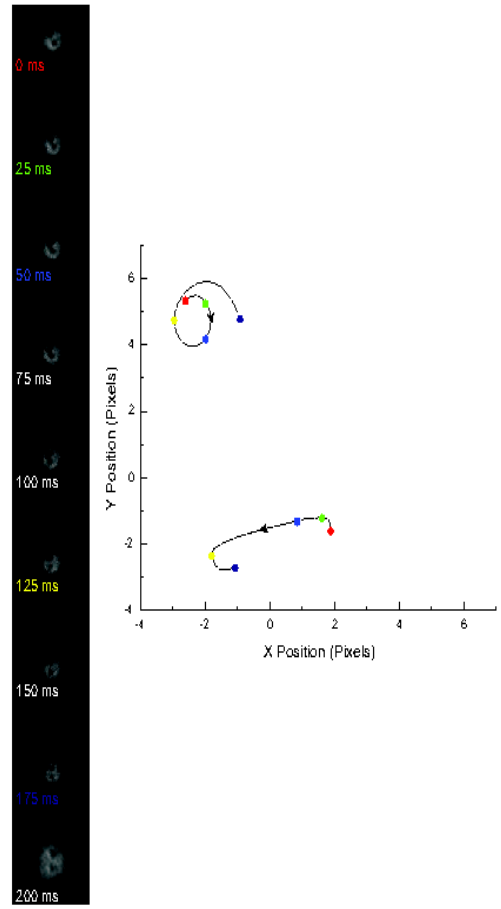


The Case of Few Vortices: What do we Know ?

1-Component, 2-dimensions: Vortex Dipoles in Amherst

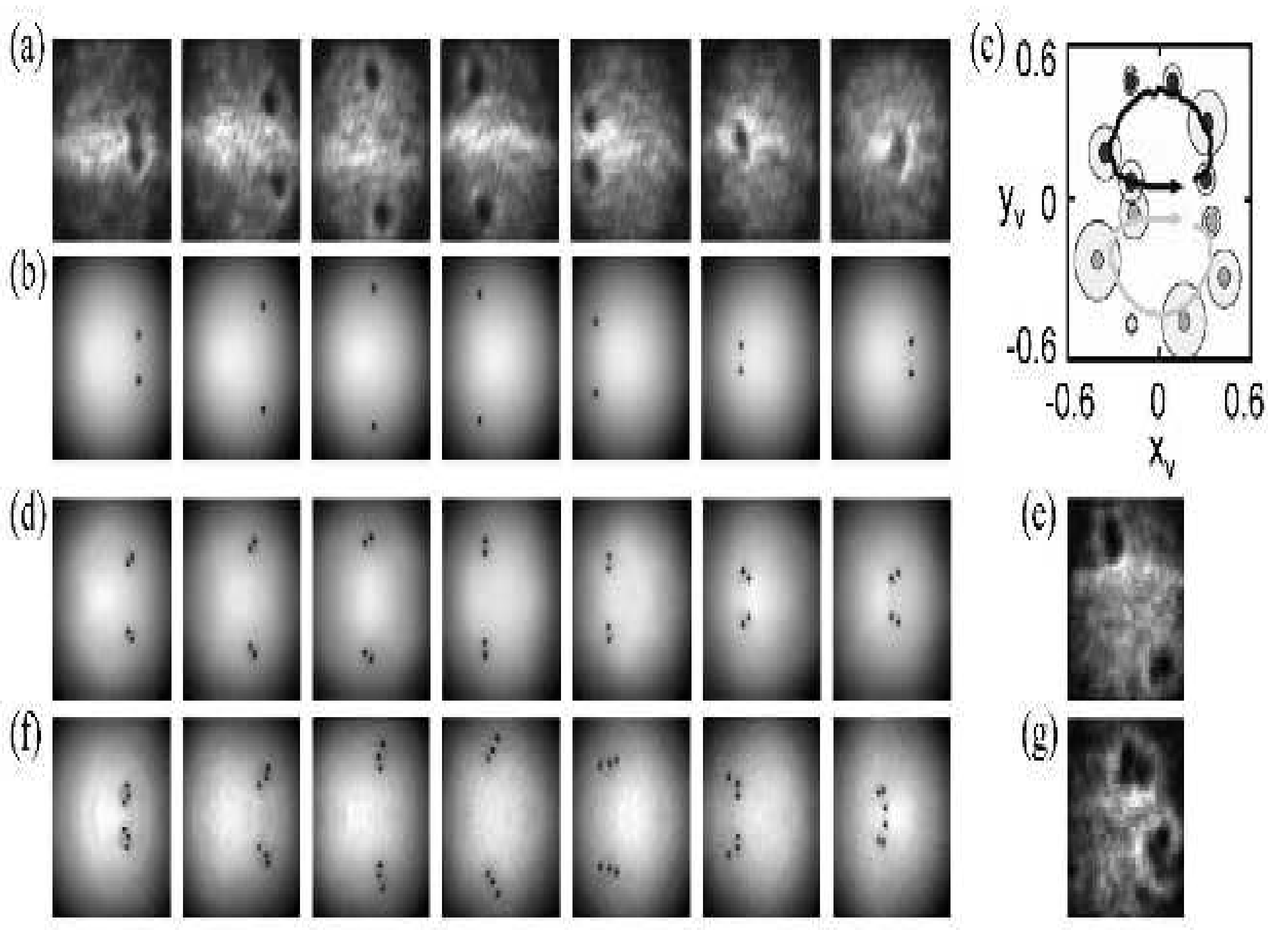






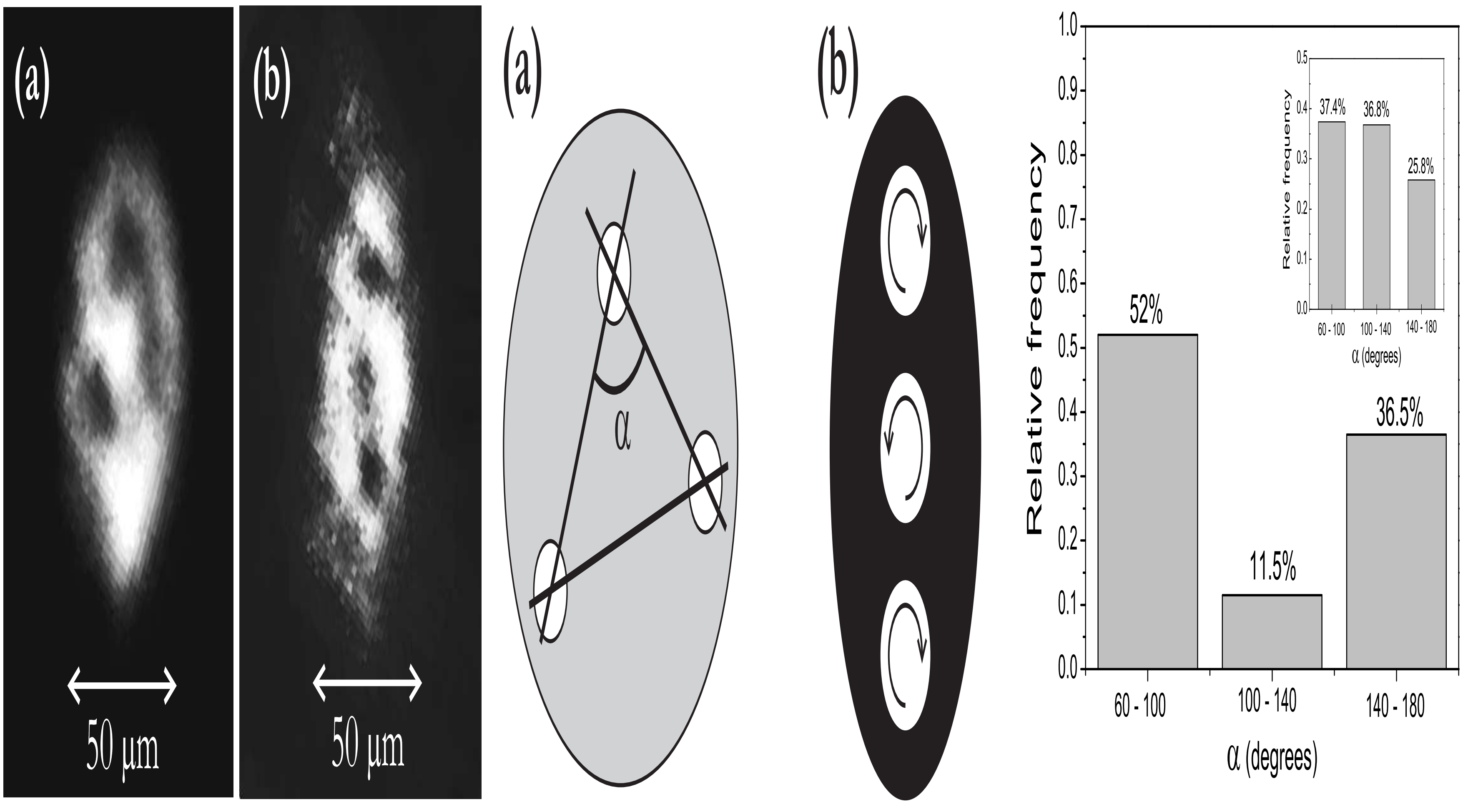
Experimentally Generating Vortex Dipoles: Part II

1-Component, 2-dimensions: Vortex Dipoles in Tucson



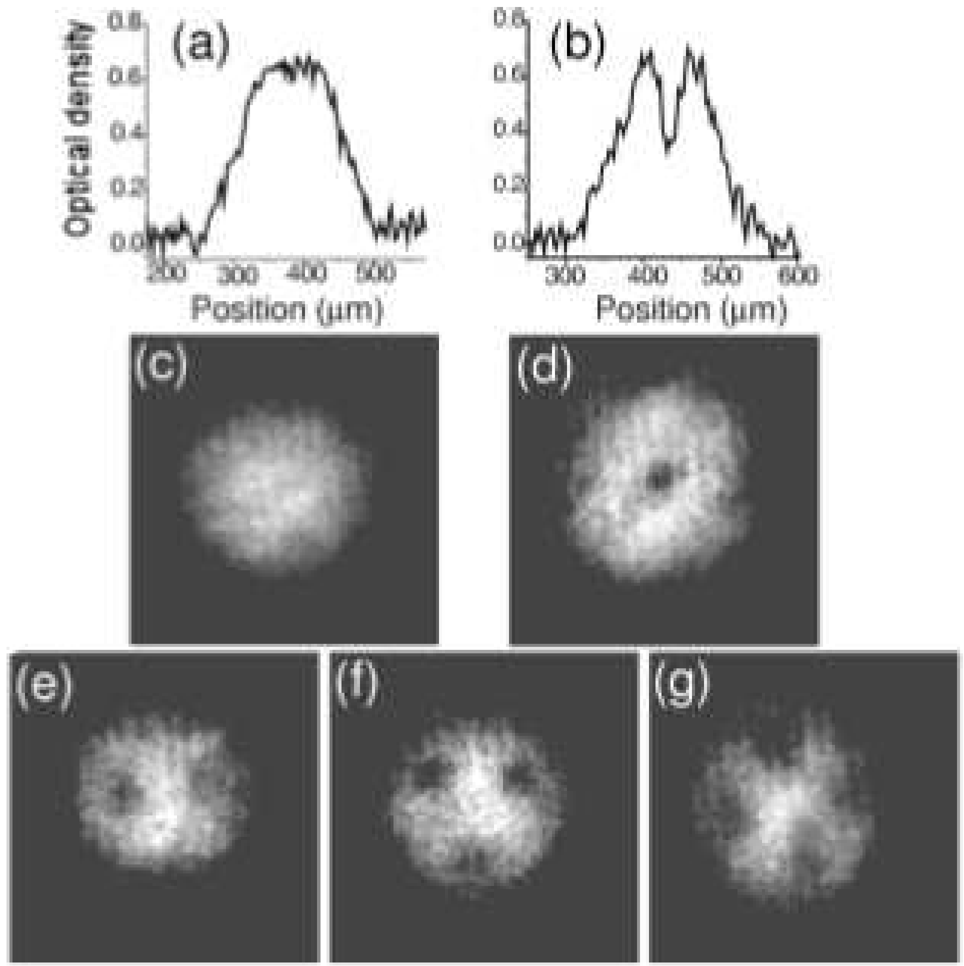
Experimentally Generating Vortex Tripoles

Connections with Work of V.S. Bagnato (PRA 82, 033616 (2010))



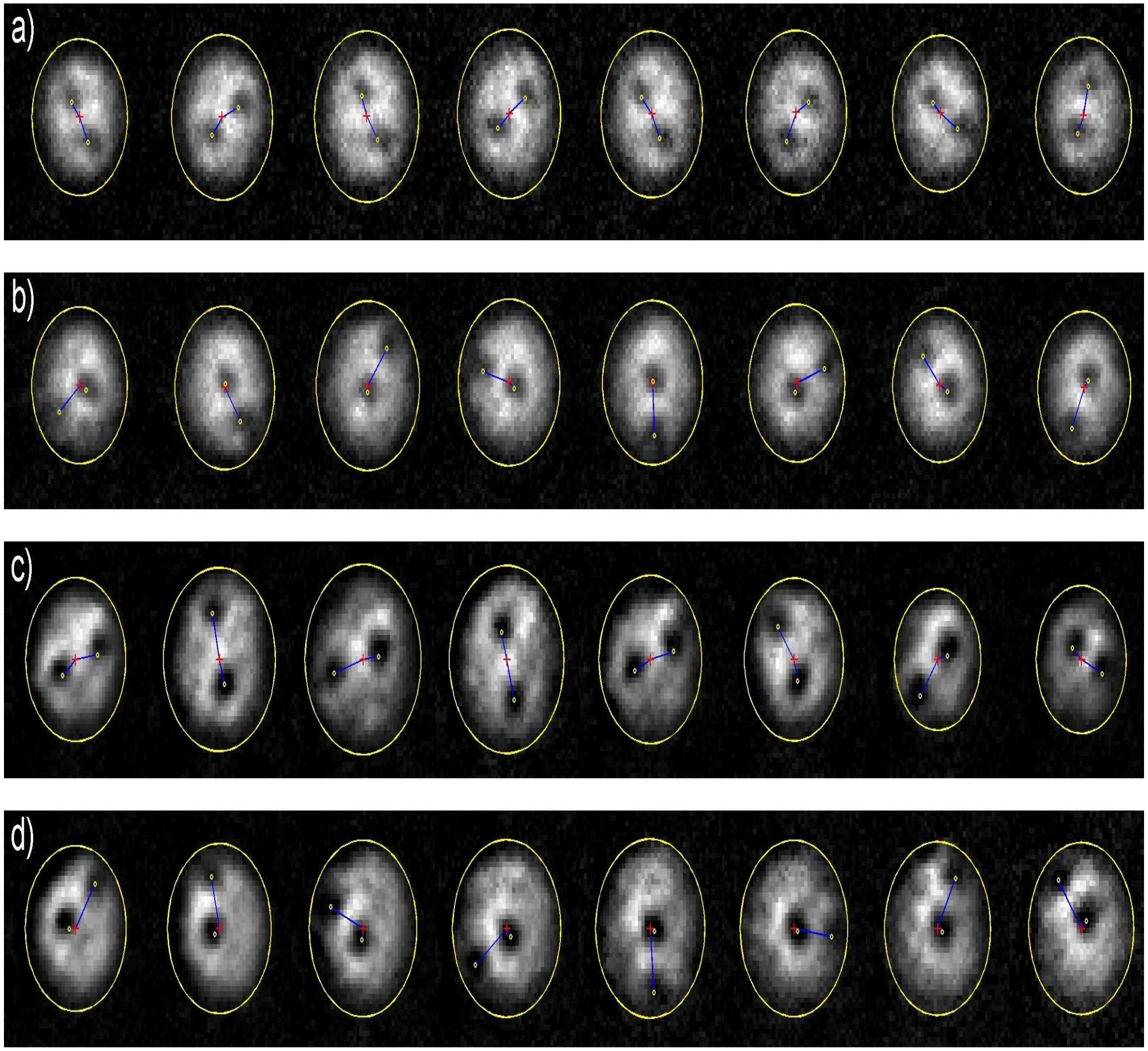
Experimental Results on Co-Rotating Vortices

Classic Work of Dalibard Group (PRL 84, 806 (2000))



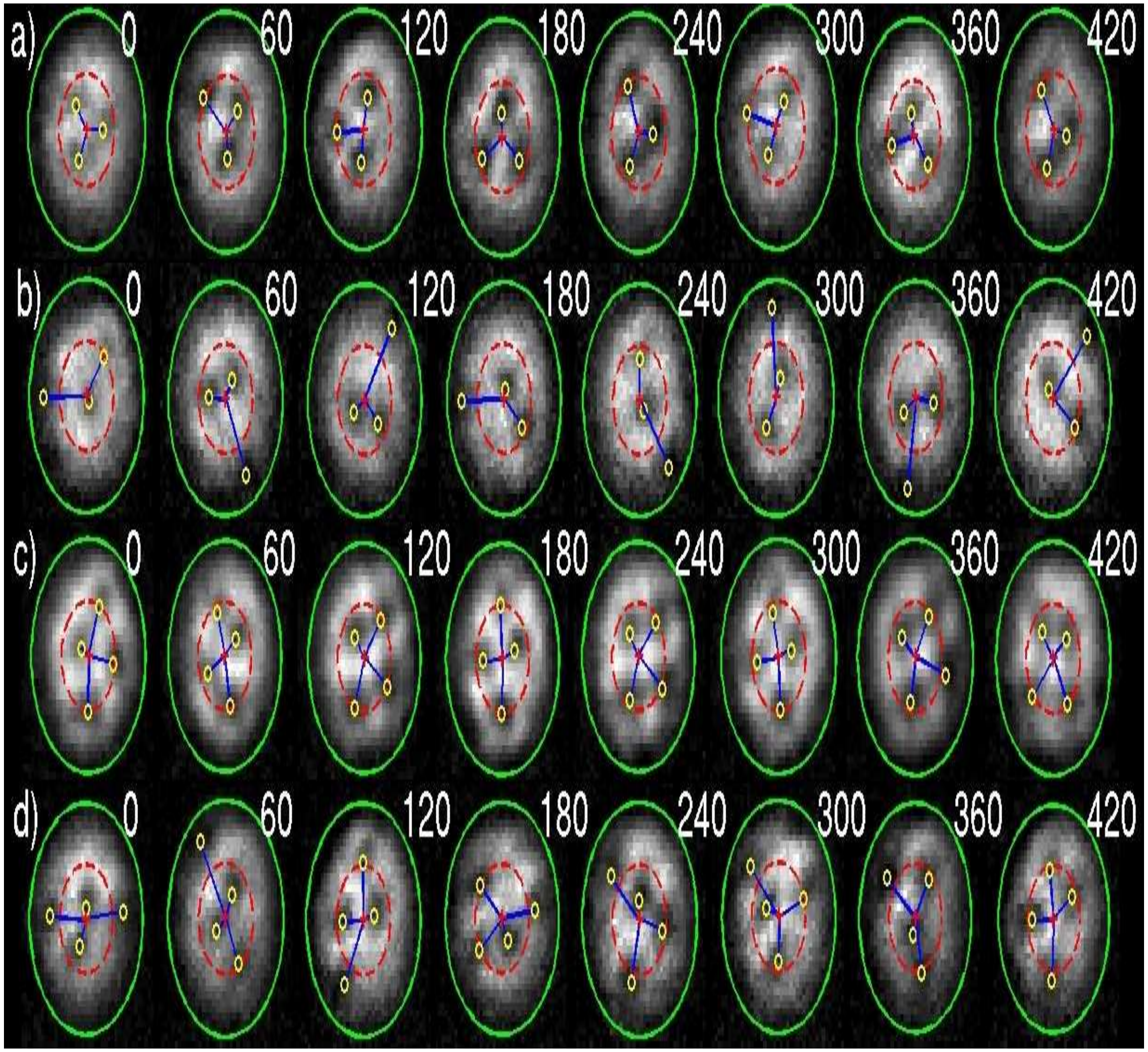
More Experimental Results on 2 Co-Rotating Vortices

