Partial Euler characteristic, normal generations and the stable $D(2)$ problem

Ji Feng (Joint with Ye Shengkui)

August 10, 2015

The classical problems

- (Normal generation conjecture) Let G be any finitely generated perfect group, *i.e.* $G = [G, G]$, the commutator subgroup of G. Then G can be normally generated by a single element.
- (Swan's problem) For a finitely presented group G, define the deficiency $def(G)$ be the maximum of $d - k$ over all (finite) presentations $\langle g_1, \dots, g_d \mid r_1, \dots, r_k \rangle$ of G.
  Consider a resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}[G]$-modules:

$$F : \cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0.$$

  If F_i is free of rank f_i, define $\mu_n(F) = f_n - f_{n-1} + f_{n-2} - \ldots + (-1)^n f_0$.
  The *partial Euler characteristic* is $\mu_n(G) = \text{infimum}_F \mu_n(F)$.
  Swan: $def(G) \leq 1 - \mu_2(G)$.
  Question: is $def(G) = 1 - \mu_2(G)$?

The classical problems

- (Wall's $D(2)$ problem, Johnson's formulation) If X is a finite 3-dimensional CW complex of cohomological dimension at most 2 (any local coefficient system), then X is homotopy equivalent to a 2-dimensional CW complex.
- (The Whitehead asphericity conjecture) A CW complex X is called *aspherical* if the universal cover $\tilde{X}$ of X is contractible.
  Any subcomplex of an aspherical 2-dimensional CW complex is aspherical.

Quillen's plus construction

- X a CW complex, $G = \pi_1(X)$, P : a perfect normal subgroup of G.
- (Quillen) There is a CW complex X^+ (unique up to homotopy equivalence) such that $\pi_1(X^+) = G/P$ and$H_n(X; f_* M) \cong H_n(X^+; M)$ for any n and local coefficient system M.

Theorem

Let X be a finite 2-dimensional CW complex. P is the perfect normal subgroup of $\pi_1(X)$ normally generated by n elements. Then the plus construction $(X \vee (S^2)^n)^+$, taken w.r.t P, is homotopy equivalent to the 2-skeleton of X^+. In particular, $(X \vee (S^2)^n)^+$ is homotopy equivalent to a 2-dimensional CW complex.

The stable $D(2)$ problem

- (The $D(2, n)$ problem) Let $n \geq 0$ be an integer. If X is a finite 3-dimensional CW complex of cohomological dimension at most 2, then $X \vee (S^2)^n$ is homotopy equivalent to a 2-dimensional CW complex.

Theorem

Normal generation conjecture $\Rightarrow$ $D(2, 1)$ problem holds for X with $\pi_1(X)$ finite.

The stable $D(2)$ problem

(Following the idea of Johnson in treating the $D(2)$ problem) Say a finitely presented group G has the $D(2, n)$ property if the $D(2, n)$ problem holds true for any finite X with $\pi_1(X) = G$.

Theorem

- G satisfies $D(2, n)$ problem $\Rightarrow def(G) \geq (1 - n) - \mu_2(G)$.
- If G is a finite group, then G has $D(2, n)$ property for $n = 2 - def(G) - \mu_2(G)$.

$\mu_n^g(G)$

- For a finitely presented group G, define a (G, n)-complex as a finite n-dimensional CW complex X with fundamental group G and $\pi_i(X) = 0, i = 2, \ldots, n-1$. In particular, a $(G, 2)$-complex is a usual 2-dimensional CW complex with $\pi_1 = G$.
- Define $\mu_n^g(G) = \min\{(-1)^n \chi(X) \mid X \text{ is a } (G, n)\text{-complex}\}$. If there is no such X, define $\mu_n^g(G) = +\infty$.

Partial Euler characteristic, deficiency and the asphericity problem

- Let G be a group having a finite n-dimensional classifying space BG. Then $\mu_n(G) = \mu_n^g(G)$. It is realized by the classifying space. In particular, $\mu_2(G) = 1 - def(G)$ if G has a finite 2-dimensional BG.
- If X is a finite aspherical 2-complex and Y is a subcomplex, then Y realizes $\mu_2^g(\pi_1(Y))$.
- The complex Y (above) is aspherical if and only if the fundamental group $\pi_1(Y)$ has a finite classifying space of dimension at most 2.

The end

Thank you.