The fixed point data and equivariant Chern numbers

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- ▶ Preliminaries.
- ▶ Main Result.

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► Proof.

NOTE : All the mfds and maps are smooth.

• G-manifold M:

 $\theta:G\times M\to M$

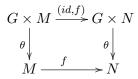
satisfying:

1. $e \in G$, $\theta(e, x) = x$, 2. $g_1, g_2 \in G$, $\theta(g_1g_2, x) = \theta(g_1, \theta(g_2, x))$.

• G-map f:

$$f:M\to N$$

satisfying the following square commute,



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• G-(complex) bundle Bundle $\xi : E \to M$ satisfied:

- 1. M is a G-manifold,
- 2. G acts linearly on the fibres (i.e. $g \in G, g : E_x \to E_{gx}$ is a G-(complex) liner map)

► G-unitary manifold M is a G-manifold and

$$\tau(M) \oplus \underline{\mathbb{R}}^l \to M$$

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is a G-complex bundle for some l. Where $\underline{\mathbb{R}}^{l}$ is trivial G-bundle with trivial G-action on the fibres \mathbb{R}^{l} .

fixed points: $M^G = \{m \in M | gm = m, \forall g \in G\}.$

Lemma

M is a G-manifold and $M^G = \coprod_F F$, where F be the connected component of the fixed point set:

- 1. F is a closed manifold with trivial G action,
- 2. $\nu_{F,M}$ is a G-bundle without trivial summand.

Lemma

M is a G-unitary manifold and $M^G = \coprod_F F$, where F be the connected component of the fixed point set:

1. F is an unitary manifold with trivial G action,

2. $\nu_{F,M}$ is a G-complex bundle without trivial summand.

We focus on the following two cases.

- $G = \mathbb{Z}_2^k$ -manifold M and the fixed point set $M^G = \coprod_F F$. $\{\nu_{F,M} \to F\}$ is called the Fixed Point Data of M.
- ► $G = T^k$ -unitary manifold M and the fixed point set $M^G = \coprod_F F. \{\nu_{F,M} \to F\}$ is called the Fixed Point Data of the unitary manifold M.

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Question

For given a family \mathbb{Z}_2^k -bundle (or T^k -complex bundle)

$$\{\nu_F \to F\},\$$

find necessary and sufficient conditions for the existence of a \mathbb{Z}_2^k -manifold (T^k -unitary manifold) with the given fixed point data.

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• Unoriented Bordism Ring: $\Omega^O_* = \sum \Omega^O_n$

$$\Omega_n^O = \{n \text{-dim closed mfds}\} / \sim .$$

• Unitary Bordism Ring: $\Omega^U_* = \sum \Omega^U_n$

$$\Omega_n^U = \{n \text{-dim closed unitary mfds}\} / \sim 1$$

For given G-mfd M_1 , M_2 , we can define $G \curvearrowright M_1 \times M_2$.

• geometric unoriented \mathbb{Z}_2^k -equivariant Bordism Ring:

$$\Omega^{O,\mathbb{Z}_2^k}_* = \sum \Omega^{O,\mathbb{Z}_2^k}_n$$

$$\Omega_n^{O,\mathbb{Z}_2^k} = \{n\text{-dim } \mathbb{Z}_2^k \text{ closed } \text{mfds}\} / \sim_{\mathbb{Z}_2^k} .$$

• geometric unitary T^k -equivariant Bordism Ring:

$$\Omega^{U,T^k}_* = \sum \Omega^{U,T^k}_n$$

 $\Omega_n^{U,T^k} = \{n\text{-dim } T^k \text{ closed unitary mfds}\} / \sim_{T^k}$.

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• $\Omega^O_* \longleftrightarrow$ Stiefel-Whitney numbers.

• $\Omega^U_* \longleftrightarrow$ Chern numbers.