Recent Work of D. Hall Group (PRL 110, 225301 (2013))



More Experimental Results on 3-4 Co-Rotating Vortices

Recent Work of D. Hall Group (PRL 110, 225301 (2013))



Theoretical Analysis: Single Vortex

- Vortices have Dynamics reminiscent of those of the Dark Solitons.
- They Bifurcate from the Linear Mode $\Psi(x,y) = \psi_0(x)\psi_1(y) + i\psi_1(x)\psi_0(y)$.
- They are dominated by an Oscillation Mode associated with their Precession Frequency. One can used Matched Asymptotics to obtain:

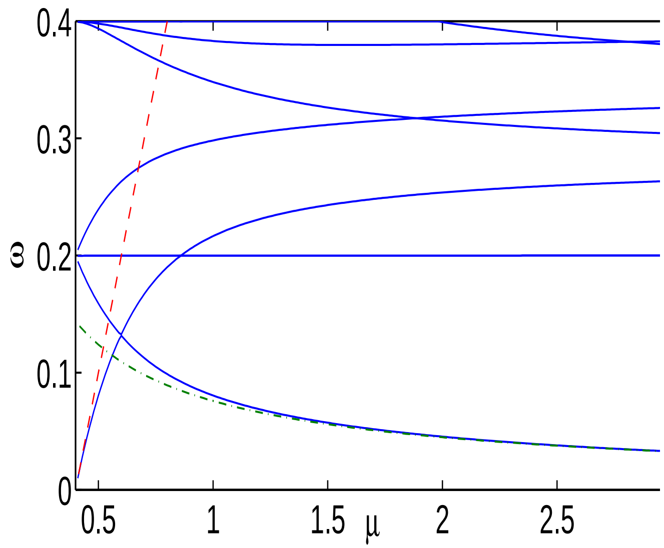
$$\dot{x}_v = \frac{\Omega^2}{2\mu} \log\left(A\frac{\mu}{\Omega}\right) y_v, \tag{22}$$

$$\dot{y}_v = -\frac{\Omega^2}{2\mu} \log\left(A\frac{\mu}{\Omega}\right) x_v, \tag{23}$$

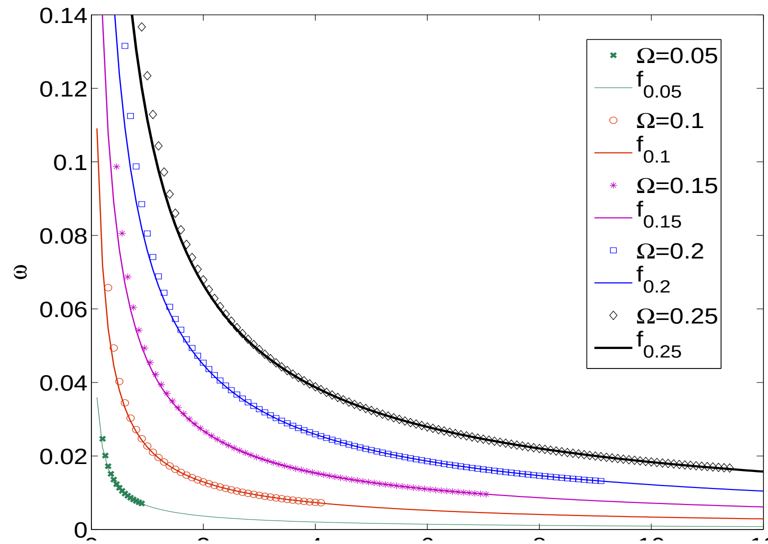
- This yields the Precession Frequency:

$$\omega_{\text{prec}} = \frac{\Omega^2}{2\mu} \log(A\frac{\mu}{\Omega}) \tag{24}$$

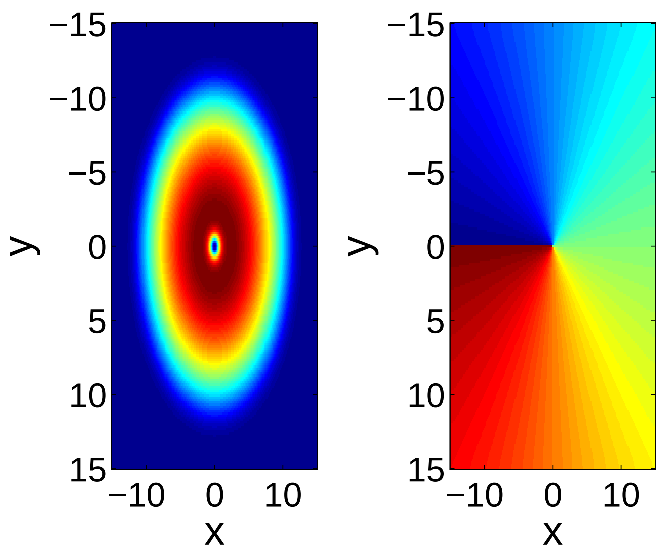
- In the presence of Finite Temperature, this mode becomes Complex giving rise to Dynamical Instability.



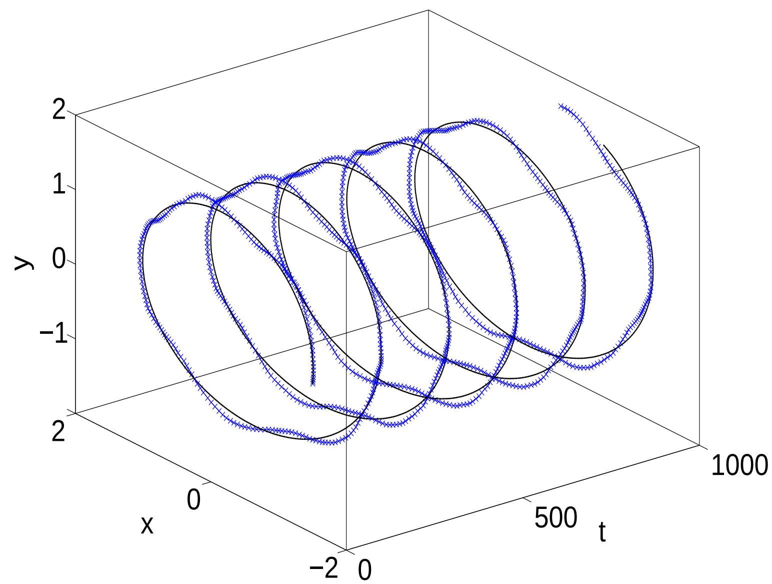
ω
μ
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
2.5



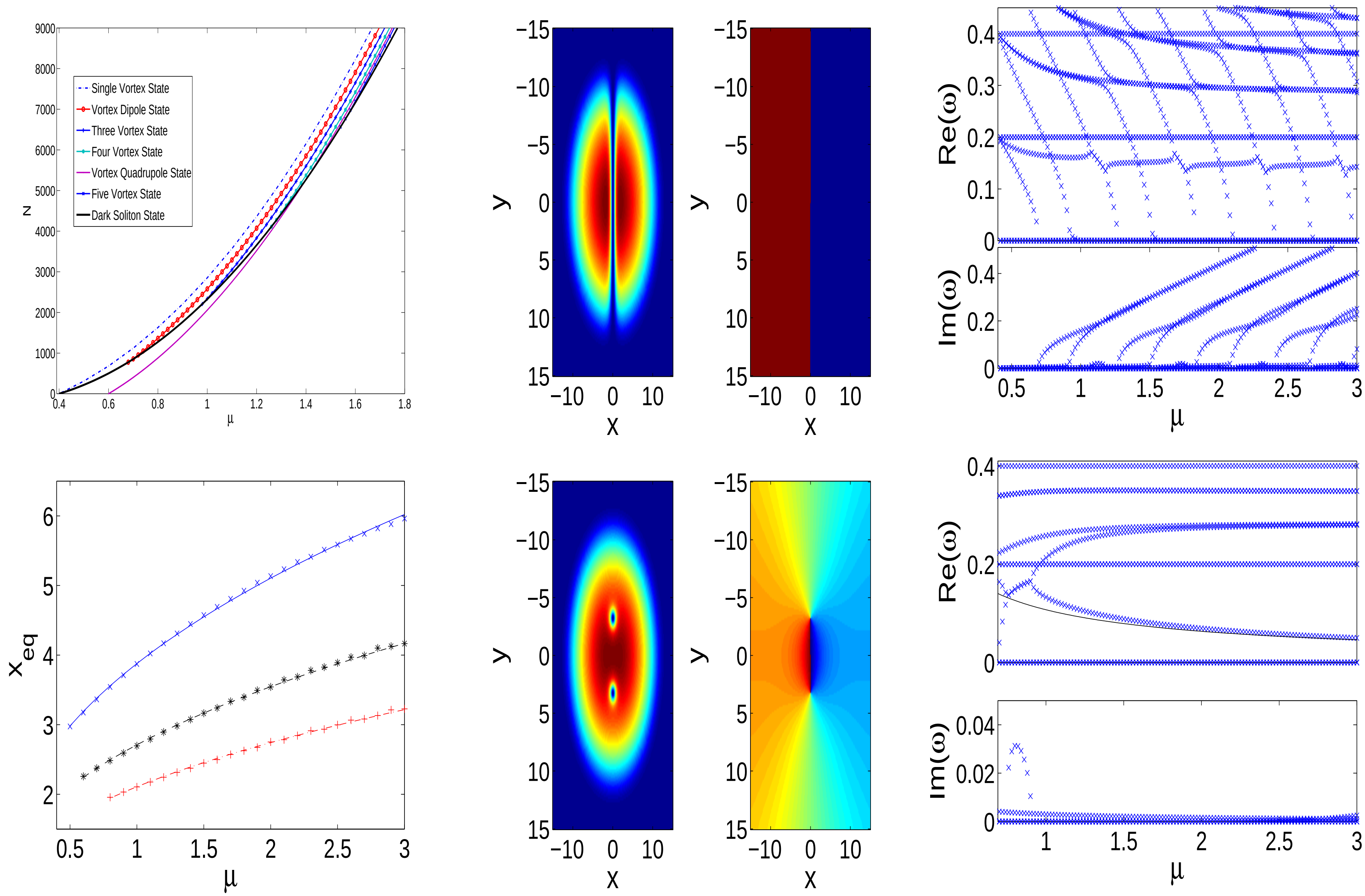
Ω=0.05
f0.05
Ω=0.1
f0.1
Ω=0.15
f0.15
Ω=0.2
f0.2
Ω=0.25
f0.25
ω
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0



y
x
−15
−10
−5
0
5
10
15
−10
0
10



y
x
t
2
1
0
−1
−2
500
1000


Single Vortex State
Vortex Dipole State
Three Vortex State
Four Vortex State
Vortex Quadrupole State
Five Vortex State
Dark Soliton State
N
μ
y
x
Re(ω)
Im(ω)
μ
x_{eq}
μ
y
x
y
x
Re(ω)
Im(ω)
μ

Supercritical Pitchfork Bifurcation & Ensuing Vortex Dynamics

- The Two-Mode Picture captures this $\pi/2$ relative phase bifurcation, yielding:

$$N_{\text{cr}} = \frac{\omega_1 - \omega_2}{B - A_0}, \quad \mu_{cr} = \omega_1 + A_0 N_{\text{cr}} \tag{25}$$

where $A_0 = \int \Psi_{10}^4 dxdy$, $B = \int \Psi_{10}^2 \Psi_{0m}^2 dxdy$. $\mu_{cr} = 10\Omega/3$ for $m = 2$ (vortex dipole), $86\Omega/19$ for $m = 3$ (vortex tripole), ...

- The resulting states are Multi-Vortex ones which can be described by a Particle Picture:

$$\dot{x}_m = - S_m \omega_{\text{pr}} y_m - B S_n \frac{y_m - y_n}{2\rho_{mn}^2} \tag{26}$$

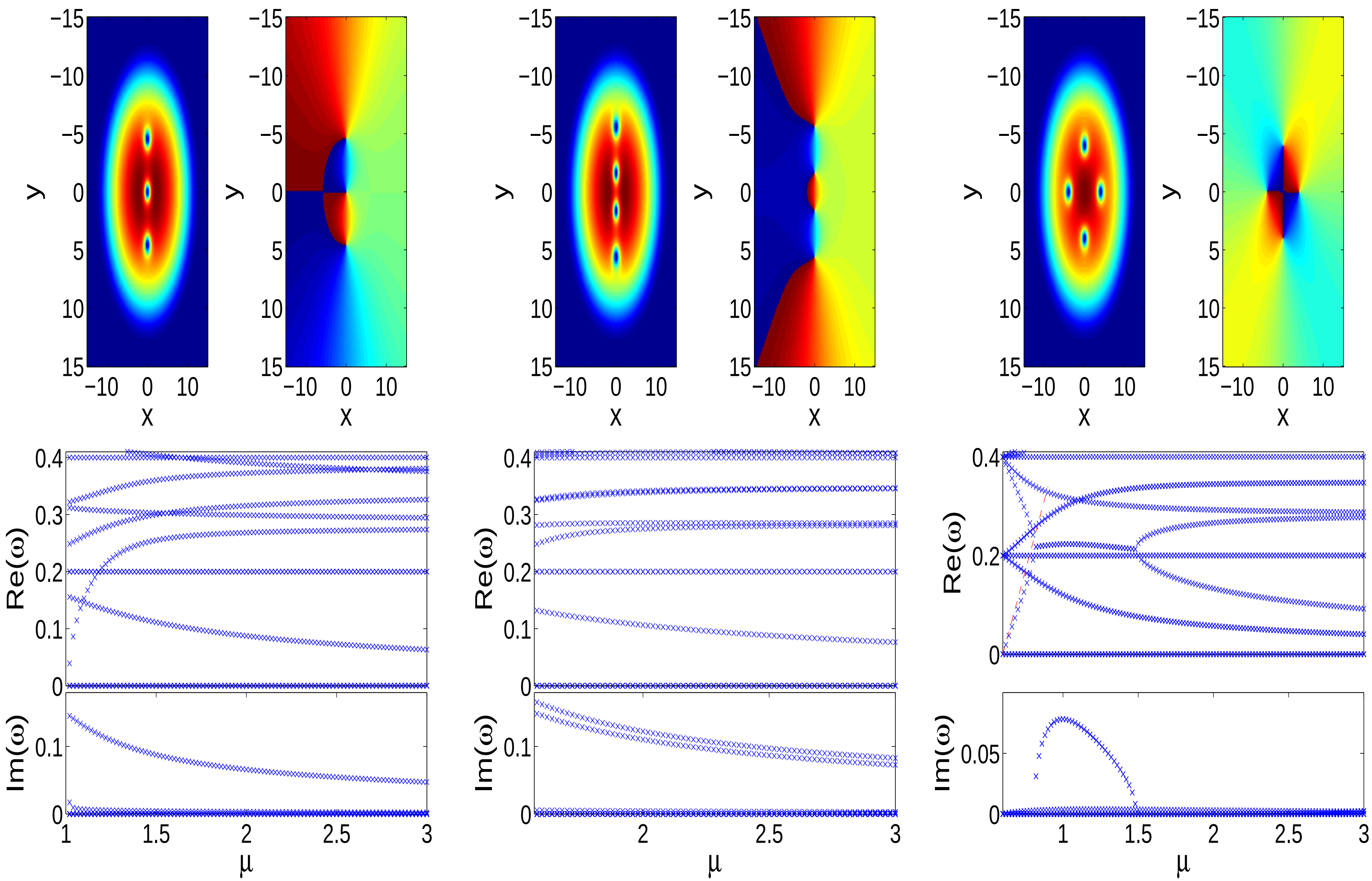
$$\dot{y}_m = \ S_m \omega_{\text{pr}} x_m + B S_n \frac{x_m - x_n}{2\rho_{mn}^2}, \tag{27}$$

- This contains Precession and Interaction (cf. Oscillation and Interaction) and makes useful predictions: $y_{1,eq} = -y_{2,eq} = \sqrt{\frac{B}{4\omega_{\text{pr}}}}$ $\quad \omega_{\text{pr}}^{\text{VD}} = \pm\sqrt{2}\omega_{\text{pr}}$

- The Dynamical Stability of the Stripe is Inherited by the Bifurcation Byproducts: Dipole, Tripole, (Aligned) Quadrupole, Quintopole, etc. E.g. for Tripole

