Stability and Instability of Standing Waves for the Nonlinear Fractional Schrödinger Equation

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1. Equation

$$iu_t - (-\triangle + k^2)^s u + (\frac{1}{|x|^\gamma} * |u|^2)u = 0, \quad x \in \mathbb{R}^N, \qquad (1)$$

$u = u(t,x) : [0,T) \times \mathbb{R}^N \to \mathbb{C}$, $0 < T \leq +\infty$;

$(-\triangle + k^2)^s u = \mathcal{F}^{-1}[(|\xi|^2 + k^2)^s \mathcal{F}[u](\xi)]$, $s > 0$;

$(\frac{1}{|x|^\gamma} * |u|^2)(x) = \int \frac{|u(y)|^2}{|x-y|^\gamma} dy$; $0 < \gamma < 4s$.

♢ Boson star, fractional quantum mechanics etc.

In particular, when $N = 3$, $s = \frac{1}{2}$ and $\gamma = 1$, Eq.(1) is the Boson star equation.

- Fröhlich, Lenzmann, Lewin(2007-2009): well-posedness, orbital stability, singularity of blow-up solutions.

- Bao and Dong(2011): efficient and accurate numerical methods to compute the ground states and dynamics.

For general $0 < s < 1$, Laskin(2000,2002): expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths.

- Cho, Hwang, Hajaiej and Ozawa(2012): local well-posedness, existence of blow-up solutions.

- Guo, Huo, Huang(2011-2012): global well-posedness.

Known Studies

Existence and stability of standing waves for the classic nonlinear Schrödinger equation(i.e.$s = 1$)

- Strauss(1977): existence of standing waves.
- Cazenave and Lions(1982): orbital stability of standing waves.
- Weinstein(1983): strongly instability of standing waves.

Existence and stability of standing waves for the fourth-order nonlinear Schrödinger equation(i.e.$s = 2$)

- Zhu, Zhang and Yang(2010): existence of standing waves and ground state.
- Fibich, Baruch and Mandelbaum(2011): numerical studies for standing waves.
- Levandosky(1998): orbital stability of standing waves.

- Problems

1. Existence of standing waves for (1).

2. Stability of these standing waves.

- Arguments

  ◇ Weinstein(1983): blow-up argument

  ◇ Gérard(1998) Hmidi & Keraani(2005): profile decomposition

  ◇ Cazenave and Lions(1982), Zhang(2000): variational argument

Denote $H^s := \{v \in S^{'}(\mathbb{R}^N) \mid \int (1+|\xi|^2)^s |\widehat{v}(\xi)|^2 d\xi < +\infty\}$ and

$$E(v) := \frac{1}{2}\int \overline{v}(-\triangle)^s v dx - \frac{1}{4}\int \overline{v}(\frac{1}{|x|^\gamma} * |v|^2) v dx.$$

Impose

$$u(0,x) = u_0 \in H^s. \tag{2}$$

Proposition 1 (Cho, Hwang, Hajaiej and Ozawa, 2012)
Let $N \geq 2$, $0 < s < 1$ and $0 < \gamma \leq 2s$. If $u_0 \in H^s$, then $\exists!$ $u(t,x)$ of (1)-(2) on $I = [0,T)$ such that $u(t,x) \in C(I; H^s) \bigcap C^1(I; H^{-s})$. Moreover, for all $t \in I$, $u(t,x)$ satisfies the following conservation laws.

(i) Conservation of mass $\|u(t)\|_2 = \|u_0\|_2$.

(ii) Conservation of energy $E(u(t)) = E(u_0)$

Profile Decomposition

Proposition 2 Let $N \geq 2$ and $0 < s < 1$. If $\{v_n\}_{n=1}^{+\infty}$ is a bounded sequence in H^s, then there exists a subsequence of $\{v_n\}_{n=1}^{+\infty}$(still denoted $\{v_n\}_{n=1}^{+\infty}$), a family $\{x_n^j\}_{j=1}^{+\infty}$ of sequences in $\mathbb{R}^N$ and a family $\{V^j\}_{j=1}^{+\infty}$ of H^s functions satisfying the following.

(i) For every $k \neq j$, $|x_n^k - x_n^j| \to +\infty$ as $n \to +\infty$.

(ii) For every $l \geq 1$ and every $x \in \mathbb{R}^N$, $v_n(x)$ can be decomposed as

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l(x)$$

with $\lim\limits_{l\to+\infty} \limsup\limits_{n\to+\infty} \|v_n^l\|_p = 0$ for every $p \in (2, \frac{2N}{(N-2s)^+})$.

