On the Lie Algebra of Braid Groups

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Brunnian Braid Groups

A braid $\beta \in B_n(M)$ is called **Brunnian** if (1) it is a pure braid and (2) it becomes trivial braid by removing any of its strands. Since the composition of any two Brunnian braids and the inverse of a Brunnian braid are still Brunnian, the set of Brunnian braids is a normal subgroup of the pure braid group which is denoted by $\mathrm{Brun}_n(M)$. For convenient, $\mathrm{Brun}_n(M)$ is denoted by Brun_n when M is the disc D^2.

Lie Algebra from Descending Central Series of Groups

For a group G, the descending central series

$$G = \Gamma_1(G) \geq \Gamma_2(G) \geq \cdots \geq \Gamma_i(G) \geq \Gamma_{i+1}(G) \geq \cdots$$

is defined by the formulas

$$\Gamma_1(G) = G, \Gamma_{i+1}(G) = [\Gamma_i(G), G] \quad (i \geq 1).$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $L(G)$:

$$L(G) = \bigoplus_{q=1}^{\infty} \Gamma_q(G)/\Gamma_{q+1}(G).$$

Tow Famous Results about the Lie Algebra of Pure Braid Groups

A presentation of the Lie algebra $L(P_n)$ for the pure braid group was done in the work of T.Kohno, and can be described as follows. It is the quotient of the free Lie algebra $L[A_{i,j} \mid 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations " or "horizontal 4T relation" given by the following three relations:

1. $[A_{i,j}, A_{s,t}] = 0 \text{ if } \{i,j\} \cap \{s,t\} = \emptyset$,
2. $[A_{i,j}, A_{i,k} + A_{j,k}] = 0 \text{ if } i < j < k$,
3. $[A_{i,k}, A_{i,j} + A_{j,k}] = 0 \text{ if } i < j < k$.

Where $A_{i,j}$ denote the projections of the $a_{i,j} \in P_n$ to $L(P_n)$.

Reference

T. Kohno, Série de Poincaré-Koszul associée aux groupes de tresses pure, Invent. Math., 82 (1985), 57-75.

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Y. Ihara gave a presentation of the Lie algebra $L(P_n(S^2))$ of the pure braid group of a sphere. It is the quotient of the free Lie algebra $L[B_{i,j} | 1 \leq i, j \leq n]$ generated by elements $B_{i,j}$ with $1 \leq i, j \leq n$ modulo the following relations:

$$\begin{cases} B_{i,j} = B_{j,i} \ \text{ for } 1 \leq i, j \leq n, \\ B_{i,i} = 0 \ \text{ for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \ \text{ if } \{i, j\} \cap \{s, t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \ \text{ for } 1 \leq i \leq n. \end{cases} \tag{1}$$

Where $B_{i,j}$ denote the projections of the $b_{i,j} \in P_n(S^2)$ to $L(P_n)(S^2)$.

Reference

- *Y. Ihara*, Galois group and some arithmetic functions. Proceedings of the International Congress of Mathematicians. Kyoto: Springer, (1991):99-C120.

Definition of the Relative Lie Algebra $L^P(\mathrm{Brun}_n(M))$

Since $\mathrm{Brun}_n(M)$ is the normal subgroup of the pure braid group $P_n(M)$, we have the following descending central series

$$\mathrm{Brun}_n(M) = \Gamma_1(P_n(M)) \cap \mathrm{Brun}_n(M) \geq \Gamma_2(P_n(M)) \cap \mathrm{Brun}_n(M) \geq \cdots$$

and **the relative Lie algebra**

$$L^P(\mathrm{Brun}_n(M)) = \bigoplus_{q=1}^{\infty} \Gamma_q(P_n(M)) \cap \mathrm{Brun}_n(M)/\Gamma_{q+1}(P_n(M)) \cap \mathrm{Brun}_n(M).$$

Property of $L^P(\mathrm{Brun}_n)$

Proposition

(Proposition 1)

$$L^P(\mathrm{Brun}_n) = \bigcap_{k=1}^{n} \ker(d_k : L(P_n) \to L(P_{n-1})).$$

Remarks.