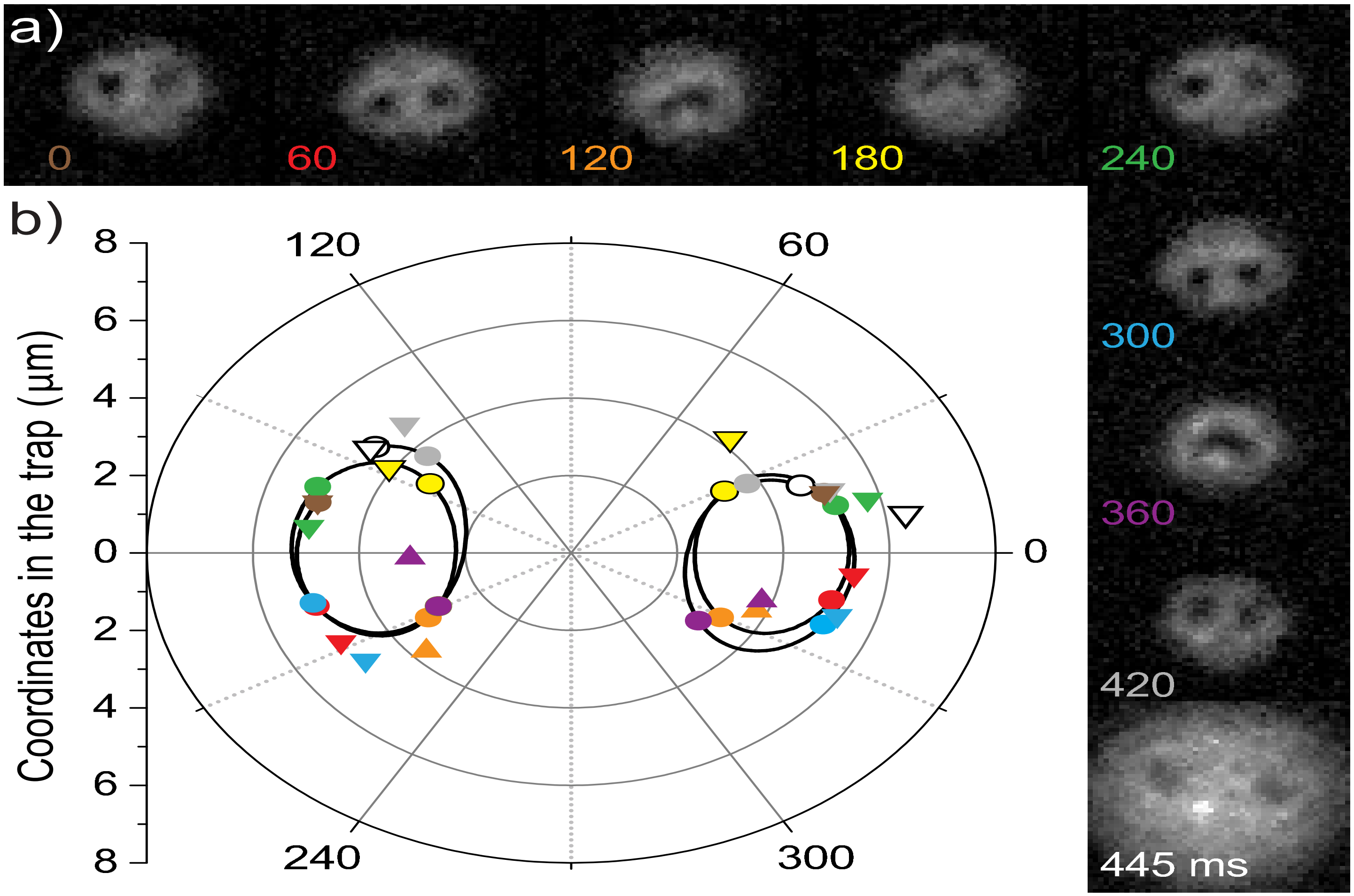
$$\omega_{\text{pr1}}^{\text{3v}} = \pm \sqrt{5}\omega_{\text{pr}}, \tag{28}$$

$$\omega_{\text{pr2}}^{\text{3v}} = \pm i\sqrt{7}\omega_{\text{pr}}. \tag{29}$$

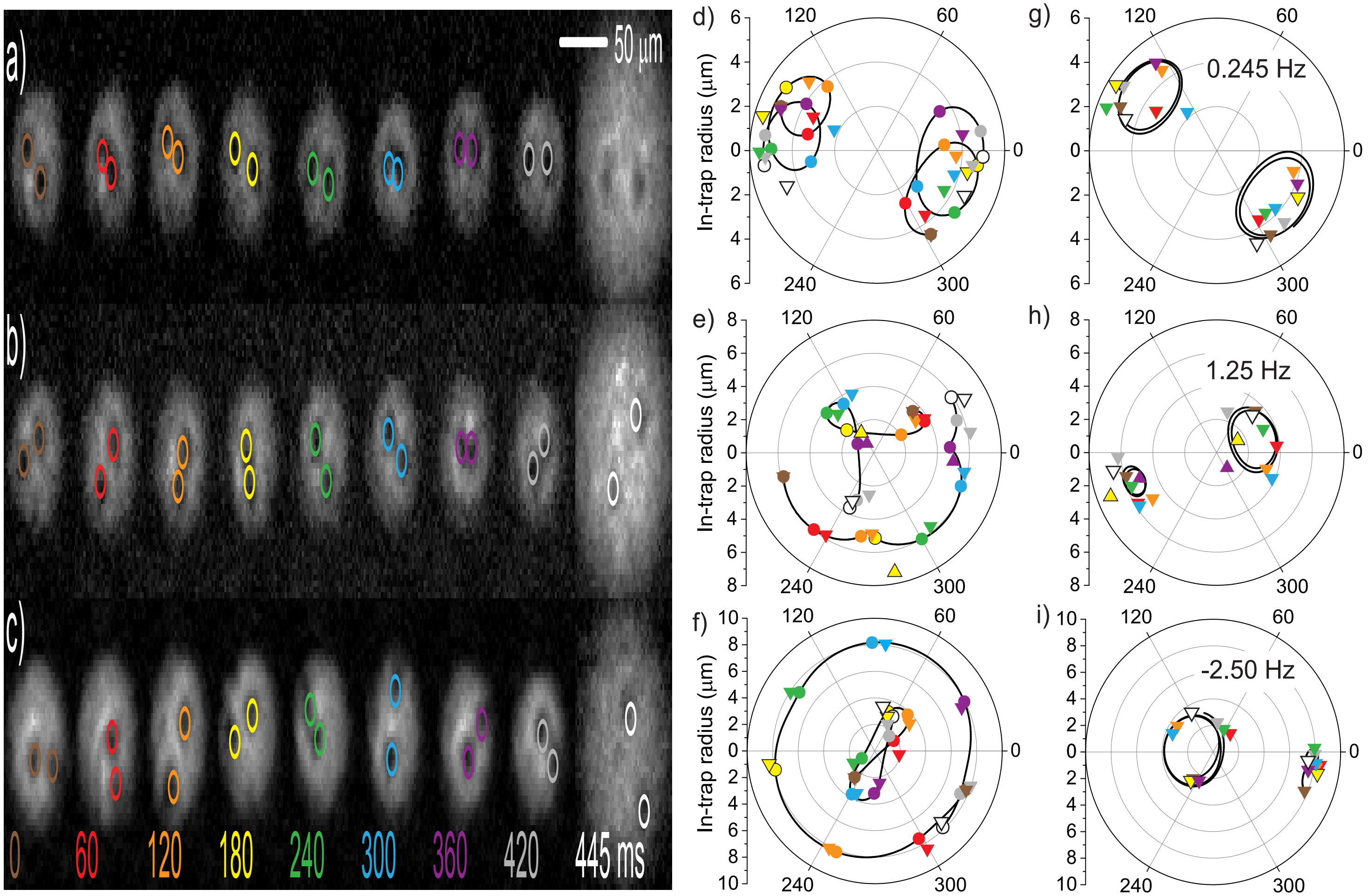


−15
−10
−5
0
5
10
15
y
−10 0 10
x
Re(ω)
Im(ω)
0.4
0.3
0.2
0.1
0
0.05
1 1.5 2 2.5 3
μ

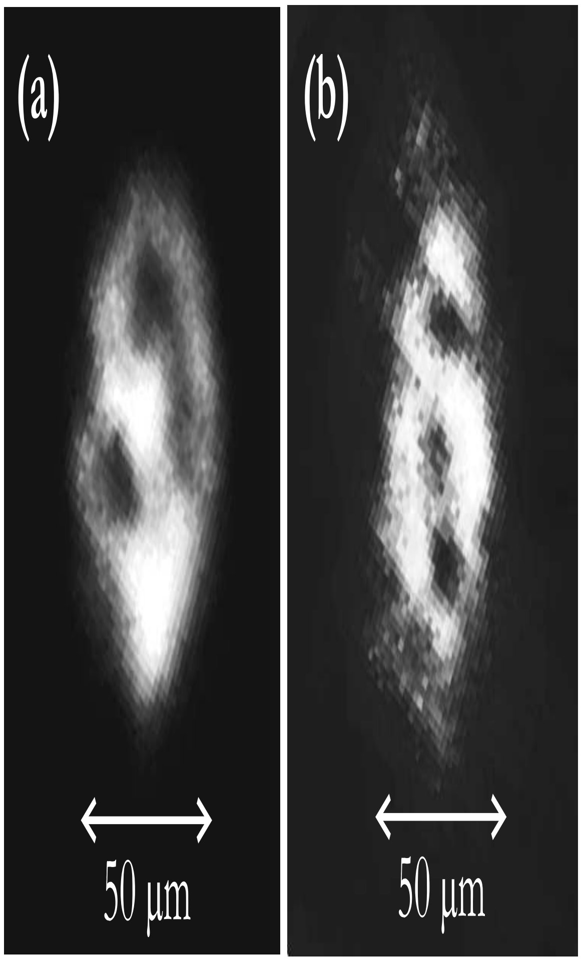
Experimental Verification Part I: Statics & Periodic Dynamics

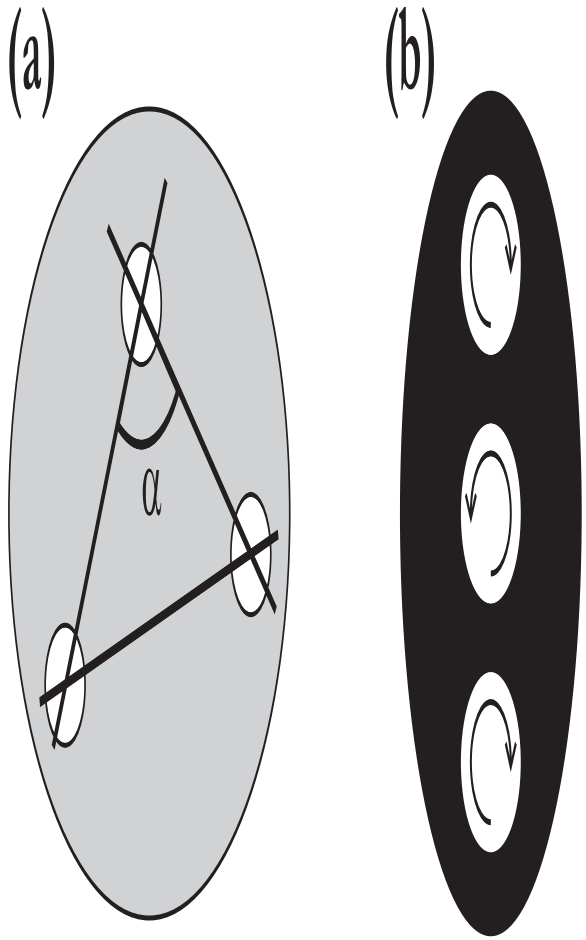


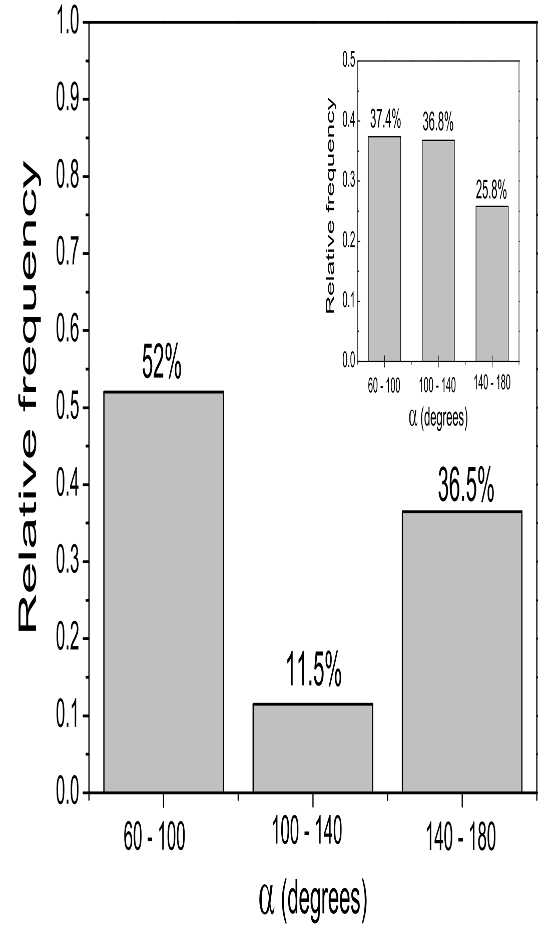
Experimental Verification Part II: General Quasi-Periodic Dynamics



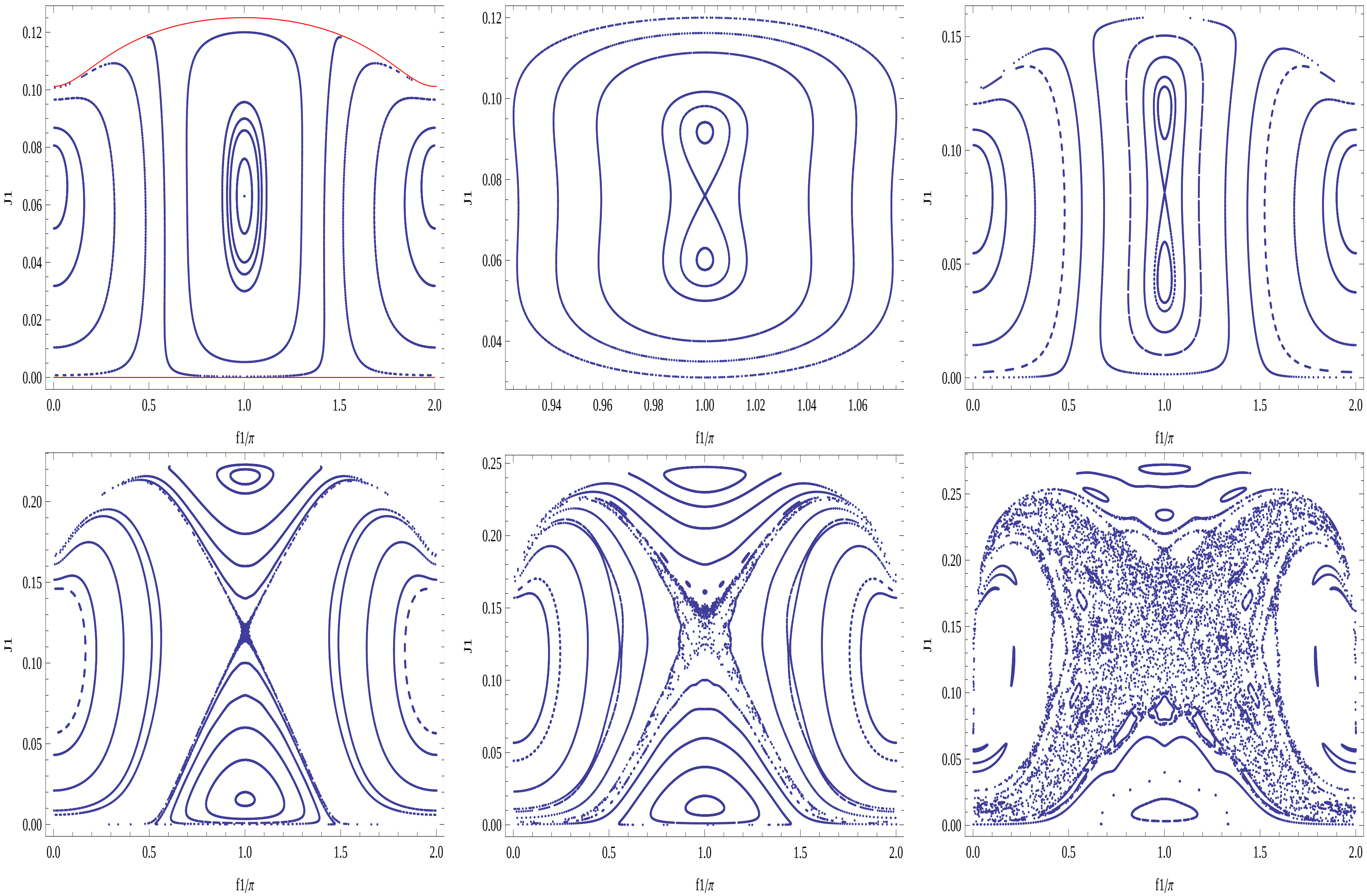
Connections with Work of V.S. Bagnato (PRA 82, 033616 (2010))



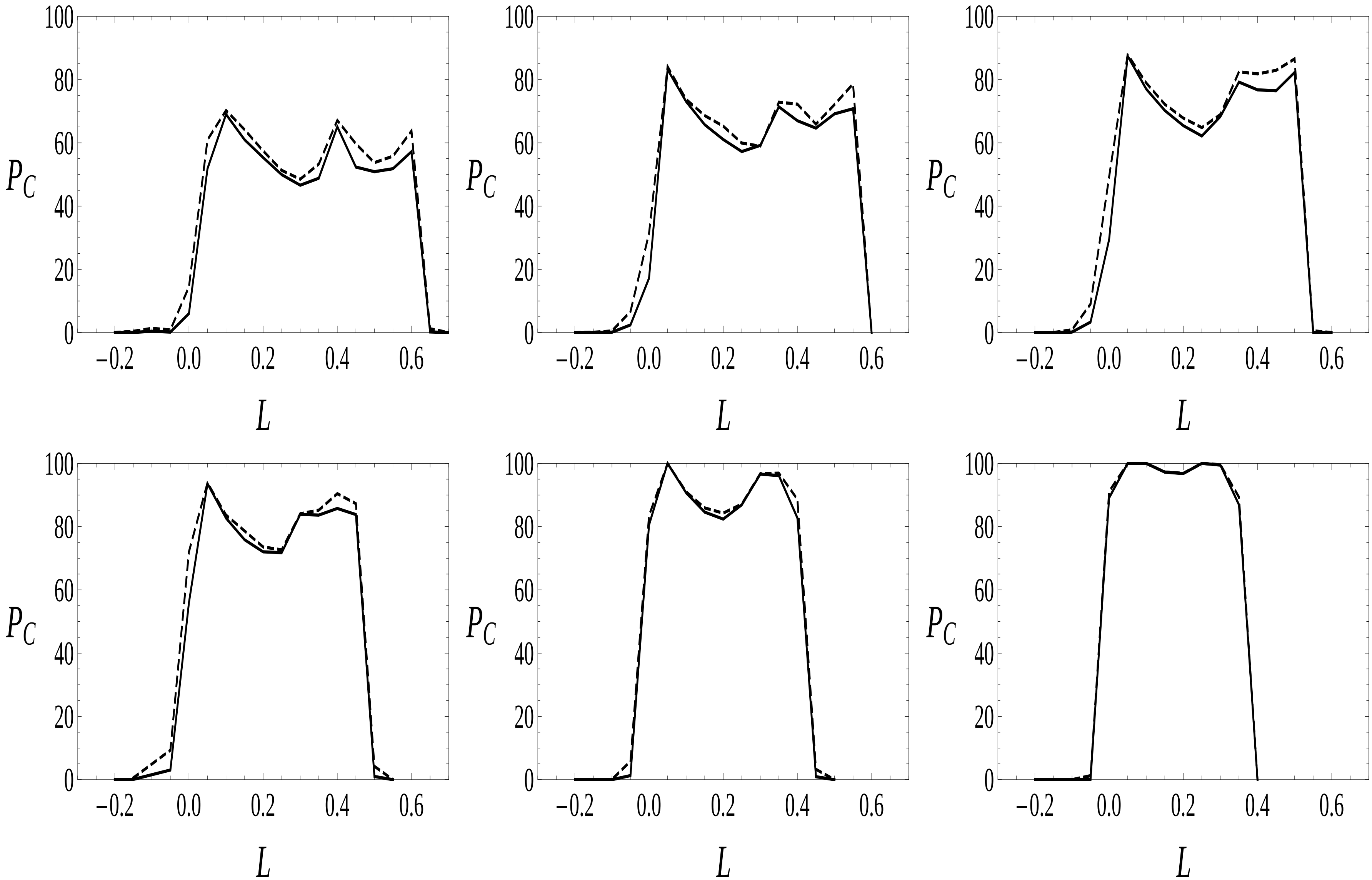




The Trimer: A Paradigm for the Birth of Chaotic Dynamics



The Trimer: Percentages of Chaoticity for Different Energies



Twist I: An Unexpected Connection: Hermite Polynomials

- For Same Charge Vortices and Constant Precession, the vortices may rotate as a rigid body. Use: $i\dot{z}_j = S_j \omega_{pr} z_j + \sum_{k \neq j} S_k/(\bar{z}_j - \bar{z}_k)$ $(S_j = S_k = 1)$.