Moreover, we have, as $n \to +\infty$,

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V^j\|_2^2 + \|v_n^l\|_2^2 + o(1),$$

$$\|v_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{l} \|V^j\|_{\dot{H}^s}^2 + \|v_n^l\|_{\dot{H}^s}^2 + o(1).$$

Orbital Stability of Standing Waves

Let $k = 0$, $N \geq 2$ and $M > 0$. For any $0 < \gamma < 2s$, we define the following variational problem

$$d_M := \inf_{\{v \in H^s | \ \|v\|_2^2 = M\}} E(v) \tag{3}$$

where $E(v) := \frac{1}{2} \int |\xi|^{2s} |\widehat{v}|^2 d\xi - \frac{1}{4} \int \int \frac{|v(x)|^2 |v(y)|^2}{|x-y|^\gamma} dxdy$ is the energy functional. Define the set

$$S_M := \{v \in H^s | \ v \text{ is the minimizer of the variational problem (3)}\}. \tag{4}$$

From the Euler-Lagrange Theorem, we see that for any $v \in S_M$, there exists $\omega \in \mathbb{R}$ such that

$$(-\triangle)^s v + \omega v - (\frac{1}{|x|^\gamma} * |v|^2) v = 0, \quad v \in H^s.$$

Theorem 1 *Let $k = 0$, $N \geq 2$ and $M > 0$. Assume $0 < s < 1$ and $0 < \gamma < 2s$. Then for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that for any $u_0 \in H^s$, if the initial data u_0 satisfies*

$$\inf_{v \in S_M} \|u_0 - v\|_{H^s} < \delta,$$

then the corresponding solution $u(t,x)$ of the Cauchy problem (1)-(2) is such that

$$\inf_{v \in S_M} \|u(t,x) - v(x)\|_{H^s} < \varepsilon$$

for all $t > 0$, where S_M is defined in (4).

Key: The existence of minimizer of Variational Problem (3)

i.e. Let $N \geq 2$, $0 < \gamma < 2s$ and $M > 0$. Then,

$$d_M := \inf_{\{v \in H^s \mid \ \|v\|_2^2 = M\}} E(v) = \min_{\{v \in H^s \mid \ \|v\|_2^2 = M\}} E(v).$$

Firstly, the variational problem (3) is well defined. Using the Hölder inequality with $1 = \frac{\gamma}{2s} + \frac{2s-\gamma}{2s}$, we deduce that

$$\int \frac{|v(x)|^2}{|x-y|^\gamma} dx \leq C \||x|^{-s} v(x+y)\|_2^{\frac{\gamma}{s}} \ \|v\|_2^{\frac{2s-\gamma}{s}} \leq C\|v\|_{\dot{H}^s}^{\frac{\gamma}{s}} \ \|v\|_2^{\frac{2s-\gamma}{s}}.$$

Thus,

$$\int (\frac{1}{|x|^\gamma} * |v(x)|^2)|v(x)|^2 dx \leq \| \int \frac{|v(x)|^2}{|x-y|^\gamma} dx \|_\infty \|v(y)\|_2^2 \leq C\|v\|_{\dot{H}^s}^{\frac{\gamma}{s}} \ \|v\|_2^{\frac{4s-\gamma}{s}}.$$

From Young inequality, $\exists\ C(\varepsilon, \gamma, s, \sqrt{M}) > 0$ such that for all $0 < \varepsilon < \frac{1}{2}$

$$E(v) \geq \frac{1}{2}\|v\|_{\dot{H}^s}^2 - C\,\|v\|_2^{\frac{4s-\gamma}{s}} \ \|v\|_{\dot{H}^s}^{\frac{\gamma}{s}} \geq (\frac{1}{2} - \varepsilon)\|v\|_{\dot{H}^s}^2 - C(\varepsilon, \gamma, s, \|v\|_2). \tag{5}$$

Secondly, there exists $C_0 > 0$,

$$d_M \leq -C_0 < 0.$$

Indeed, take $v_n = \rho_n^{\frac{N}{2}} R(\rho_n x)$, where $\rho_n > 0$ and R is a function such that $\|v_n\|_2^2 = \|R\|_2^2 = M$. We deduce that

$$\begin{aligned} E(v_n) &= \frac{1}{2}\int |\xi|^{2s}|\widehat{v_n}|^2 d\xi - \frac{1}{4}\int\int \frac{|v_n(x)|^2|v_n(y)|^2}{|x-y|^\gamma} dxdy \\ &= \frac{\rho_n^{2s}}{2}\int |\xi|^{2s}|\widehat{R}|^2 d\xi - \frac{\rho_n^\gamma}{4}\int\int \frac{|R(x)|^2|R(y)|^2}{|x-y|^\gamma} dxdy. \end{aligned}$$

Since $0 < \gamma < 2s$, we can choose $\rho_n > 0$ sufficiently small such that there exists $C_0 > 0$ such that $E(v_n) \leq -C_0 < 0$.