Remark

The relative Lie algebra $\mathrm{L}^{\mathrm{P}}(\mathrm{Brun}_{\mathrm{n}})$ has better features: (1) it is freely generated; (2) it is of finite type; (3) it has connection to the theory of Vassiliev invariants.

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Definition of $\mathcal{K}(n)_k$

We recursively define the sets $\mathcal{K}(n)_k$, $1 \leq k \leq n$, in the reverse order as follows:

1) Let $\mathcal{K}(n)_n = \{A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}\}$.
2) Suppose that $\mathcal{K}(n)_{k+1}$ is defined as a subset of Lie monomials on the letters $A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}$ with $k < n$. Let

$$\mathcal{A}_k = \{W \in \mathcal{K}(n)_{k+1} \mid W \text{ does not contain } A_{k,n} \text{ in its entries}\}.$$

Define

$$\mathcal{K}(n)_k = \{W' \text{ and } [\cdots[[W', W_1], W_2], \ldots, W_t]\}$$

for $W' \in \mathcal{K}(n)_{k+1} \setminus \mathcal{A}_k$ and $W_1, W_2, \ldots, W_t \in \mathcal{A}_k$ with $t \geq 1$. Note that $\mathcal{K}(n)_k$ is again a subset of Lie monomials on letters $A_{1,n}, A_{2,n}, \cdots, A_{n-1,n}$.

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Free Generators of Lie Algebra $L^P(\mathrm{Brun}_n)$

Theorem

(Theorem 2) The relative Lie algebra $L^P(\mathrm{Brun}_n)$ is a free Lie algebra generated by $\mathcal{K}(n)_1$ as a set of free generators.

Example

Let $n = 4$. The set $\mathcal{K}(4)_1$ is constructed by the following steps:

1) $\mathcal{K}(4)_4 = \{A_{1,4}, A_{2,4}, A_{3,4}\}$.
2) $\mathcal{K}(4)_3 = \{[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \mid 1 \leq j_1, \cdots, j_t \leq 2,\ t \geq 0\}$, where, for $t = 0$, $[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] = A_{3,4}$.
3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \in \mathcal{K}(4)_3$. If W does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [[A_{3,4}, A_{1,4}], \cdots, A_{1,4}]$. Let

$$\mathrm{ad}^t(b)(a) = [[a, b], \cdots, b]$$

with t entries of b, where $\mathrm{ad}^0(b)(a) = a$. Then W does not contain $A_{2,4}$ if and only if

$$W = \mathrm{ad}^t(A_{1,4})(A_{3,4})$$

for $t \geq 0$.

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3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \in \mathcal{K}(4)_3$. If W does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [[A_{3,4}, A_{1,4}], \cdots, A_{1,4}]$. Let
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3) For constructing $\mathcal{K}(4)_2$, let $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \in \mathcal{K}(4)_3$. If W does not contain $A_{2,4}$, then $W = A_{3,4}$ or $W = [[A_{3,4}, A_{1,4}], \cdots, A_{1,4}]$. Let

$$\mathrm{ad}^t(b)(a) = [[a, b], \cdots, b]$$

with t entries of b, where $\mathrm{ad}^0(b)(a) = a$. Then W does not contain $A_{2,4}$ if and only if

$$W = \mathrm{ad}^t(A_{1,4})(A_{3,4})$$

for $t \geq 0$.

From the definition, $\mathcal{K}(4)_2$ is given by

$$[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}] \quad \text{and}$$

$$[[[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})],$$

where $1 \leq j_1, \cdots, j_t \leq 2$ with at least one $j_i = 2$, $s_1, \cdots, s_q \geq 0$ and $q \geq 1$.