- For $z_j = e^{-i\tilde{\Omega}t} x_j$, $\Omega = \tilde{\Omega} - \omega_{pr}$, the Equation of Motion becomes:

$$\Omega x_i = \sum_{j=1, j \neq i}^{n} \frac{1}{x_i - x_j}. \tag{30}$$

- Define the Generating Function $P(x) = \prod_{i=1}^{n}(x - x_i) = (x - x_1)\ldots(x - x_n)$ that satisfies:

$$P'(x) = P(x) \sum_{i=1}^{n} \frac{1}{x - x_i}, \quad P''(x) = P(x) \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{1}{(x - x_i)(x - x_j)} \tag{31}$$

- Observing that $1/((x - x_i)(x - x_j)) = [\frac{1}{x - x_i} - \frac{1}{x - x_j}]\frac{1}{x_i - x_j}$, we get that $P''(x) = 2P(x)\sum \frac{x_i}{x - x_i} = 2P(x)\sum \frac{x_i - x + x}{x - x_i} = 2P(x)\sum -1 + x\sum \frac{1}{x - x_i} \Rightarrow$:

$$P''(x) = -2nP(x) + 2xP'(x) \tag{32}$$

Hence, this Generating Function satisfies the Hermite Polynomial ODE and the roots should be the vortex equilibrium positions.

Twist I: Generalizing the Connection to Counter-Rotating Vortices

- For $\pm$-Charges, the Equations of Motion become:

$$x_{2k+1} = \sum_{l=1}^{n_+} \frac{1}{x_{2k+1} - x_{2l}} - \sum_{l=1, l\neq k}^{n_-} \frac{1}{x_{2k+1} - x_{2l+1}}, x_{2k} = - \sum_{l=1, l\neq k}^{n_+} \frac{1}{x_{2k} - x_{2l}} + \sum_{l=1}^{n_-} \frac{1}{x_{2k} - x_{2l+1}}$$

- Define now Two Generating Functions $P(x) = \prod_{l=1}^{n_+}(x - x_{2l})$ and $Q(x) = \prod_{l=1}^{n_-}(x - x_{2l+1})$ to obtain:

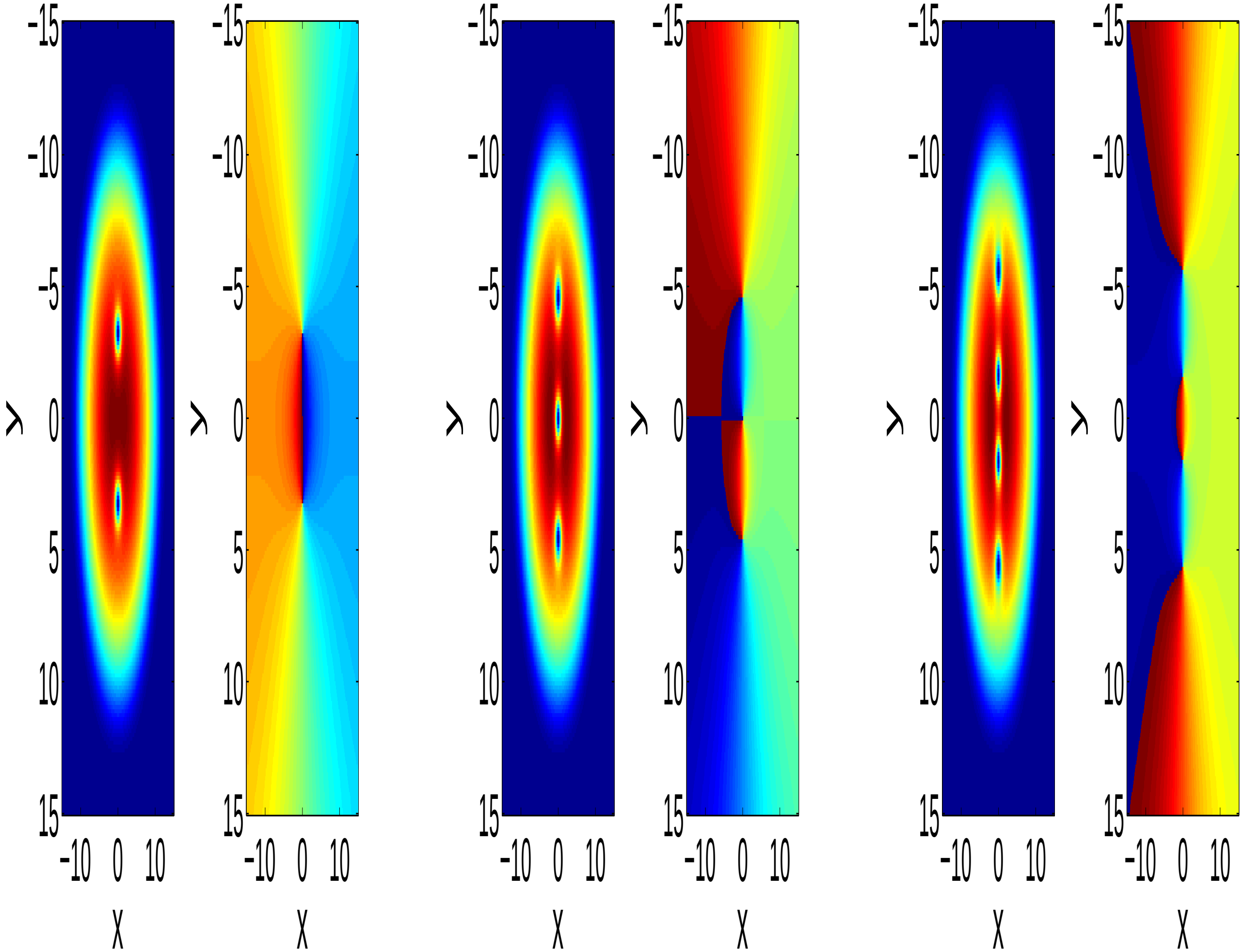
$$P''(x) = 2P(x) \left[\sum_{l=1}^{n_+} \frac{-x_{2l}}{x - x_{2l}} + \sum_{l=1}^{n_+} \sum_{k=1}^{n_-} \frac{1}{(x - x_{2l})(x_{2l} - x_{2k+1})} \right] \tag{33}$$

$$Q''(x) = 2Q(x) \left[\sum_{k=1}^{n_-} \frac{-x_{2k+1}}{x - x_{2k+1}} + \sum_{l=1}^{n_+} \sum_{k=1}^{n_-} \frac{1}{(x - x_{2k+1})(x_{2k+1} - x_{2l})} \right]. \tag{34}$$

- Multiplying $Q\times$ (33) $+P\times$ (34), we obtain the Tkachenko Equation

$$PQ'' + QP'' = 2P'Q' + 2(n_+ + n_-)PQ - 2x(PQ)'. \tag{35}$$

- Now all the above Steady States can be obtained: For Dipole $P(x) = (x - a)$ and $Q(x) = (x + a)$, yielding the 2nd Hermite Polynomial and $a = \pm 1/\sqrt{2}$. For Tripole $P(x) = x$ and $Q(x) = x^2 - a^2$, we get the same $a = \pm 1/\sqrt{2}$. For Aligned Quadrupole $P(x) = (x - a)(x - b) = Q(-x)$. Then $a^2 + b^2 = 1$, $ab = (1 \pm \sqrt{2})/2$.



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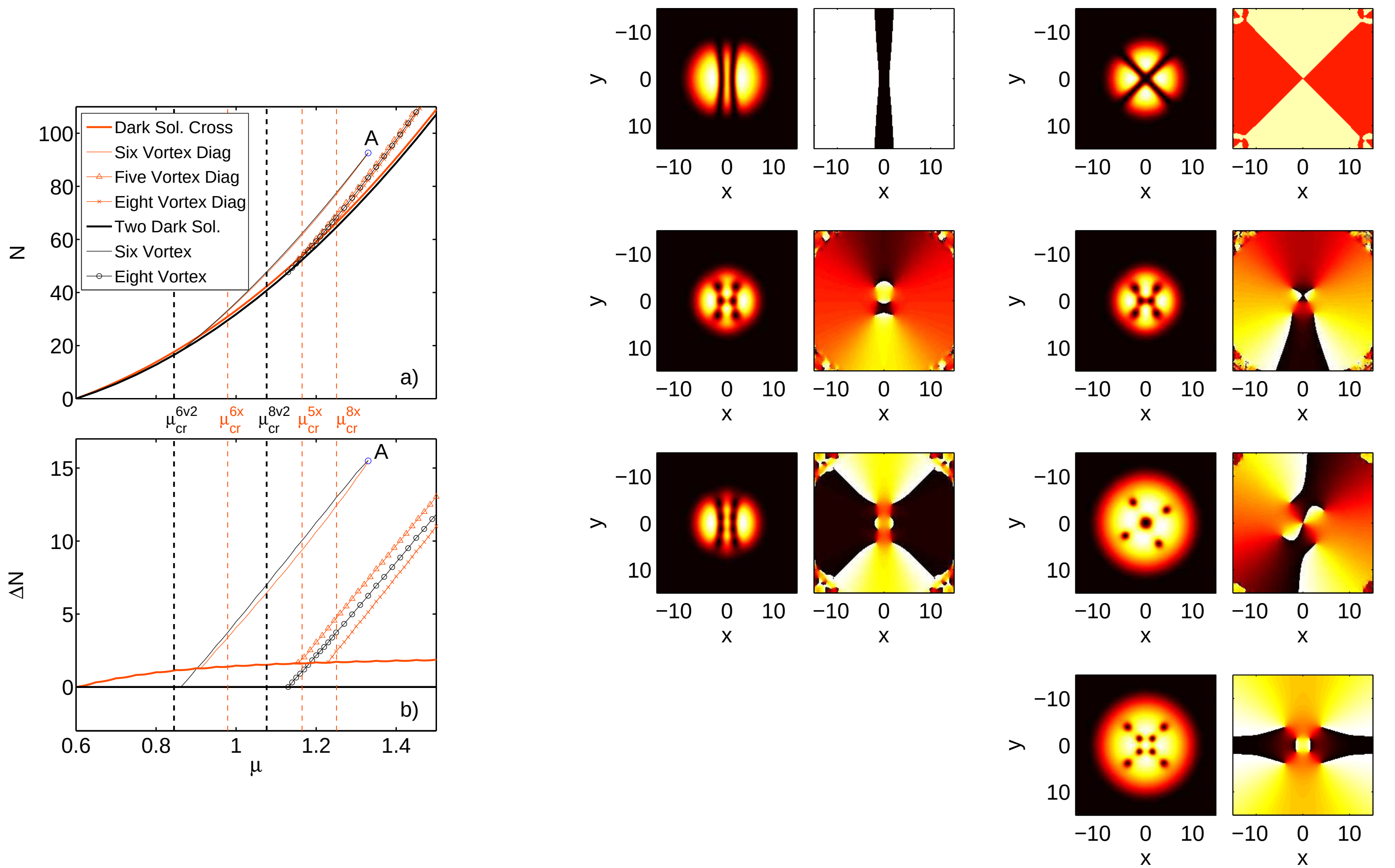
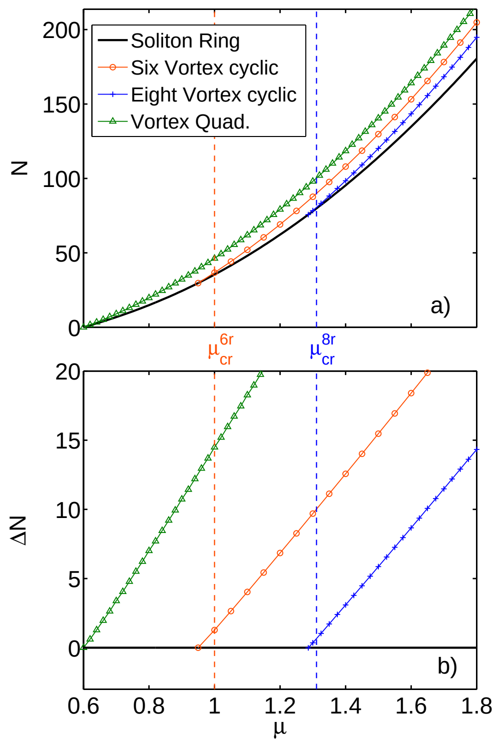
Twist II: Generalizing the Bifurcation Picture

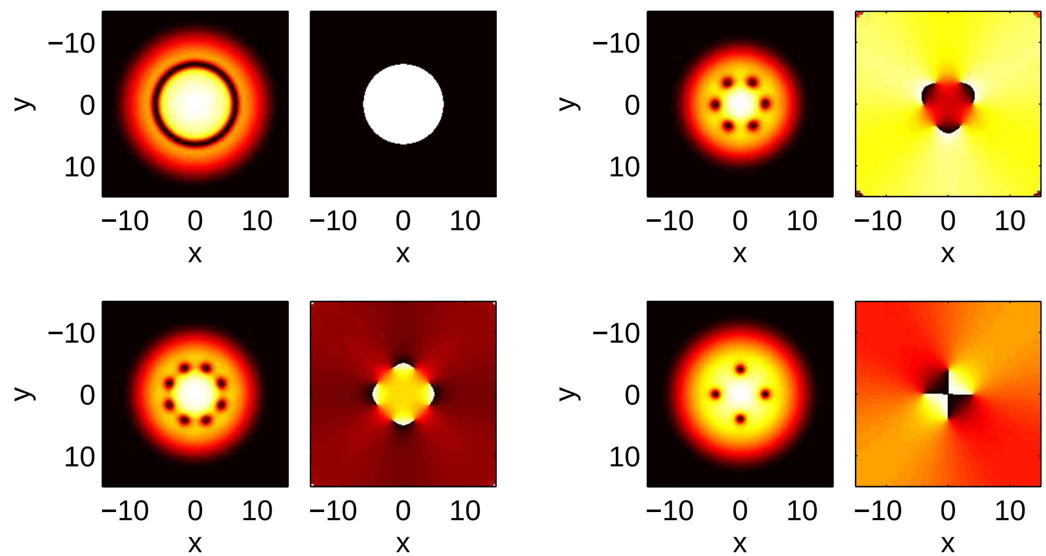
Part a: Rectangular States



Twist II: Generalizing the Bifurcation Picture

Part b: Radial States





Generating Functions on the Complex Plane: Part I

- Steady Equations of Motion generalize as:

$$S_j z_j = -\sum_{k=1, k\neq j}^{n_+ + n_-} S_k \frac{z_j - z_k}{|z_j - z_k|^2}. \tag{36}$$

- For Generating Functions $P(z) = \prod_{i=1}^{n_+}(z - z_i)$ and $Q(z) = \prod_{j=1}^{n_-}(z - z_j)$, we have:

$$P''(z) = P(z) \sum_{i=1}^{n_+} \frac{2}{z - z_i}\left[-\bar{z}_i + \sum_{k=1}^{n_-} \frac{1}{z_i - z_k}\right] \tag{37}$$

$$Q''(z) = Q(z) \sum_{j=1}^{n_-} \frac{2}{z - z_j}\left[-\bar{z}_j + \sum_{k=1}^{n_+} \frac{1}{z_j - z_k}\right] \tag{38}$$

- Then, Tkachenko Equation Becomes:

$$P''Q + PQ'' = 2P'Q' + 2(n_+ + n_-)PQ - 2z(PQ)' + 4iPQ \sum_{j=1}^{n_+ + n_-} \frac{y_j}{z - z_j} \tag{39}$$

- Seek Solutions of the form: $P(z) = z^n - R^n$ and $Q(z) = z^n - R^n e^{in\phi}$:

$$R^2 = \frac{1}{2}, \quad e^{in\phi} = -1 \tag{40}$$

Generating Functions on the Complex Plane: Part II

- Steady Equations of Motion for Single Vortex S_0, surrounded by n Vortices S

$$S_0 z_0 = -\sum_{j=1}^{n} \frac{S}{\bar{z}_0 - \bar{z}_j} \tag{41}$$

$$S z_k = -\sum_{j=1, j\neq k} \frac{S}{\bar{z}_k - \bar{z}_j} - \frac{S_0}{\bar{z}_k - \bar{z}_0} \tag{42}$$

- We define the usual Generating Function $P(z) = \prod_{j=1}^{n}(z - z_j)$, obtaining:

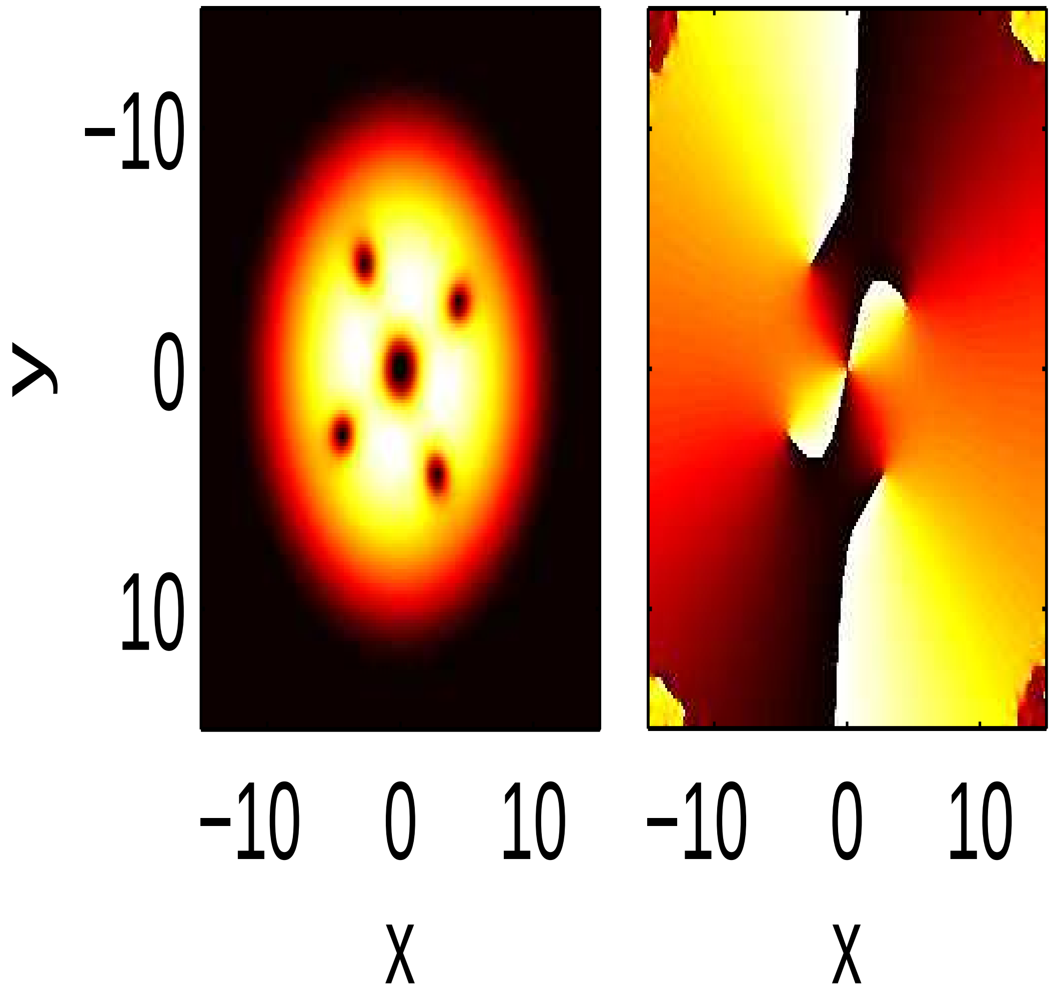
$$P''(z) = 2P(z) \sum_{i=1}^{n} \sum_{j=1, j\neq i} \frac{1}{z - z_i} \left[-\frac{S_0}{S} \frac{1}{z_i - z_0} - \bar{z}_i \right] \tag{43}$$