- $\pi: EG \to BG$ is the universal principal G-bundles.
- The Borel construction gives us $EG \times_G \tau_M$ over $EG \times_G M$.
 - \blacktriangleright G equivariant Stiefel-Whitney class

$$w^G(M) := w(EG \times_G \tau_M).$$

► G equivariant Stiefel-Whitney number The constant map gives $p_!: H^*_G(M, \mathbb{Z}_2) \to H^*(BG, \mathbb{Z}_2).$ Then

$$w^G_{\omega}[M] := p_!(w^G_{\omega}(M))$$

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- $\pi : EG \to BG$ is the universal principal G-bundles. The Borel construction gives us $EG \times_G \tau_M$ over $EG \times_G M$.
 - \blacktriangleright G equivariant Chern class

$$c^G(M) := c(EG \times_G \tau_M).$$

► G equivariant Chern number The constant map gives $p_!: H^*_G(M) \to H^*(BG)$. Then

$$c^G_{\omega}[M] := p_!(c^G_{\omega}(M))$$

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- ▶ $\Omega^{O,\mathbb{Z}_2^k}_* \longleftrightarrow \mathbb{Z}_2^k$ -equivariant Stiefel-Whitney numbers. (tom Dieck in 1971 lnventiones math)
- ▶ $\Omega^{U,T^k}_* \longleftrightarrow T^k$ -equivariant Chern number. (Guillemin-Ginzburg-Karshon's conjecture, answered by Lü-Wang)

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Unoriented \mathbb{Z}_2^k -manifold with only isolated fixed-points:

Theorem (tom Dieck)

For given $G = \mathbb{Z}_2^k$ representation W^1, \ldots, W^s , they are the fixed point data of a closed G-manifold if and only if for any symmetric homogeneous polynomial $f(x_1, \ldots, x_n)$ over \mathbb{Z}_2 ,

$$\sum_{i=1}^{s} \frac{f(x_1^r, \dots, x_n^r)}{x_1^r \cdots x_n^r} \in H^*(BG, \mathbb{Z}_2).$$

where $W^r = \bigoplus_{i=1}^n W_i^r$ and $x_i^r = w_1^G(W_i^r)$.

Unoriented \mathbb{Z}_2 -case:

Theorem (Stong and Kosniowski)

For given \mathbb{Z}_2 -bundle $\{\nu_F^{n-r} \to F^r\}$, they are the fixed point data of a \mathbb{Z}_2 -manifold, if and only if for any symmetric polynomial $f(x_1, \ldots, x_n)$ over \mathbb{Z}_2 of degree at most n,

$$\sum_{r} \frac{f(1+y_1,\ldots,1+y_{n-r},z_1,\ldots,z_r)}{\prod(1+y_i)} \ [F] = 0,$$

where $w(F^r) = \prod (1 + z_i)$ and $w(\nu_F) = \prod (1 + y_i) \in H^*(F; \mathbb{Z}_2).$

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Remark

p is an isolated fixed point of the unitary T^k -manifold M. The normal bundle $\nu_{p,M}$ has two orientations:

- 1. induced by the orientation of M which comes from the unitary structure
- 2. induced by the orientation of $\nu_{p,M}$.

Then we can define the sign of the isolated fixed point p:

$$\zeta(p) := \begin{cases} +1, & \text{two orientations are same,} \\ -1, & \text{otherwise.} \end{cases}$$

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Question (B, P and R in [Toric genera])

For any set of signs $\zeta(x)$ and complex representation W_x , and necessary and sufficient conditions for the existence of a tangentially stably complex T^k manifold with the given fixed point data.

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Preliminaries.

► Main Result.

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► Proof.

n is even.

- 1. isolated fixed points x, n/2-dim complex representation W_x , and the sign $\zeta(x)$.
- 2. $G = T^k$ complex bundle $\nu_F \to F$ and $\dim F = r > 0$, ν_F is (n-r)/2-dim *G*-complex bundle over *F*. (*l* is large enough to stablize all the τ_F .)

