### Toric topology and applications

# Victor M. Buchstaber

# Steklov Mathematical Institute

### Lomonosov Moscow State University

#### e-mail: buchstab@mi.ras.ru

This 5-lecture course is an introduction to toric topology and its applications.

In the focus of interest there will be connections between toric topology and combinatorics of polytopes. As an example of applications we will discuss mathematical results and unsolved problems about fullerenes. Fullerenes are carbon molecules discovered in 1985 by chemists-theorists Robert Curl, Harold Kroto and Richard Smalley (Nobel Prize 1996). Nowadays the problem of classification of fullerenes is well-known and is vital due to the applications in chemistry, physics, biology and nanotechnology.

Lectures.

1. Introduction. Toric topology and fullerenes.

- 2. Combinatorics of simple polytopes. Constructions of polytopes with given properties.
- 3. Moment-angle complexes and manifolds.
- 4. Quasitoric manifolds.

5. Cohomology of moment-angle manifolds and applications.

One of the main objects of the toric topology is the moment-angle functor  $K \to \mathcal{Z}_K$ .

It assigns to each simplicial complex K with m vertices a moment-angle complex  $\mathcal{Z}_K$  with an action of a compact torus  $T^m$ , whose orbit space  $\mathcal{Z}_K/T^m$  can be identified with the cone CK over K.

In the case when  $K = \partial P^*$ , where P is an n-dimensional convex simple polytope with m facets, the momentangle complex  $\mathcal{Z}_K$  has the structure of a smooth (m + n)-dimensional manifold  $\mathcal{Z}_P$  with a smooth action of  $T^m$ , and the orbit space  $\mathcal{Z}_P/T^m$  can be identified with P itself.

For certain conditions on the combinatorics of a simple polytope P there exists a free smooth action of the torus  $T^{m-n}$  on the manifold  $\mathcal{Z}_P$ . The manifold of orbits  $M^{2n} = \mathcal{Z}_P/T^{m-n}$  is called a quasitoric manifold. We discuss combinatorial operations on polytopes producing quasitoric manifolds with important geometric properties.

A mathematical fullerene is a three dimensional convex simple polytope with all 2-faces being pentagons and hexagons. In this case the number  $p_5$  of pentagons is 12. The number  $p_6$  of hexagons can be arbitrary except for 1. Nonequivalent fullerenes with the same  $p_6$  are called isomers. The number of isomers grows fast as a function of  $p_6$ .

Thanks to the toric topology, we can assign to each fullerene P its moment-angle manifold  $\mathcal{Z}_P$  and a family of quasitoric manifolds  $M^6 = M^6(P, \Lambda)$ , where  $\Lambda$  determines a free action of the torus  $T^{m-3}$  on  $\mathcal{Z}_P^{m+3}$ .

The cohomology ring  $H^*(\mathcal{Z}_P)$  is a combinatorial invariant of the fullerene P.

We shall focus upon results on the rings  $H^*(\mathcal{Z}_P)$  and give applications based on geometric interpretation of cohomology classes and their products. The multigrading in the ring  $H^*(\mathcal{Z}_P)$ , coming from the construction of  $\mathcal{Z}_P$ , and the multigraded Poincare duality plays an important role here.

The talks is based on joint works with Taras Panov and Nikolay Erokhovets.

#### References.

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