- The Last Term yields $2nP(z) - 2zP'(z) + 4iP(z)\sum_n y_i/(z - z_i)$.
- For the First Term, assuming a Ring of radius R, $P(z) = z^n - R^n$ and $z_0 = 0$, we obtain:

$$-2P(z)\frac{S_0}{S} \sum_{i=1}^{n} \frac{1}{z - z_i} \frac{1}{z_i - z_0} = -2P(z)\frac{S_0}{S}\frac{n z^{n-2}}{z^n - R^n} = -2\frac{S_0}{S} n z^{n-2} \tag{44}$$

- Finally, substituting everything, we get:

$$R^2 = -\frac{S_0}{S} + \frac{1-n}{2} \tag{45}$$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Generating Functions on the Complex Plane: Part II

- Use in Tkachenko Equation the Ansatz

$$P(z) = (z^n - R_1^n)\left(z^n - R_2^n e^{in\phi}\right) \tag{46}$$

$$Q(z) = (z^n - R_2^n)\left(z^n - R_1^n e^{in\phi}\right). \tag{47}$$

- Obtain the Solvability Conditions, Specifically for $\phi = \pi/n$

$$1 = R_1^2 + R_2^2 \tag{48}$$

$$0 = 4nR_1^n R_2^n + (-1 + 2R_1^2)R_2^{2n} + R_1^{2n}(-1 + 2R_2^2). \tag{49}$$

- Use in Tkachenko Equation the Ansatz

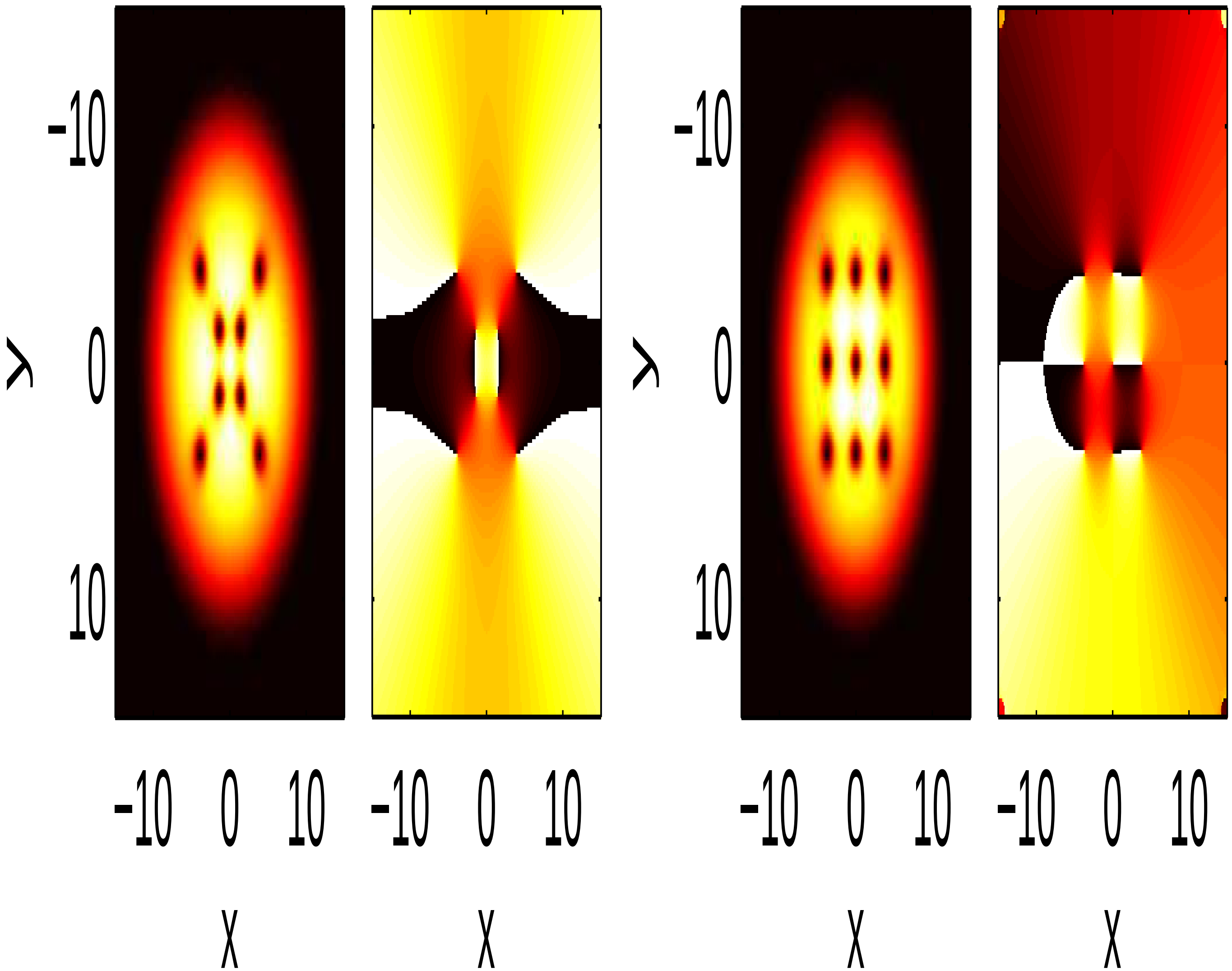
$$P(z) = z(z^n - R_1^n) \tag{50}$$

$$Q(z) = \left(z^n - R_2^n e^{in\phi}\right). \tag{51}$$

- Obtain the Solvability Conditions, Specifically for $\phi = \pi/n$

$$1 = R_1^2 + R_2^2 \tag{52}$$

$$0 = -e^{in\phi}(1 + n + 2R_1^2)R_2^n - R_1^n(-3 + n + 2R_2^2). \tag{53}$$



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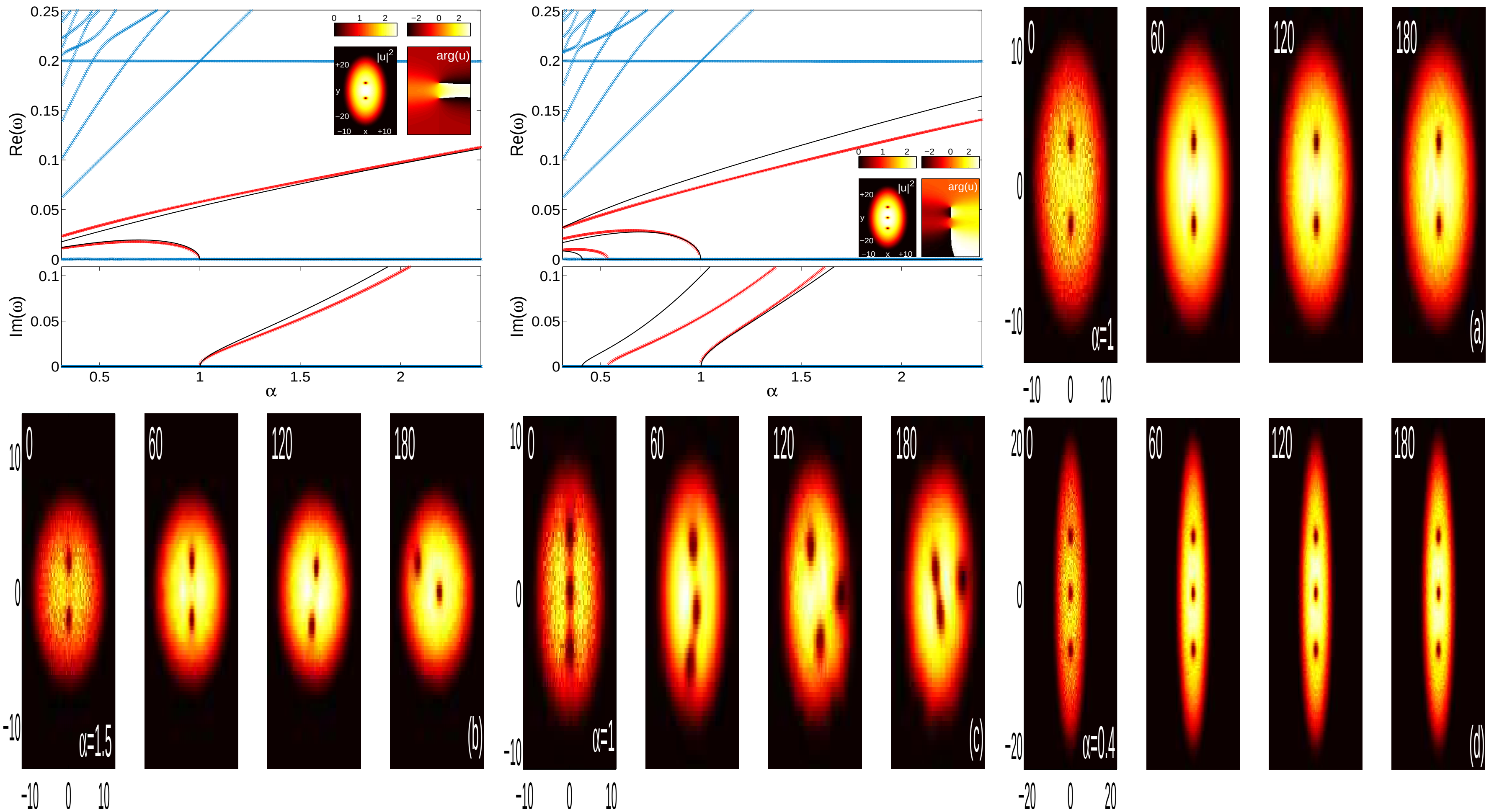
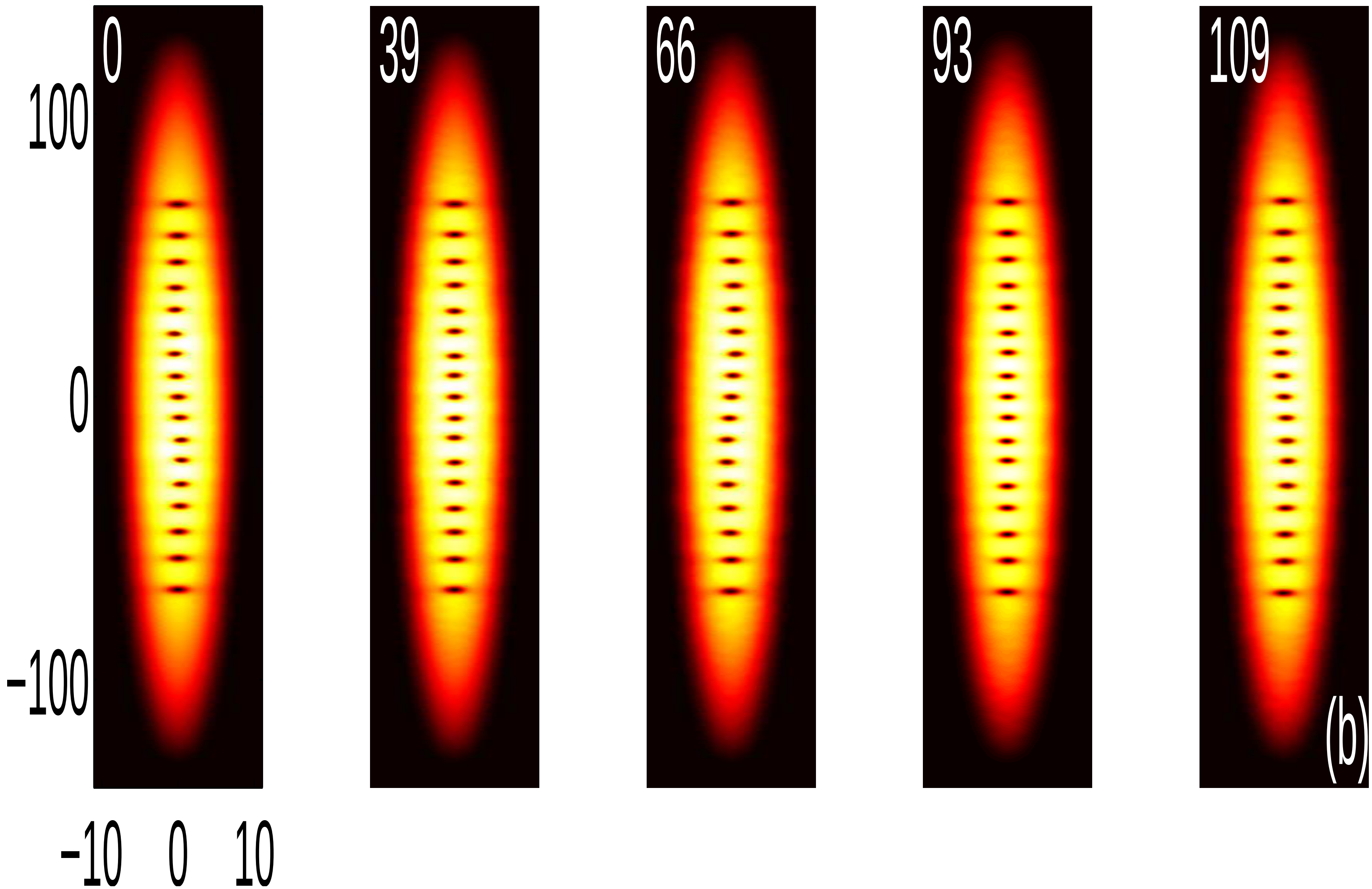
Twist III: The role of Anisotropy

- $\dot{x} = -\omega_y^2 Q y$ and $\dot{y} = \omega_x^2 Q x$, where $Q = \ln(A\mu/\omega_{\text{eff}})/(2\mu)$, $A \approx 2\sqrt{2}\pi$ and $\omega_{\text{eff}} = \sqrt{(\omega_x^2 + \omega_y^2)/2}$



Stabilizing Vortex States at Will



Twist IV: More on the Co-Rotating Vortices

- One can write Equations of Motion for 2 Vortices in the form:

$$\dot{r}_m = -\frac{c r_n \sin(\theta_m - \theta_n)}{\rho_{mn}^2}, \quad \dot{\theta}_m = -\frac{c r_n \cos(\theta_m - \theta_n)}{r_m \rho_{mn}^2} + \frac{c}{\rho_{mn}^2} + \frac{1}{1 - r_m^2}. \quad (54)$$

- Stationary Solutions then satisfy: $r_1 = r_2 = r_*$ and $\theta_1 - \theta_2 = \pi$, while: $\dot{\theta}_1 = \dot{\theta}_2 = \omega_{orb} = \frac{c}{2r_*^2} + \frac{1}{1-r_*^2}$.

- Linearization around this Stationary State yields the Epicyclic Frequency: $\omega_{\text{ep}}^2 = \frac{c^2}{2r_*^4} - \frac{2c}{\left(1-r_*^2\right)^2}$.

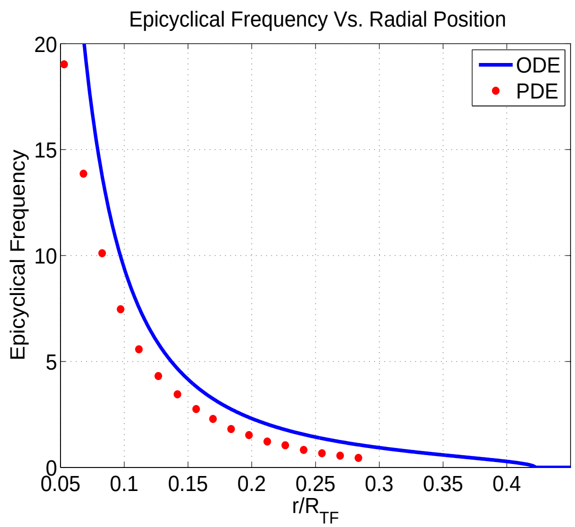
- This Changes Sign, causing an Instability at: $r_{\text{cr}}^2 = \sqrt{c}/(\sqrt{c} + 2)$.