4) For constructing $\mathcal{K}(4)_1$, let W denote

$$[[[[A_{3,4}, A_{j_1,4}], \ldots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})] \in \mathcal{K}(4)_2,$$

where, for $q = 0$, $W = [[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}]$. Then W does not contain $A_{1,4}$ if and only if $q = 0$ and $W = [[A_{3,4}, A_{2,4}], \cdots, A_{2,4}]$, namely

$$W = \mathrm{ad}^t(A_{2,4})(A_{3,4})$$

for $t \geq 1$.

Thus $\mathcal{K}(4)_1$, which is a set of free generators for $\mathrm{L}^P(\mathrm{Brun}_4)$, is given by

$$W \text{ and } [[W, \mathrm{ad}^{l_1}(A_{2,4})(A_{3,4})], \cdots, \mathrm{ad}^{l_p}(A_{2,4})(A_{3,4})],$$

where $l_i \geq 1$ for $1 \leq i \leq p$ with $p \geq 1$ and

$$W = [[[[A_{3,4}, A_{j_1,4}], \cdots, A_{j_t,4}], \mathrm{ad}^{s_1}(A_{1,4})(A_{3,4})], \cdots, \mathrm{ad}^{s_q}(A_{1,4})(A_{3,4})],$$

so that each of $A_{2,4}$ and $A_{1,4}$ appears in W at least once.

The Symmetric Bracket Sum of Ideals

Let L be a lie algebra and $I_1, \cdots, I_n$ are its ideals. The fat bracket sum $[[I_1, I_2, \cdots, I_n]]$ of these ideals is defined to be the sub Lie ideal of L generated by all of the commutators

$$\beta^t(a_{i_1}, \cdots, a_{i_t}),$$

where

1) $1 \leq i_s \leq n$;
2) $\{i_1, \cdots, i_t\} = \{1, \cdots, n\}$, that is each integer in $\{1, 2, \cdots, n\}$ appears as at least one of the integers i_s;
3) $a_j \in I_j$;
4) β^t runs over all of the bracket arrangements of weight t (with $t \geq n$).

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The symmetric bracket sum of these ideals is defined as

$$[I_1, \ldots, I_l]_S := \sum_{\sigma \in \Sigma_n} [[I_{\sigma(1)}, I_{\sigma(2)}], \cdots, I_{\sigma(n)}],$$

where Σ_n is the symmetric group of degree n.

Lemma

(Lemma 3) Let I_j be any Lie ideals of a Lie algebra L with $1 \leq j \leq n$. Then

$$[[I_1, I_2, \cdots, I_n]] = [[I_1, I_2], \cdots, I_n]_S.$$

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Let us denote the ideal

$$L[A_{k,n}, [\cdots[A_{k,n}, A_{j_1,n}], \cdots, A_{j_m,n}] \mid j_i \neq k, n; i \leq m; \ m \geq 1]$$

by I_k. Then we have the following theorem.

Theorem

(Theorem 4)

$$L^P(\mathrm{Brun}_n) = [[I_1, I_2], \cdots, I_{n-1}]_S.$$

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Proof

It is evident that the symmetric bracket sum $[[I_1, I_2], \cdots, I_{n-1}]_S$ *lies in the kernels of all* d_i*. On the other hand, from lemma 3 and theorem 2,* $L^P(\mathrm{Brun}_n)$ *is given as "fat bracket sum" of* $I_1, \cdots, I_{n-1}$ *because each element in* $\mathcal{K}(n)_1$ *is a Lie monomial containing each of* $A_{1,n}, \cdots, A_{n-1,n}$*. we know that*

$$\mathcal{K}(n)_1 \subseteq [[I_1, \cdots, I_{n-1}]] = [[I_1, I_2], \cdots, I_{n-1}]_S.$$

The Rank of $L_q^P(\mathrm{Brun}_n)$

Proposition

(Proposition 5) There is a decomposition

$$L_q(P_n) = \bigoplus_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 0 \le k \le n-1}} d^{i_k} d^{i_{k-1}} \cdots d^{i_1} (L_q^P(\mathrm{Brun}_{n-k}))$$

for each n and q. □

Proposition

(Proposition 6) There is a formula

$$\mathrm{rank}(L_q(P_n)) = \sum_{k=0}^{n-1} \binom{n}{k} \mathrm{rank}(L_q^P(\mathrm{Brun}_{n-k}))$$

for each n and q.

