Simple Polytopes with Small Chromatic Number

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Abstract

In the talk we present recent results about quasitoric manifold and over simple polytopes having law chromatic number. We focus on their interesting geometrical and topological properties with emphasis on their applications in combinatorics and theory of fixed point. Our constructions are guided by combinatorial properties of the coloring. Using the methods of toric topology we calculate the characteristic classes and the cup lengths of the manifolds and obtain the theorems regarding immersions and embeddings and nice generalizations of the Knaster-Kuratowski-Mazurkiewicz theorem and the Lebesgue theorem from theory of set covering.

Summary

The facets of a n-dimensional simple polytope P^n are properly colored in k colors if no two facets having nonempty intersection are colored by the same color. Davis and Januszkiewicz observed that the polytopes with the chromatic number equal to n or $n+1$ are always the orbit space of a quasitoric manifold M^{2n} i. e. there exists the canonical characteristic map associated with the coloring. However, for n-colored simple polytopes there are infinitely many characteristic maps producing the quasitoric manifolds with fascinating topological and geometrical properties. Actually, they are wealthy source for examples of manifolds having classical properties of total parallelizability or having high cup length and rank of the characteristic classes. We present explicit constructions of such manifolds and obtain interesting new results about their immersions and embeddings into Euclidean spaces. Also, we get the similar results for corresponding small covers i. e. their counterparts in real case.

Even more striking result is that the polytopes with small chromatic number are natural objects for generalizing the classical theorems in set covering, the Knaster-Kuratowski-Mazurkiewicz theorem (on covers of simplex) and the Lebesgue theorem (on covers of cubes). The latter theorems are closely related to Sperner's lemma in combinatorics and Brower fixed point theorem in the fixed point theory. We also show that the n-dimensional Hex theorem admits a generalization where the n-dimensional cube is replaced by a n-colorable simple polytope. Toric topology approach offers new transparent generalization of the famous theorems and opens possibility for further research and implementation of these methods in geometric combinatorics.

Remarks

The proposed talk is based on two preprints:

1. Dj. Baralić and V. Grujić : *Immersions and Embeddings of Small Covers and Quasitoric Manifolds over n-Colored Simple Polytopes*, http://arxiv.org/abs/1406.4722 accepted for publication in Sbornik Mathematics

2. Dj. Baralić and R. Živaljević: *Colorful versions of the Lebesgue and KKM theorem*, http://arxiv.org/abs/1412.8621

The key ingredients in the proofs have pure combinatorial nature. The main focus of my research is the impact of combinatorics of polytopes and simplicial complexes on topology of naturally associated toric spaces such as quasitoric manifolds, the polyhedral product functors, small covers, etc. Research I wish to address is connected with the problems of combinatorial rigidity (and the rigidity problems in toric topology) and determination of the classes of simple polytopes having common topological properties (in the sense defined *RIGIDITY PROBLEMS IN TORIC TOPOLOGY, A SURVEY* S. Choi, M. Masuda, and D. Y. Suh) and the classical problems in toric topology such as the classification problem of quasitoric manifolds and the determination of generators in complex cobordism ring. Natalia Dobrinskaya, Jesper M. Møller and Dietrich Notbohm in the paper *Vertex colorings of simplicial complexes* (and in their another publications) obtained very interesting results combining methods of combinatoric and toric topology and explained the connection among topology of DJ spaces and moment-angle complexes and the chromatic polynomials of simplicial complexes.