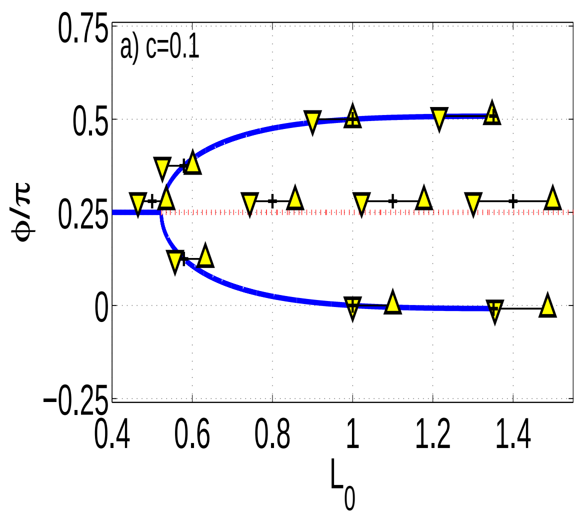
- A New Asymmetric State emerges, satisfying: $-r_1^* r_2^* (r_1^* + r_2^*)^2 + c\left(1 - r_1^{*2}\right)\left(1 - r_2^{*2}\right) = 0$,

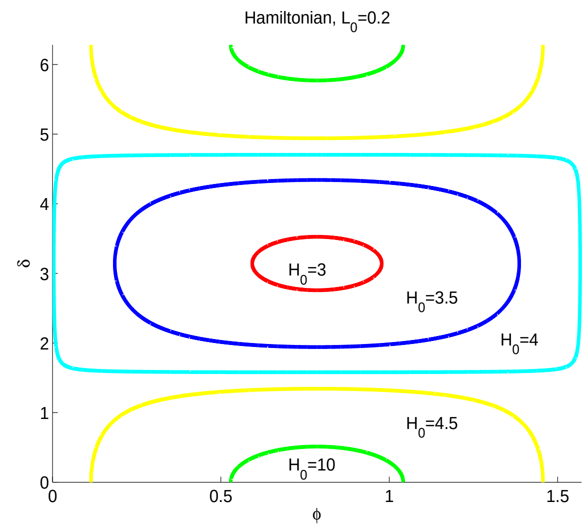
- Visualize the Instability using $L_0^2 = \sum_i r_i^2$ and

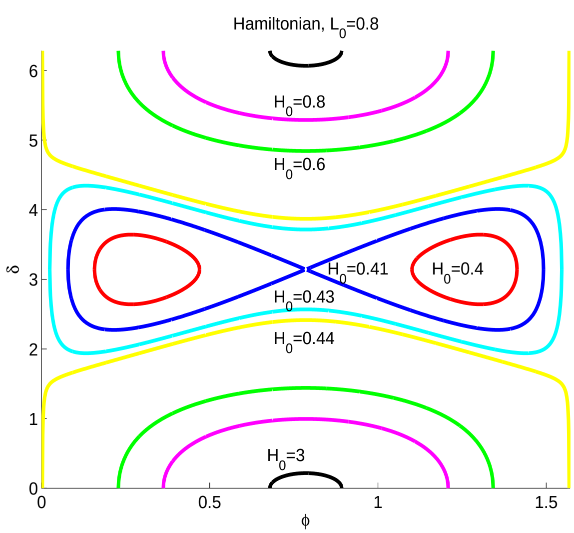
$$H = \frac{1}{2} \ln\left[\left(1 - r_1^2\right)\left(1 - r_2^2\right)\right] - \frac{c}{2} \ln\left[r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta_2 - \theta_1)\right].$$

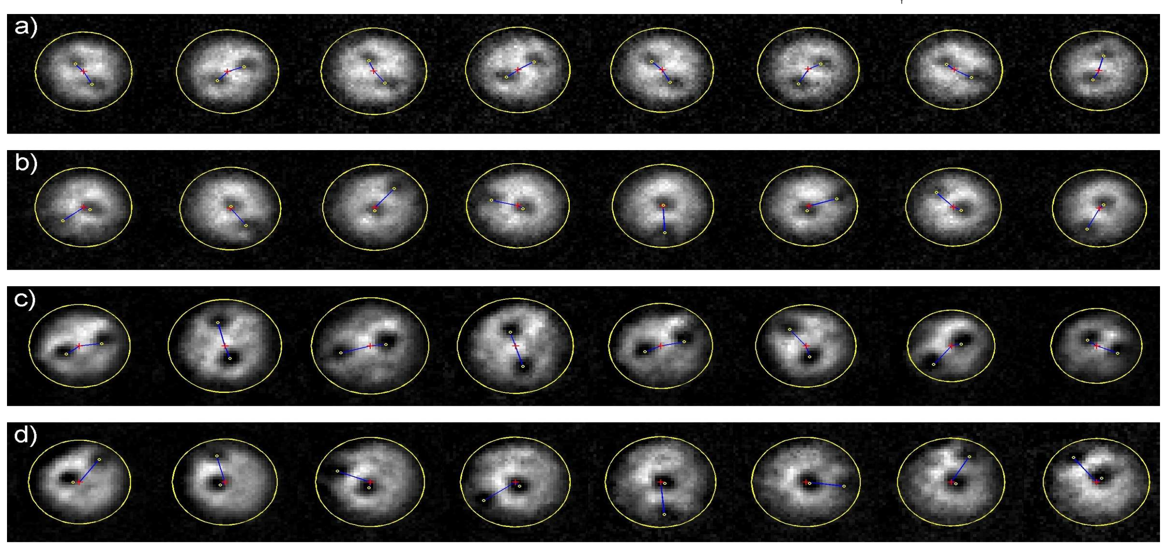
Twist IV: Symmetry Breaking For 2 Co-rotating Vortices











Twist IV: Symmetry Breaking For $N>2$ Co-rotating Vortices

- Similar Instabilities arise for $N=3,\dots,6$ vortices, respectively with: $r_1^2 \equiv \frac{\sqrt{c}}{\sqrt{c}+2}$, $r_2^2 \equiv \frac{\sqrt{c}}{\sqrt{c}+\sqrt{2}}$, and $L_{cr,N=2}=2\,r_1^2$, $L_{cr,N=3}=3\,r_2^2$, $L_{cr,N=4}=4\,r_2^2$, $L_{cr,N=5}=5\,r_2^2$, $L_{cr,N=6}=6\,r_1^2$.

- Using a Perturbation Theory of the form: $x_k(t)=R\exp\left(2\pi ik/N\right)\left(1+h_k(t)\right)$ with $|h_k|\ll 1$, we obtain:

$$\frac{dh_k}{dt}=\left(f'(R)\frac{R}{2}+f(R)-\omega\right)h_j+f'(R)\frac{R}{2}\bar{h}_j+c\sum_{k\neq j}\frac{\exp\left(2\pi\left(k-j\right)/N\right)\bar{h}_j-\bar{h}_k}{4R^2\sin^2\left(\frac{\pi(k-j)}{N}\right)},$$

- Decomposing into Fourier Modes:

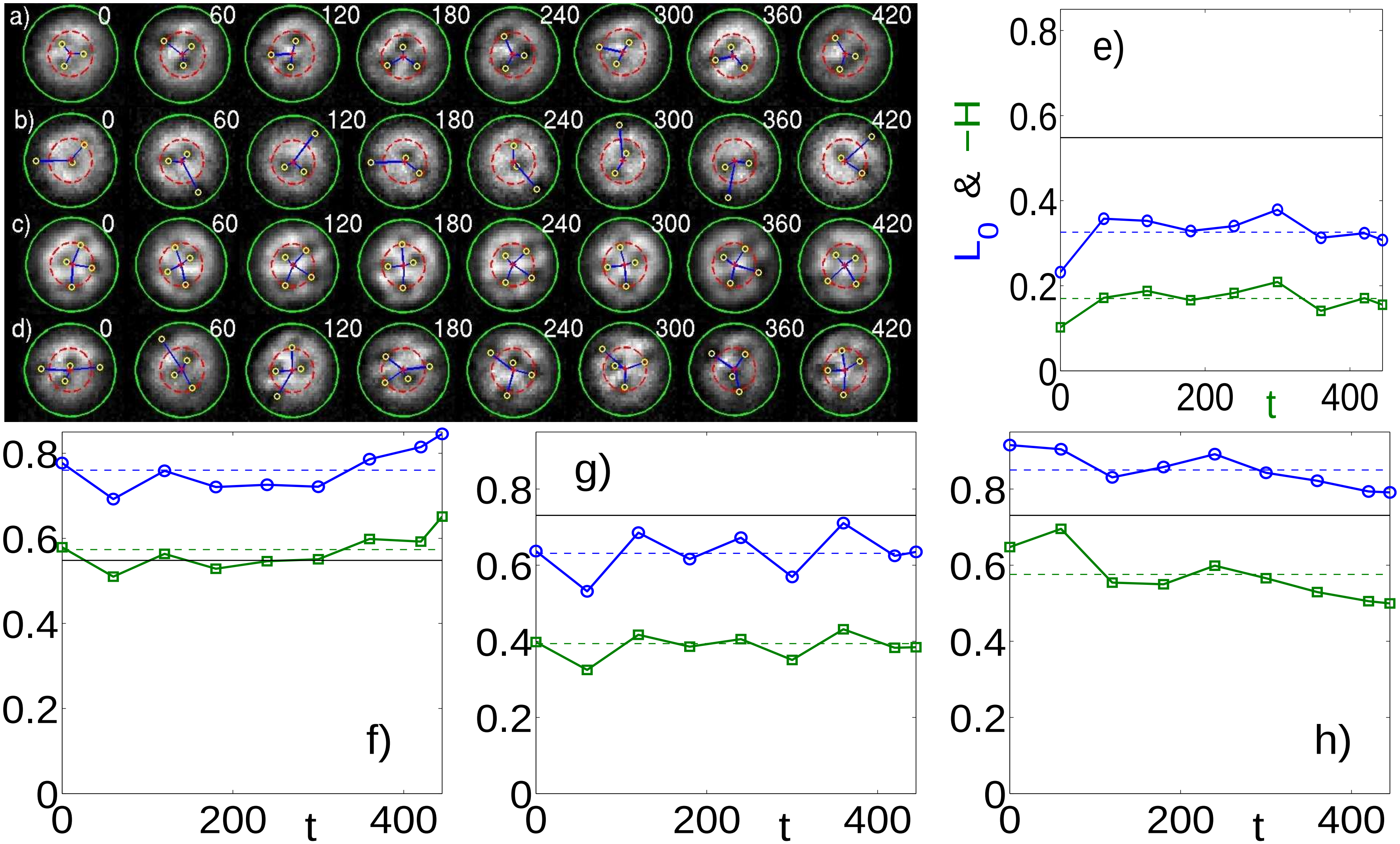
$$h_j(t)=\xi_+(t)e^{im2\pi j/N}+\xi_-(t)e^{-im2\pi j/N},\qquad m\in\mathbb{N}, \tag{55}$$

and evaluating the eigenvalues, we find:

$$\begin{aligned}\lambda_-(m) &= \frac{c}{2R^2}\left[-(N-1)-(m-1)\left(N-m-1\right)\right],\\ \lambda_+(m) &= f'(R)R+\frac{c}{2R^2}\left[(m-1)\left(N-m-1\right)-(N-1)\right].\end{aligned} \tag{56}$$

- For general Precession Term $f(r)$ (instead of $1/(1-r^2)$), Instability of Ring of Vortices occurs when $f'(R)R+c(N-1)(N-7)/(8R^2)>0$. At Equilibrium $(f(R)-\omega)R+c(N-1)/(2R)=0$.

Twist IV: Symmetry Breaking For 3, 4 Co-rotating Vortices



Twist V: The (Continuum) Limit of Many Co-rotating Vortices

- Assume a Coarse-Graining of the Vortex Density $\rho(x) = \sum_{k=1\ldots N} \delta(x - x_k)$. Then, the Aggregation Model can be written as $\dot{x}_j = v(x_j)$ with:

$$v(x) \equiv (f(r) - \omega)\, x + c \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \rho(y)\, dy.$$

- Using $\int_{\mathbb{R}^2} \rho(x) dx = N$, the corresponding Continuum PDE (Conservation Law) reads:

$$\rho_t + \nabla \cdot (v\rho) = 0. \tag{57}$$

- Using the CDF $u(r) = \int_0^r \rho(s) s ds$, and $V = r\,(f(r) - \omega) + \frac{2\pi c}{r} u$, the Continuity PDE can be written:

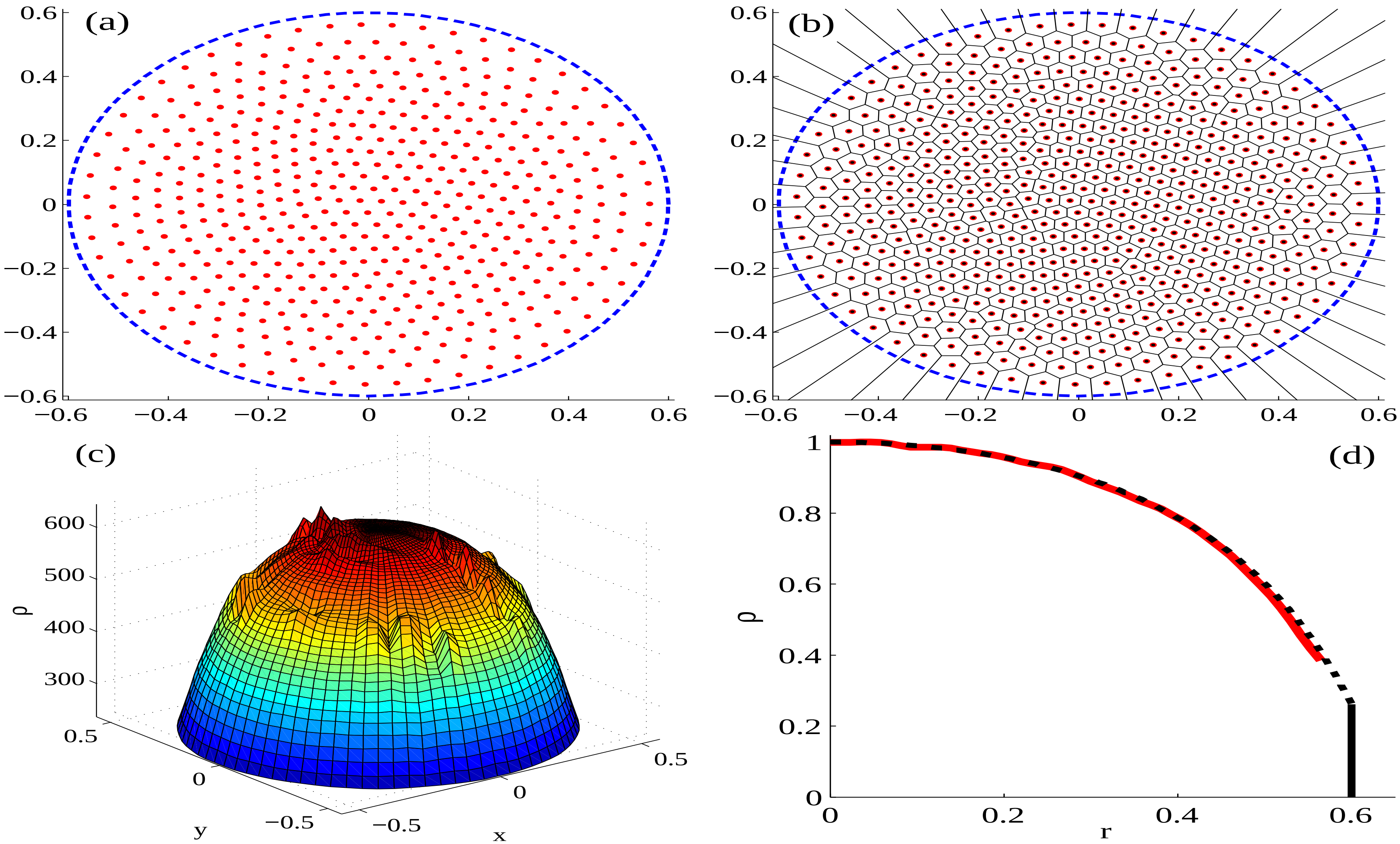
$$u_t + V r u_r = 0. \tag{58}$$

- Then Characteristic Lines evolve according to: $\frac{dr}{dt} = r\,(f(r) - \omega) + \frac{2\pi c}{r} u$, which means that at steady state $u = -r^2\,(f(r) - \omega)\,/(2\pi c)$ and finally:

$$\rho = \frac{1}{\pi c} \left(\omega - \frac{a}{(1 - r^2)^2} \right). \tag{59}$$

- This Distribution is a Good Approximation of the corresponding Numerics.

Twist V: The (Continuum) Limit of Many Co-rotating Vortices



Twist VI: 3-dimensions and Vortex Rings

- A Single Vortex Ring travels according to ($r_c = |m|\xi$):

$$c \approx -\frac{am}{R}\left(\ln\frac{8R}{r_c} + L_0(m) - 1\right), \tag{60}$$

- It is sought as Exact (Numerical) Solution with:

$$\Psi_0 = \Psi(r,z,\theta,0) = g(r,z)\ \exp[im(\phi^-(r,z) - \phi^+(r,z))]$$

where $g(r,z) = f(\rho(r,z))$ with $\rho(r,z) = \sqrt{(r-R)^2 + z^2}$ and $\phi^\pm(r,z) = \arctan\left(\frac{z}{r\mp R}\right)$. A Galilean Boost is also applied to make it steady: $\Psi = U(r,Z,\theta)\ \exp\left[i\left(\frac{c}{2a}z + \left[\Omega - \frac{c^2}{4a}\right]t\right)\right]$, where $Z = z - ct$

- For Ring Scattering we find: $\theta_s \approx -\frac{1}{2}\frac{q}{R} + \frac{\pi}{4}$ if $q/R \in (1/2, 2)$ and $\theta_s \approx -2\frac{q}{R} + \frac{\pi}{2}$ for $q/R \in [0, 1/2)$.

Twist VI: 3-dimensions and Vortex Rings

- The Velocity Field induced by a Vortex Ring around is given by Biot-Savart:

$$\mathbf{u}_v(\mathbf{x}) = \frac{1}{4\pi}\int \frac{\boldsymbol{\omega}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d\mathbf{x}', \tag{61}$$

- Finally, the Particle Equations here read (combining the Translation of Single Ring and their Pairwise Interaction in the Local Induction Approximation:

$$\dot{Z}_i = \frac{\kappa_i}{4\pi R_i}\left(\ln\frac{8R_i}{r_{c,i}} - C\right) + \frac{1}{\kappa_i R_i}\frac{\partial U}{\partial R_i}, \tag{62}$$

