The Homotopy Type of Configuration Spaces

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but that would be a bit extreme.)

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(I will say more about Sullivan models soon.)

Answers to Q1 (The good news)

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There exist lense spaces $L(p,q), L(p,q')$ *such that*
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There exist lense spaces $L(p,q), L(p,q')$ such that $L(p,q), \simeq L(p,q')$ but $F(L(p,q),2) \not\simeq F(L(p,q'),2)$

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Fact: $H(A) \cong H^*(X)$ in a functorial way.

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Theorem (Lambrechts-S)

If B is a CDGA such that $H(B)$ satisfies Poincare duality then there is a quasi-isomorphic PDCDGA A.

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determines $M \times M \setminus \Delta = F(M,2)$.

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This implies (using general results of Lambrechts-S) that

$$A \otimes A \oplus_{\Delta} s^{n-1} A = A^2[g_{12}]/ \simeq$$

is a model for $F(M,2)$.

Ideas from the proofs 3-points I

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Lemma (*)

$$\begin{array}{ccc} M^3 \setminus \Delta_{12} \cup \Delta_{13} & \longrightarrow & M^3 \setminus \Delta_{12} \\ \downarrow & & \downarrow \\ M^3 \setminus \Delta_{13} & \longrightarrow & M^3 \end{array}$$

Is a pullback and a homotopy pullback.

Ideas from the proofs 3-points II

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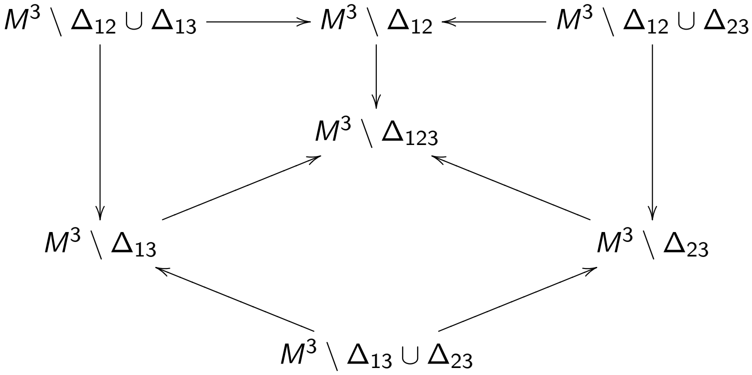
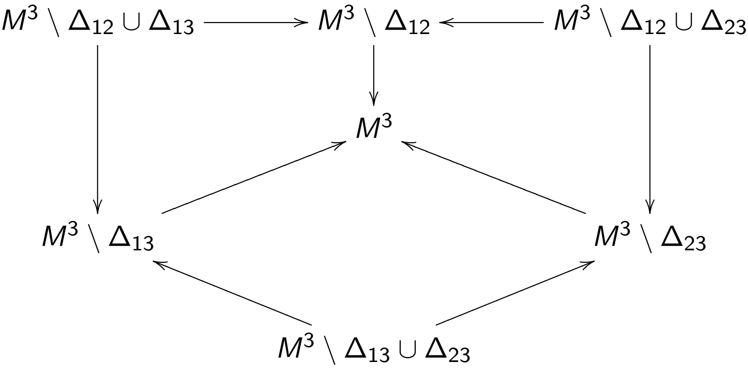
The natural map $F(M,3)$ into the *holim* of



Is $3n-4$ connected.

Ideas from the proofs 3-points III

Lemma (*) allows us to compare diagrams of the form



Ideas from the proofs 3-points III

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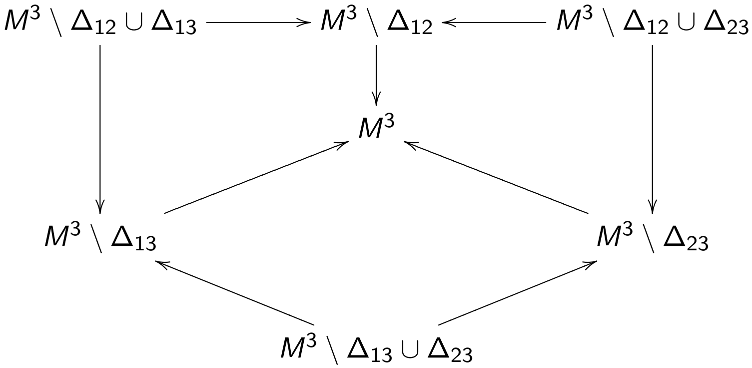

But still need to extend over whole diagrams.