Thirdly, we prove the existence of minimizers of (3). Take the minimizing sequence $\{v_n\}_{n=1}^{+\infty} \subset H^s$ of (3) satisfying that as $n \to +\infty$,

$$E(v_n) \to d_M \quad \text{and} \quad \|v_n\|_2^2 \to M. \tag{6}$$

Then, for n large enough, $E(v_n)$ satisfies $E(v_n) < d_M + 1$. From (5), for any $0 < \varepsilon < \frac{1}{2}$ and $n \geq 1$ large enough, we deduce that

$$(\frac{1}{2} - \varepsilon)\|v_n\|_{\dot{H}^s}^2 \leq d_M + 1 + C(\varepsilon, \gamma, s, M).$$

Hence, $\{v_n\}_{n=1}^{+\infty}$ is bounded in H^s. Apply Proposition 2, we see that

$$v_n(x) = \sum_{j=1}^{l} V^j(x - x_n^j) + v_n^l, \tag{7}$$

$$\|v_n\|_2^2 = \sum_{j=1}^{l} \|V_n^j\|_2^2 + \|v_n^l\|_2^2 + o(1), \quad \|v_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{l} \|V_n^j\|_{\dot{H}^s}^2 + \|v_n^l\|_{\dot{H}^s}^2 + o(1),$$

$$\int\int \frac{|v_n(x)|^2|v_n(y)|^2}{|x-y|^\gamma}dxdy = \sum_{j=1}^{l} \int\int \frac{|V_n^j(x)|^2|V_n^j(y)|^2}{|x-y|^\gamma}dxdy + \int\int \frac{|v_n^l(x)|^2}{|x -}$$

Thus, we have

$$E(v_n) = \sum_{j=1}^{l} E(V^j(x - x_n^j)) + E(v_n^l) + o(1). \tag{8}$$

Take the scaling transformation $V_{\rho_j}^j = \rho_j V^j(x - x_n^j)$ with $\rho_j = \frac{\sqrt{M}}{\|V^j(x-x_n^j)\|_2} \geq$ 1. We have $\|V_{\rho_j}^j\|_2^2 = M$. We deduce that as $n \to +\infty$ and $l \to +\infty$

$$\begin{aligned} d_M &\geq E(v_n) \\ &= \sum_{j=1}^{l} \left(\frac{E(V_{\rho_j}^j)}{\rho_j^2} + \frac{\rho_j^2 - 1}{4} \int\int \frac{|V^j(x-x_n^j)|^2 |V^j(y-x_n^j)|^2}{|x-y|^\gamma} dxdy \right) + E(v_n^l) + o(\\ &\geq \sum_{j=1}^{l} \frac{d_M}{\rho_j^2} + \inf_{j \geq 1} \frac{\rho_j^2 - 1}{4} \left(\sum_{j=1}^{l} \int\int \frac{|V^j(x-x_n^j)|^2 |V^j(y-x_n^j)|^2}{|x-y|^\gamma} dxdy \right) + \frac{\|v_n^l\|_2^2}{M} d \\ &\geq d_M + C_0 \left(\frac{M}{\|V^{j_0}\|_2^2} - 1 \right) + o(1), \end{aligned}$$

where $C_0 > 0$. Therefore, there exists only one term $V^{j_0} \neq 0$ in the decomposition (7) such that $\|V^{j_0}\|_2^2 = M$. V^{j_0} is the minimizer of (3).

Theorem 2 *Let $k = 0$ and $N \geq 2$. Assume $0 < \frac{\gamma}{2} = s < 1$. Then, the ground state solitary waves $e^{i\omega t}Q(x)$ of the fractional nonlinear Schrödinger equation (1) are unstable in the following sense: For arbitrary $\varepsilon > 0$, there exist the radial initial data $\{u_{0,n}\}_{n=1}^{+\infty} \subset H^{s_0}$ with $s_0 = \max\{2s, \frac{2s+1}{2}\}$ satisfying $|x|u_{0,n} \in L^2$, $x \cdot \nabla u_{0,n} \in L^2$,*

$$\|u_{0,n} - Q\|_{H^s} < \varepsilon \tag{9}$$

and the corresponding solution $\{u_n(t,x)\}_{n=1}^{+\infty}$ blows up in the finite time, where Q is the ground state solution of

$$(-\triangle)^s Q + Q - \left(\frac{1}{|x|^{2s}} * |Q|^2\right)Q = 0. \tag{10}$$

Proposition 3 *Let $N \geq 2$ and $0 < s < 1$. Then,*

$$\int |v|^2 dx \leq \frac{2}{N} \left(\int \overline{v} x (-\triangle)^{1-s} x v dx \right)^{\frac{1}{2}} \left(\int \overline{v} (-\triangle)^s v dx \right)^{\frac{1}{2}}, \quad \forall\, v \in H^s.$$

Proof.