$$\dot{R}_i = -\frac{1}{\kappa_i R_i}\frac{\partial U}{\partial Z_i}, \tag{63}$$

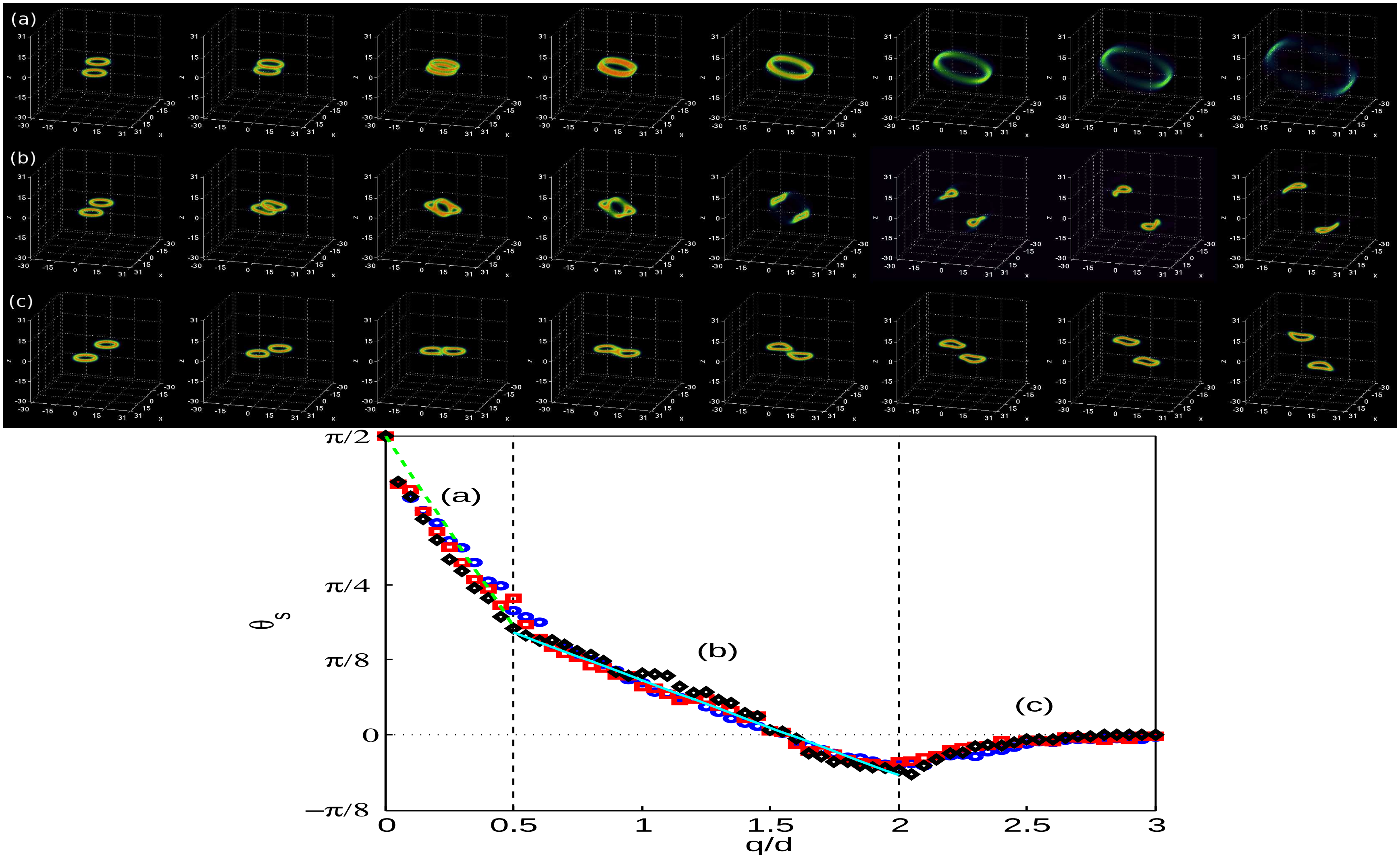
where

$$U = \frac{1}{2\pi}\sum_{i=1}^{N}\sum_{j>i}^{N}\kappa_i\kappa_j I_{ij},$$

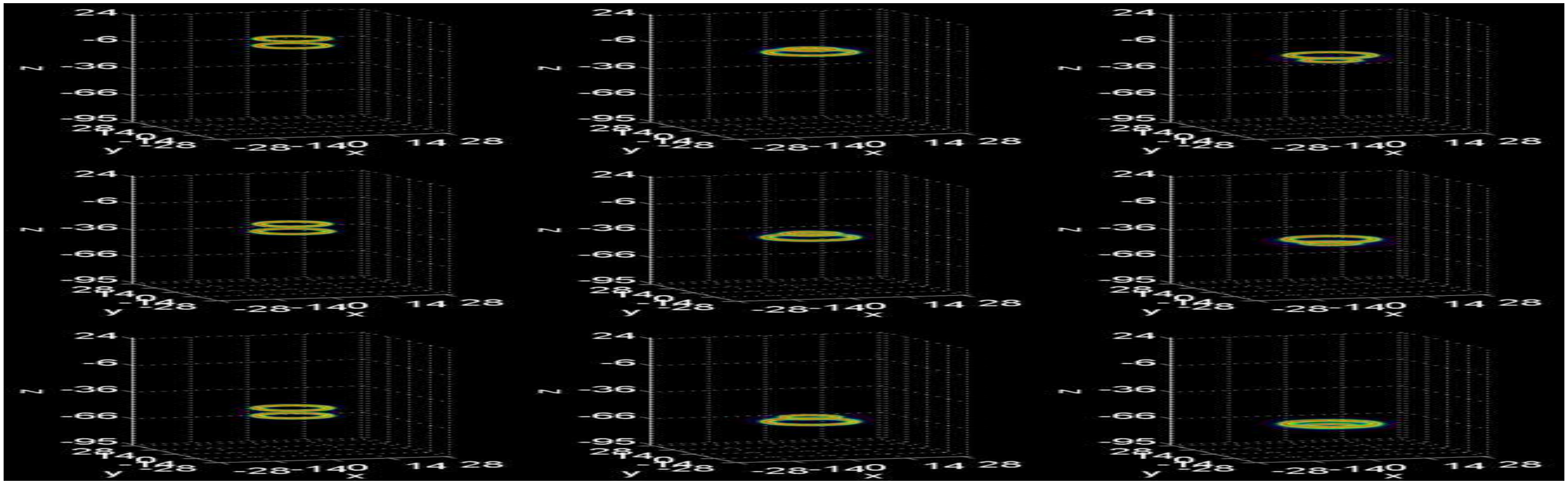
and

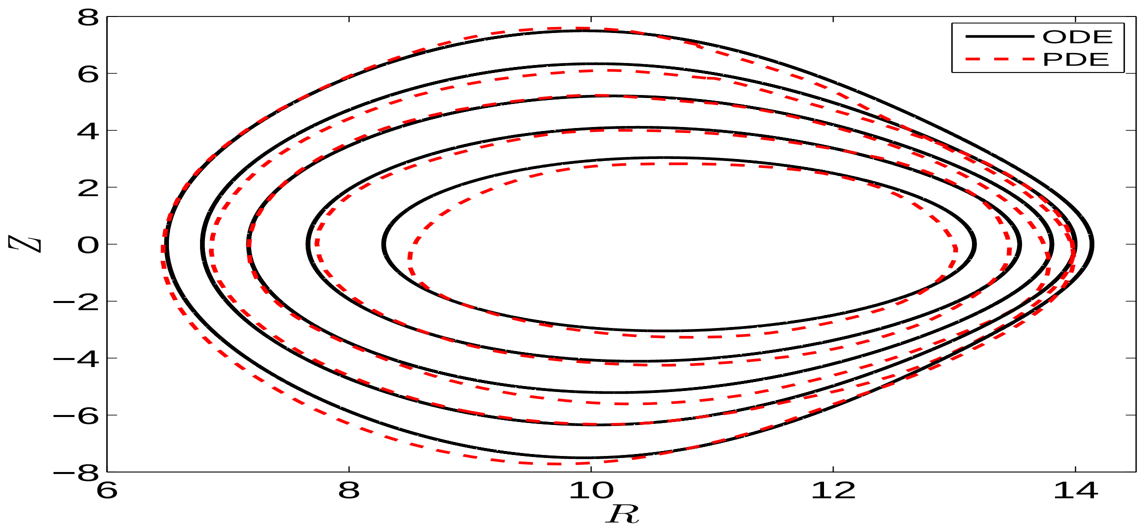
$$I_{ij} = \int_0^{\pi} \frac{R_i R_j \cos\theta \, d\theta}{\sqrt{(Z_i - Z_j)^2 + R_i^2 + R_j^2 - 2R_iR_j\cos\theta}},$$

Twist VI: Vortex Ring Scattering

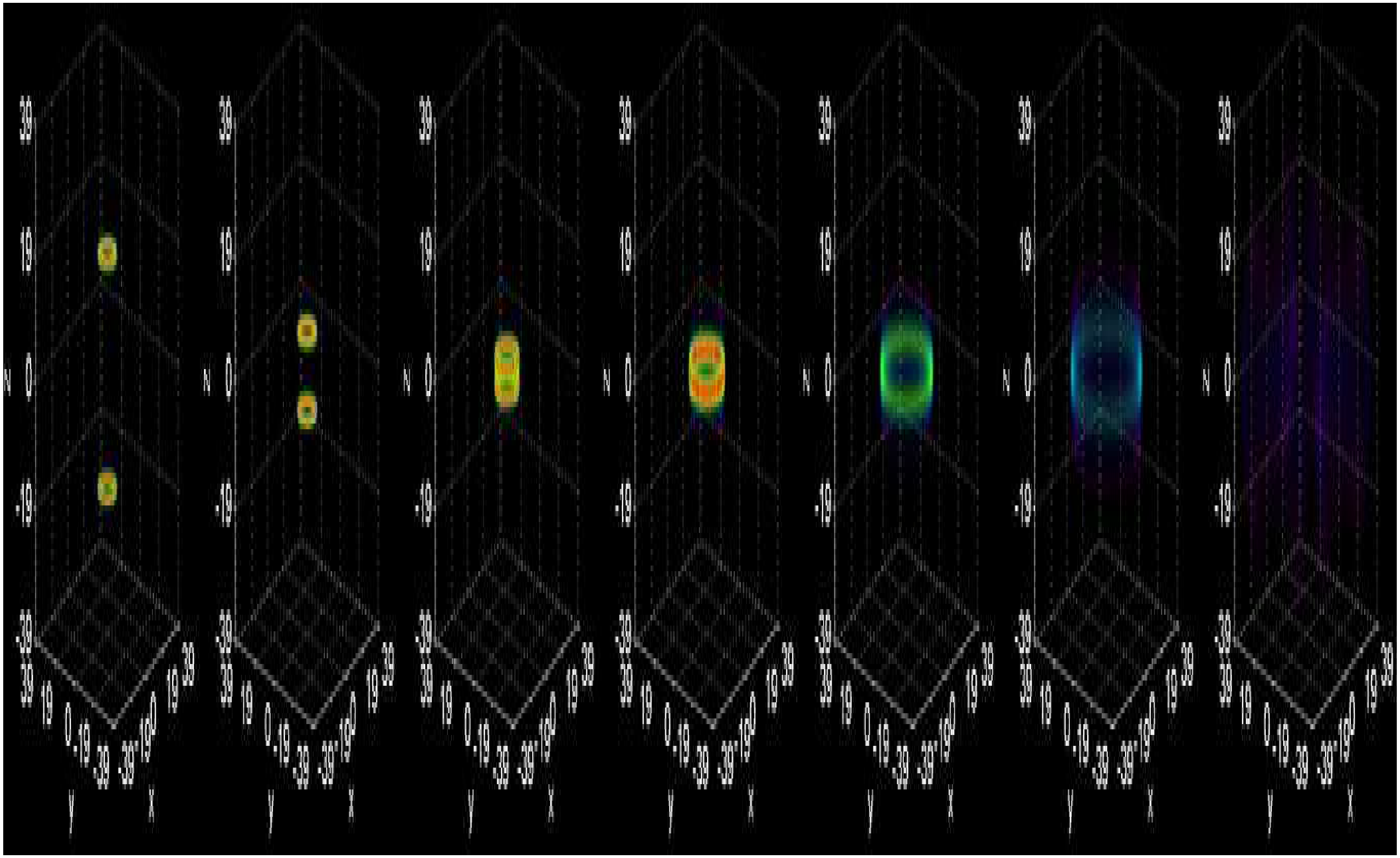


Twist VI: Vortex Ring Leapfrogging





Twist VI: Vortex Ring Annihilation



Summary of Results

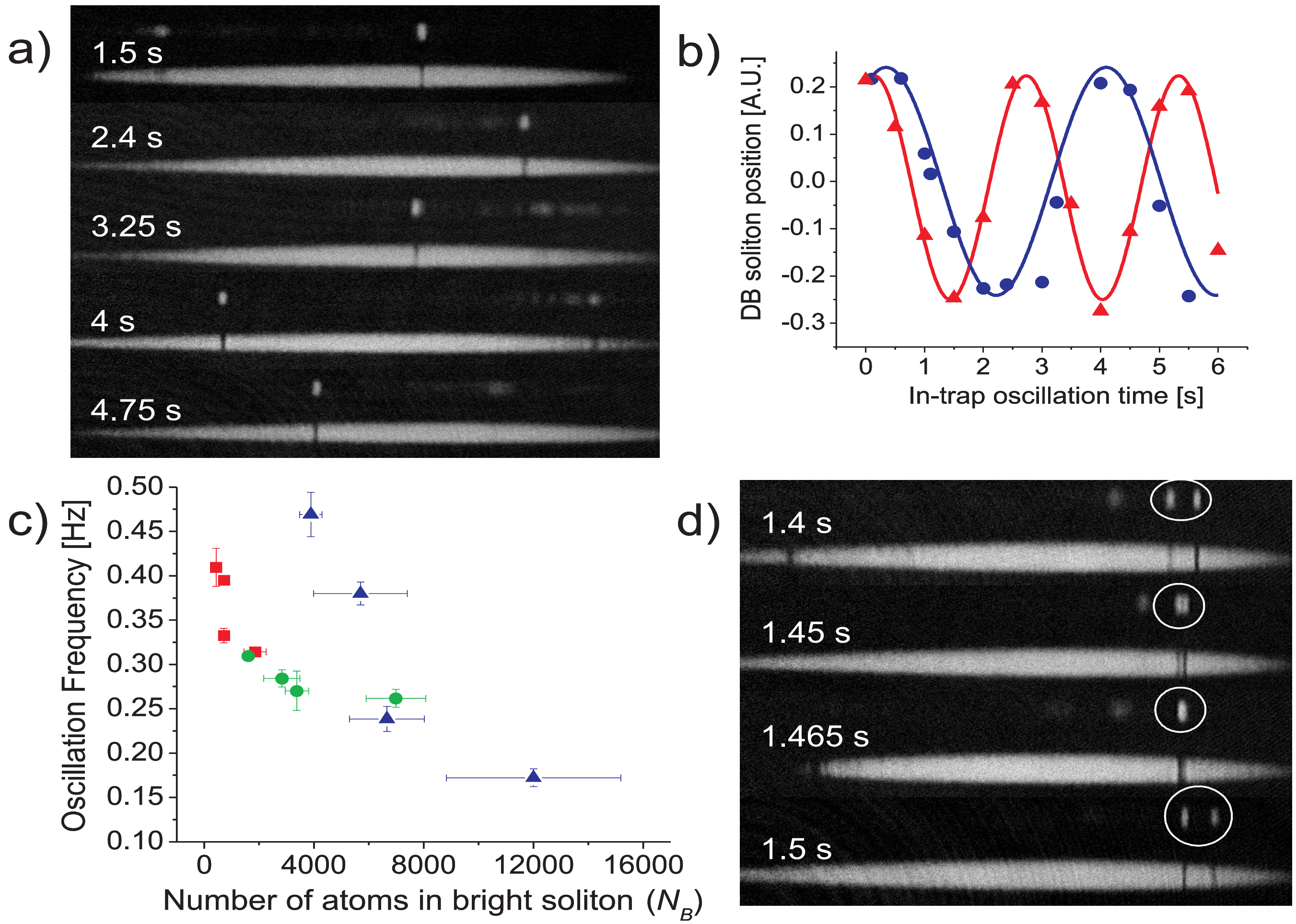
- The Single Vortex stably Precesses in a 2d Trap.
- The 2d Generalization of Dark Solitons becomes Progressively More Unstable.
- Out of this Symmetry Breaking Emanate Multi-Vortex States
- Properties such as the Equilibrium Positions and Epicyclic Dynamics of such states can be Experimentally Monitored.
- For the Vortices similarly to the Dark Solitons, Particle Based Methods can be developed to monitor their Complex Dynamics.
- Generating Function Methods can be used to Identify Equilibria and provide Intriguing Connections to Classical Polynomials.
- Finally, for Vortex Rings, we characterized their Translation and Pairwise Interactions, observing interesting Scattering, Leapfrogging, Annhilation etc. Events.

Present/Future Challenges

- Vortices
  - Generalization of the Picture → Including Effects of Temperature, N-Vortex States
  - Connections to Quantum Superfluid Turbulence
  - Can Arbitrary Vortex Configurations be generated in a trap ?
  - Configurations with an Intermediate Number of Vortices
  - How do the Vortices get generated/inserted in the trap via rotation ?
- For Vortex Rings
  - Identify Trap Effects, Include them in Dynamics, Spectral Picture, etc.
  - Understand their Interaction Dynamics in the Trap/Multi VR states

Another Generalization: (Pseudo-) Spinor Condensates

2-Components, 1-dimension: Dark-Bright Solitons in Pullman



Extensions in 2-Components, 1-Dimension: Dark-Bright Solitons

- Model reads: $i\hbar\partial_t\psi_j = \left(-\frac{\hbar^2}{2m}\partial_x^2\psi_j + V(x) - \mu_j + \sum_{k=1}^{2} g_{jk}|\psi_k|^2\right)\psi_j.$

- Dark-Bright Soliton Solutions:

$$\psi_1(x,t) = \cos\phi \tanh\left[D(x - x_0(t))\right] + i\sin\phi, \tag{64}$$

$$\psi_2(x,t) = \eta\,\mathrm{sech}\left[D(x - x_0(t))\right] \exp\left[ikx + i\theta(t)\right], \tag{65}$$

- Interaction between 2 DB has 3 pieces (Stationary State Exists) + Restoring Force (Trap):

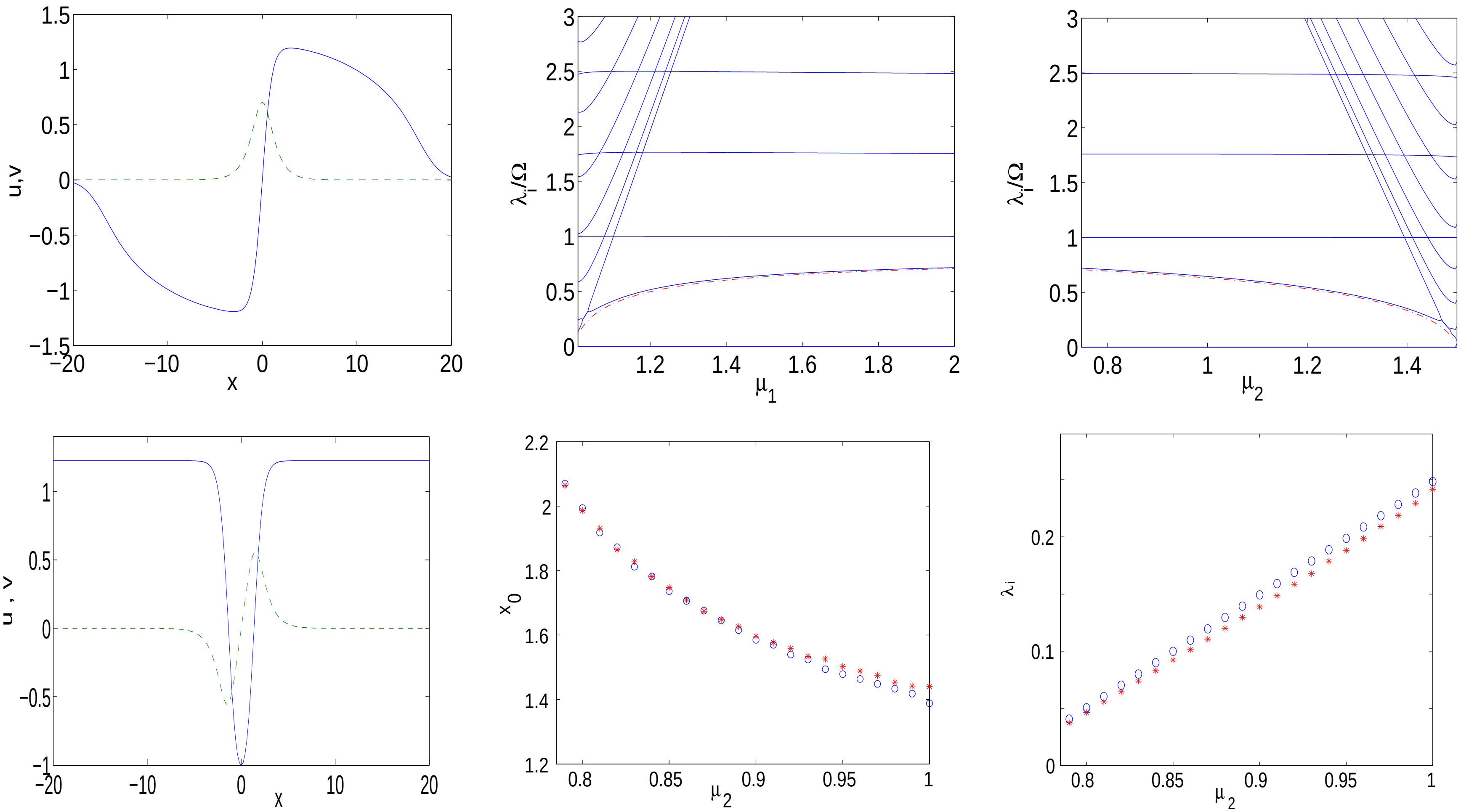
$$F_{\mathrm{DD}} = \frac{1}{\chi_o}\left[\frac{1}{3}(544 - 352D_0^2) + 128D_0\left(D_0^2 - 1\right)x_0\right] e^{-4D_0x_0},$$

$$F_{\mathrm{BB}} = \frac{\chi}{\chi_o}\left[\left(4 - 2\chi D_0 - 6D_0^2\right)D_0 + 4D_0^2\left(D_0^2 + 1\right)x_0\right]\cos\Delta\theta e^{-2D_0x_0} - 8\frac{\chi^2}{\chi_o}D_0^3x_0\cos^2\Delta\theta$$