Let $b_q(P_n) = \mathrm{rank}(L_q(P_n))$ and $b_q^P(\mathrm{Brun}_n) = \mathrm{rank}(L_q^P(\mathrm{Brun}_n))$.
we have

$$\begin{pmatrix} b_q(P_n) \\ b_q(P_{n-1}) \\ b_q(P_{n-2}) \\ \vdots \\ b_q(P_1) \end{pmatrix} = \begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-1} \\ 0 & 1 & \binom{n-1}{1} & \cdots & \binom{n-1}{n-2} \\ 0 & 0 & 1 & \cdots & \binom{n-2}{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} b_q^P(\mathrm{Brun}_n) \\ b_q^P(\mathrm{Brun}_{n-1}) \\ b_q^P(\mathrm{Brun}_{n-2}) \\ \vdots \\ b_q^P(\mathrm{Brun}_1) \end{pmatrix}$$

Let A_n be the coefficient matrix of the above linear equations. Then

$$A_n^{-1} = \begin{pmatrix} 1 & -\binom{n}{1} & \binom{n}{2} & -\binom{n}{3} & \cdots & (-1)^{n-1}\binom{n}{n-1} \\ 0 & 1 & -\binom{n-1}{1} & \binom{n-1}{2} & \cdots & (-1)^{n-2}\binom{n-1}{n-2} \\ 0 & 0 & 1 & -\binom{n-2}{1} & \cdots & (-1)^{n-3}\binom{n-2}{n-3} \\ 0 & 0 & 0 & 1 & \cdots & (-1)^{n-4}\binom{n-3}{n-4} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Theorem

(Theorem 7)

$$\mathrm{rank}(L_q^P(\mathrm{Brun}_n)) = \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}\mathrm{rank}(L_q(P_{n-k}))$$

for each n and q, where $P_1 = 0$ and, for $m \geq 2$,

$$\mathrm{rank}(L_q(P_m)) = \frac{1}{q}\sum_{k=1}^{m-1}\sum_{d|q}\mu(d)k^{q/d}$$

with μ the Möbis function.

Property of $L^P(\mathrm{Brun}_n(S^2))$

The removing-strand operation on braids induces an operation

$$d_k\colon L(P_n(S^2)) \longrightarrow L(P_{n-1}(S^2)).$$

Proposition

(Proposition 8) There is an inclusion of Lie algebras

$$\mathrm{L^P(Brun_n(S^2))} \subset \bigcap_{i=1}^{n} \ker(d_i : L(P_n(S^2)) \to L(P_{n-1}(S^2))). \quad \square$$

The Homotopy Group of a Lie Algebra

Let $L = \{L_n\}_{n\geq 0}$ denote a simplicial Lie algebra with faces $d_i : L_n \longrightarrow L_{n-1}$ for $0 \leq i \leq n$. The Moore complex $N(L) = \{N_n(L)\}_{n\geq 0}$ of L is defined by

$$N_n(L) = \bigcap_{i=1}^{n} \ker(d_i : L_n \to L_{n-1}).$$

Then $N(L)$ with d_0 is a chain complex of Lie algebra. The Moore cycle and Moore boundary of L are defined by

$$Z_n(L) = \ker(d_0 : N_n(L) \longrightarrow N_{n-1}(L)) = \bigcap_{i=0}^{n} \ker(d_i : L_n \longrightarrow L_{n-1}),$$