For all $v \in H^s$, we have

$$\begin{aligned}
\int |v|^2 dx &= \tfrac{2}{N} \int \mathcal{F}[\nabla \cdot (xv)] \mathcal{F}^{-1}[\overline{v}] d\xi \\
&= \tfrac{2}{N} \int \overline{\mathcal{F}[v]}\ \xi \cdot \nabla_\xi \mathcal{F}[v] d\xi \\
&\leq \tfrac{2}{N} \int |\xi|^s |\overline{\mathcal{F}[v]}|\ |\xi|^{1-s} |\nabla_\xi \mathcal{F}[v]| d\xi \\
&\leq \tfrac{2}{N} \left(\int \overline{v} (-\triangle)^s v dx \right)^{\frac{1}{2}} \left(\int \overline{v} x (-\triangle)^{1-s} x v dx \right)^{\frac{1}{2}}.
\end{aligned}$$

□

Proposition 4 (see [Cho, Hwang, Kwon and Lee, 2012]) Let $k = 0$, $N \geq 2$, $0 < s < 1$ and $\gamma = 2s$. Assume that $u_0 \in H^{s_0}$ with $s_0 = \max\{2s, \frac{\gamma+1}{2}\}$ is radial symmetric, and $|x|u_0 \in L^2$ and $x \cdot \nabla u_0 \in L^2$. If $u(t,x)$ is the solution of the Cauchy problem (1)-(2), then for all $t \in I$ (the maximal time interval), $\int \overline{u}x(-\Delta)^{1-s}udx$ is nonnegative and

$$\int \overline{u}x(-\Delta)^{1-s}xudx \leq 2sE(u_0)t^2 + Ct + C.$$

Sketch of the Proof of Theorem 2

Let $\{c_n\}_{n=1}^{+\infty} \subset \mathbb{C}\backslash\{0\}$ be such that $|c_n| > 1$ and $\lim_{n\to+\infty} |c_n| = 1$, and $\{\rho_n\}_{n=1}^{+\infty} \subset \mathbb{R}^+$ be such that $\lim_{n\to+\infty} \rho_n = 1$. We take the initial data

$$u_{0,n} = c_n \rho_n^{\frac{N}{2}} Q(\rho_n x),$$

where Q is the ground state solution of (10). We see that for all $n \geq 1$, $\|u_{0,n}\|_2 > \|Q\|_2$ and

$$\lim_{n\to+\infty} \|u_{0,n}\|_2 = \lim_{n\to+\infty} |c_n| \; \|Q\|_2 = \|Q\|_2$$

and

$$\lim_{n\to+\infty} \|u_{0,n}\|_{\dot{H}^s} = \lim_{n\to+\infty} |c_n| \; \rho_n^s \; \|Q\|_{\dot{H}^s} = \|Q\|_{\dot{H}^s}.$$

Hence, from the Brézis-Lieb Lemma, $\forall\ \varepsilon > 0$, $\|u_{0,n} - Q\|_{H^s} < \varepsilon$ as n is sufficiently large.

On the other hand, we see that

$$E(u_{0,n}) = \frac{(|c_n|^2 - |c_n|^4)\rho_n^{2s}}{2} \|Q\|_{\dot{H}^s}^2 \leq -C_0 < 0.$$

Then,

$$\int \overline{u_n} x(-\Delta)^{1-s} x u_n \, dx \leq -2sC_0 t^2 + Ct + C,$$

which implies that there exists $0 < T < +\infty$ such that

$$\lim_{t \to T} \int \overline{u_n} x(-\Delta)^{1-s} x u_n \, dx = 0.$$

Finally, using the conservation of mass and applying the inequality in Proposition 3 to u_n, we see that for all time t

$$\begin{aligned} \|u_{0,n}\|_2^2 = \|u_n(t)\|_2^2 \quad &\leq \frac{2}{N} \left(\int \overline{u_n} x(-\triangle)^{1-s} x u_n \, dx \right)^{\frac{1}{2}} \left(\int \overline{u_n} (-\triangle)^s u_n \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{2}{N} \left(\int \overline{u_n} x(-\triangle)^{1-s} x u_n \, dx \right)^{\frac{1}{2}} \|u_n(t)\|_{\dot{H}^s}. \end{aligned}$$

Therefore, there exists $0 < T < +\infty$ such that

$$\lim_{t \to T} \|u_n(t)\|_{\dot{H}^s} = +\infty.$$

Thanks!