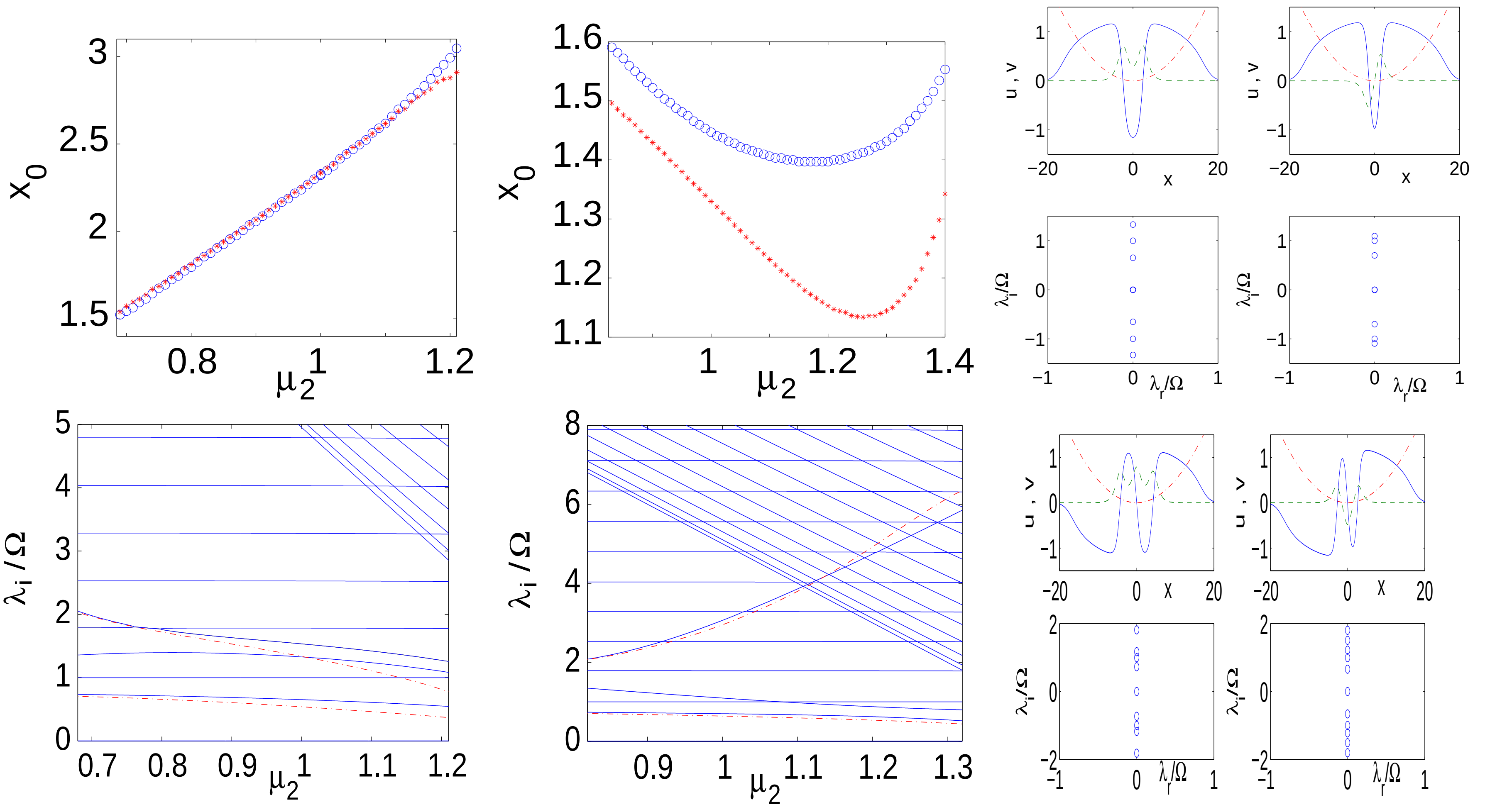
$$F_{\mathrm{DB}} = \frac{\chi}{2\chi_o}\left(6\chi D_0^2 + 12\chi D_0^2\cos\Delta\theta - \frac{214}{3}D_0 + 8\left(8D_0^2 - \chi D_0^3\right)x_0\right)e^{-4D_0x_0},$$

$$F_{Trap} = -\Omega_{DB}^2 x_0 = -\Omega^2\left(\frac{1}{2} - \frac{\chi}{\chi_o}\right)x_0$$

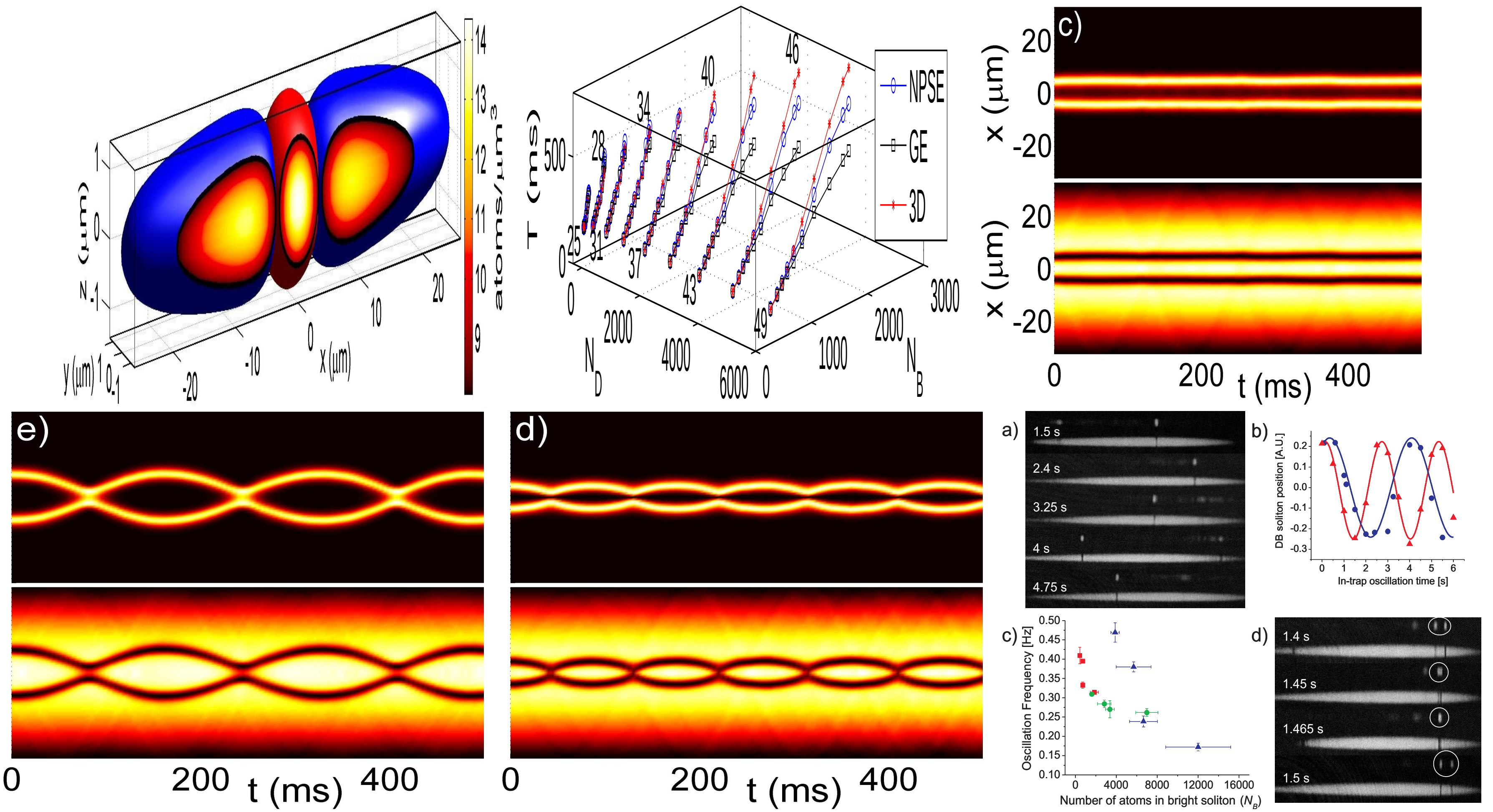
1 and 2 DBs: Analysis vs. Numerics



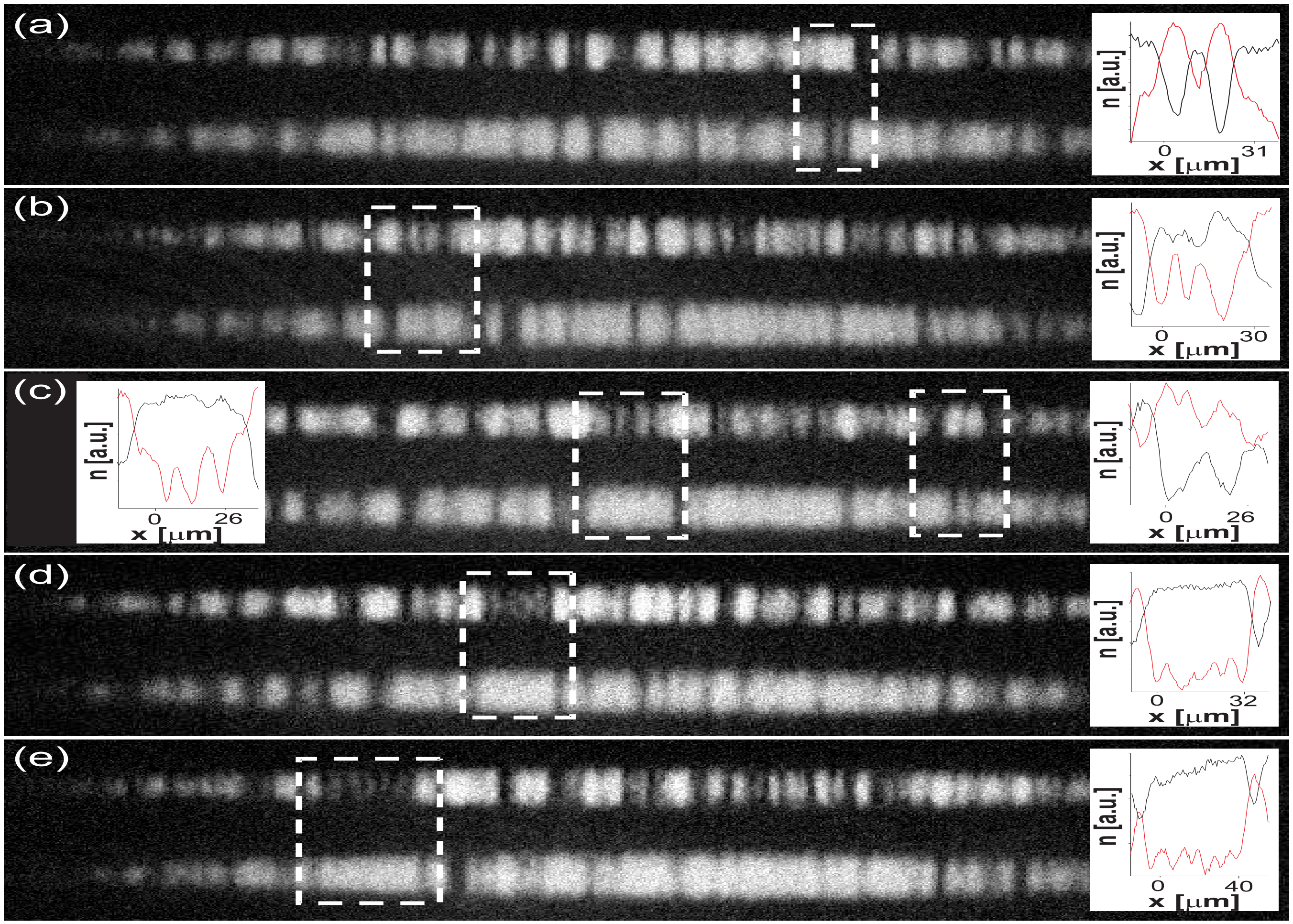
Multiple DBs: Analysis vs. Numerics



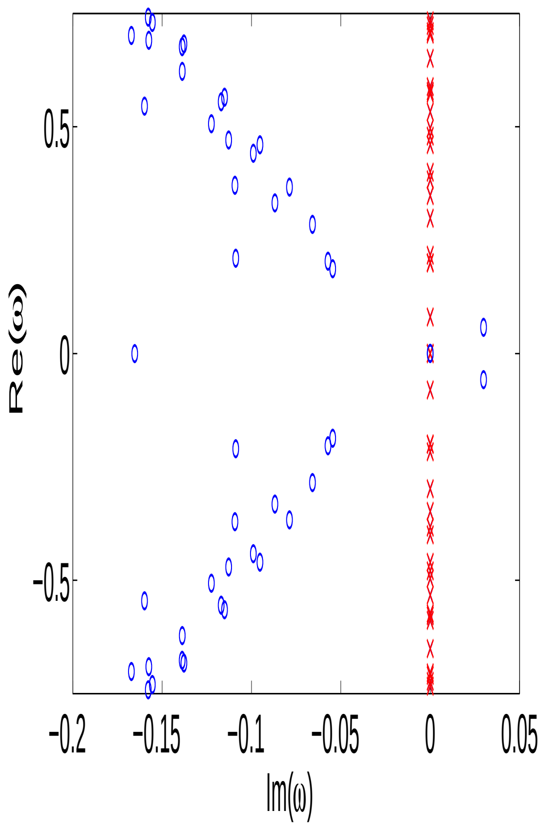
3d Version of Quasi-1d Results and Connection with P. Engels' Experiments

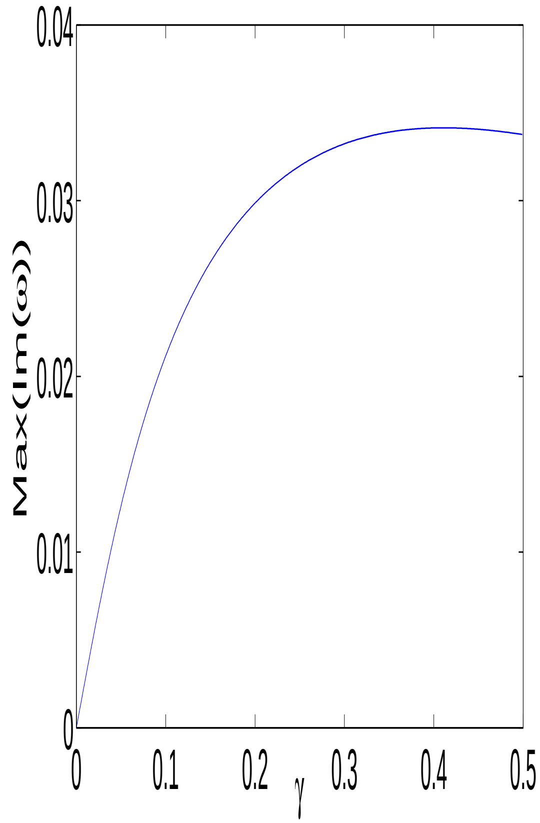


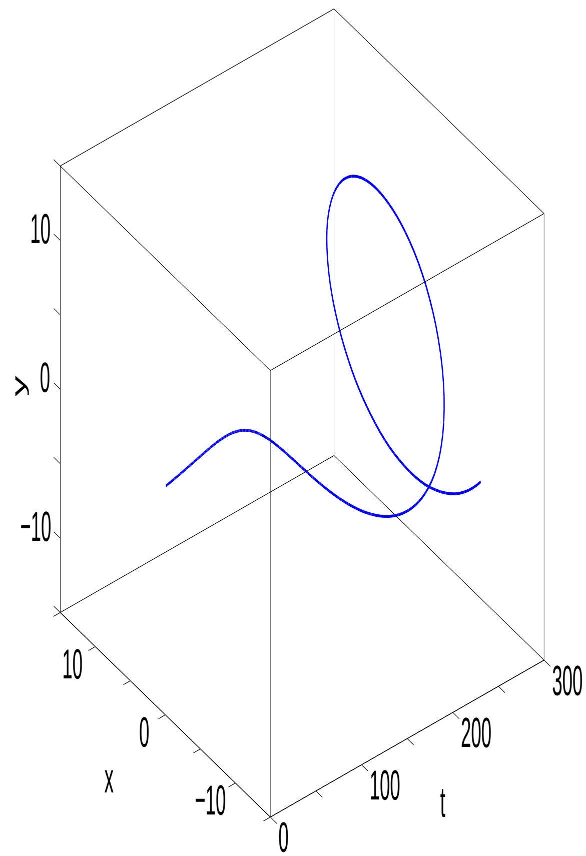
Connection with Peter Engels Experiments: Multi-DBs



Thermal Effects on Vortices







Connection of Dark Solitons and Vortices: The Bifurcation Picture

- In 2d, Stable Dark Solitons bifurcate from the Linear Mode $\Psi_{10}(x,y)=\psi_1(x)\psi_0(y)$.
- Subsequently, they become Unstable due to a Cascade of Symmetry-Breaking Bifurcations.
- The Symmetry-Breaking stems from the mixing of $\Psi_{0m}(x,y)=\psi_0(x)\psi_m(y)$ with $\Psi_{10}(x,y)=\psi_1(x)\psi_0(y)$.
- Use Galerkin Truncation to study the Symmetry Breaking as a Bifurcation Problem:

$$u(x,z)=c_0(z)\Psi_{10}+c_1(z)\Psi_{0m}, \tag{66}$$

$$i\dot{c}_0=(\mu+\omega_0)c_0-a_{00}|c_0|^2c_0-a_{01}\left(2|c_1|^2c_0+c_0^\star c_1^2\right), \tag{67}$$

$$i\dot{c}_1=(\mu+\omega_1)c_1-a_{11}|c_1|^2c_1-a_{01}\left(2|c_0|^2c_1+c_1^\star c_0^2\right), \tag{68}$$

$$\dot{\rho}_0=a_{01}\rho_1^2\rho_0\sin(2\Delta\phi), \tag{69}$$

$$\dot{\Delta\phi}=-\Delta\omega+a_{11}\rho_1^2-a_{00}\rho_0^2+a_{01}\left(2+\cos(2\Delta\phi)\right)\left(\rho_0^2-\rho_1^2\right), \tag{70}$$

A Side Note: Symmetry Breaking in Double Wells in Optics/BEC