and

$$B_n(L) = d_0(N_{n+1}(L))$$

respectively. The nth homotopy group is defined to be the quotient of

$$\pi_n(L) = Z_n(L)/B_n(L).$$

Lie Algebra $L(\widehat{F})$

Let $\widehat{F}_{n+1}$ be the quotient of the free group $F(x_0, x_1, \cdots, x_n)$ subject to the single relation $x_0 x_1 \cdots x_n = 1$. Let $\widehat{x}_j$ be the image of x_j in $\widehat{F}_{n+1}$. The group $\widehat{F}_{n+1}$ is written $\widehat{F}(\widehat{x}_0, \widehat{x}_1, \cdots, \widehat{x}_n)$ in case the generators $\widehat{x}_j$ are used. Clearly

$$\widehat{F}_n \cong F(\widehat{x}_0, \widehat{x}_1, \cdots, \widehat{x}_{n-1})$$

is a free group of rank n. Define the faces $\widehat{d}_i : \widehat{F}_{n+1} \longrightarrow \widehat{F}_n$ and degeneracies $\widehat{s}_i : \widehat{F}_n \longrightarrow \widehat{F}_{n+1}$ on $\widehat{F} = \{\widehat{F}_{n+1}\}_{n \geq 0}$ as follows:

$$\widehat{d}_i \widehat{x}_j = \begin{cases} \widehat{x}_j, & \text{if } j < i, \\ 1, & \text{if } j = i, \\ \widehat{x}_{j-1}, & \text{if } j > i. \end{cases} \qquad \widehat{s}_i \widehat{x}_j = \begin{cases} \widehat{x}_j, & \text{if } j < i, \\ \widehat{x}_j \widehat{x}_{j+1}, & \text{if } j = i, \\ \widehat{x}_{j+1}, & \text{if } j > i. \end{cases} \tag{2}$$

It is straightforward to check that the sequence of groups $\widehat{F} = \{\widehat{F}_{n+1}\}_{n\geq 0}$ is simplicial group under $\widehat{d}_i$ and $\widehat{s}_i$ defined as above.
Let $L(\widehat{F}) = \{L(\widehat{F}_{n+1})\}_{n\geq 0}$ denote the free simplicial Lie algebra generated by $\widehat{F}$.

The Intersection of the Kernel $d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2)))$

Proposition

(Proposition 9) The intersection of the kernel $d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2)))$ is the Moore cycle of $L(\widehat{F})$, i.e.

$$\bigcap_{i=1}^{n+1} \ker(d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2))) = \bigcap_{i=0}^{n-1} \ker(\widehat{d}_i : L(\widehat{F}_n) \to L(\widehat{F}_{n-1})).$$

Let us denote the ideal

$$L[B_{k,n+1},[\cdots[B_{k,n+1},B_{j_1,n+1}],\cdots,B_{j_m,n+1}]\mid j_i\neq k,n+1;\, i\leq m;\ m\geq 1]$$

by J_k. Then we have the following theorem.

Proposition

(Proposition 10) For $n \geq 4$, there is an isomorphism of groups:
$\bigcap_{i=1}^{n+1} \ker(d_i : L(P_{n+1}(S^2)) \to L(P_n(S^2)))/[[J_1, J_2], \cdots, J_{n-1}]_S$
$\cong \pi_{n-1}(L(\hat{F})) \cong \pi_{n-1}(L(F[S^1]) \cong \pi_{n-1}(L(G(S^2)).$

Remarks.

Remark

$\pi_*(L(G(S^2)))$ can be computed by using Λ- algebra.

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$$L[B_{k,n+1},[\cdots[B_{k,n+1},B_{j_1,n+1}],\cdots,B_{j_m,n+1}]\mid j_i\neq k,n+1;i\leq m;\ m\geq 1]$$

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Proposition

(Proposition 10) For $n\geq 4$, there is an isomorphism of groups:
$\bigcap_{i=1}^{n+1}\ker(d_i:L(P_{n+1}(S^2))\to L(P_n(S^2)))/[[J_1,J_2],\cdots,J_{n-1}]_S$
$\cong\pi_{n-1}(L(\widehat{F}))\cong\pi_{n-1}(L(F[S^1])\cong\pi_{n-1}(L(G(S^2)).$

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Thanks for your attention!