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Theorem

They are the fixed point data of an unitary G-manifold if and only if for any symmetric homogeneous polynomial f(x) over \mathbb{Z} in (n+l)/2 variables,

$$\sum_{F} \frac{f(y,z)}{\prod y_i} \ [F] + \sum_{x} \zeta(x) \frac{f(u)}{\prod u_i} \in H^*(BG;\mathbb{Z}),$$

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where $c^G(F) = \prod (1 + z_i), c^G(\nu_F) = \prod (1 + y_j)$ and $c^G(W_x) = \prod (1 + u_i).$

n is even, W^1, \ldots, W^s are the n/2-dim complex representations and $\zeta(r)$ is the sign of W^r .

Remark

They are the fixed point data of an unitary G-manifold if and only if for any symmetric homogeneous polynomial f(x) over \mathbb{Z} in n/2 variables,

$$\sum_{i=1}^{s} \zeta(r) \frac{f(x_1^r, \dots, x_n^r)}{x_1^r \cdots x_n^r} \in H^*(BG),$$

where $W^r = \bigoplus_{i=1}^n W_i^r$, and $x_i^r = c_1^G(W_i^r)$.

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For given $\{\nu_F \to F\}$,

Theorem

They are the fixed point data of a G-manifold if and only if for any symmetric homogeneous polynomial f(x) over \mathbb{Z}_2

$$\sum_{F} \frac{f(y,z)}{\prod y_i} \ [F] \in H^*(BG;\mathbb{Z}_2),$$

where

$$w^{G}(F) = \prod (1+z_{i}) \in H^{*}_{G}(F; \mathbb{Z}_{2}),$$
$$w^{G}(\nu_{F}) = \prod (1+y_{i}) \in H^{*}_{G}(F; \mathbb{Z}_{2}).$$

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Preliminaries.

▶ Main Result.

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► Proof.

Theorem (tom Dieck)

$\Omega^{O,G}_* \xrightarrow{PT} MO^*_G \xrightarrow{\alpha} MO^*(BG) \xrightarrow{B} H^*_G(BG) \otimes \mathbb{Z}_2[[a]].$

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All the maps are injective.

 $\xi: E \to X$ is G-vector bundle. There is the G-equivariant classifying map:

$$\begin{array}{c} E \longrightarrow EO_G(n) \\ \downarrow \\ \downarrow \\ X \xrightarrow{f} BO_G(n) \end{array}$$

► Thom class:

$$t(\xi) := \lim \{Th(f) : Th(\xi) \to MO_G(n)\} \in MO_G^n(Th(\xi))$$
$$s : X \hookrightarrow Th(\xi)$$

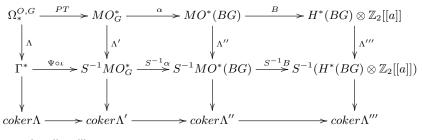
Euler class:

$$e(\xi) := s^*(t(\xi)) \in MU^n_G(X)$$

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Theorem (tom Dieck)

$G = \mathbb{Z}_2^k$ equivariant cobordism theory:



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 $\Lambda, \Lambda', \Lambda'', \Lambda'''$ and all the horizontal maps are injective.

Theorem (tom Dieck, Sinha, Hanke)

$$\begin{array}{ccc} \Omega^{U,G}_{*} & \xrightarrow{PT} & MU_{G}^{*} & \xrightarrow{\alpha} & MU^{*}(BG) \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & & \\ & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & &$$

The diagram is a pull back square. And

$$S^{-1}MU^G_* \cong MU_*(B) \otimes \mathbb{Z}[e_1(V)^{-1}, e_1(V)]$$
$$\Gamma = MU_*(B) \otimes \mathbb{Z}[e_1(V)^{-1}] \subset S^{-1}MU^G_*.$$

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• $\eta: E \to F$ is a G equivariant complex bundle where F is compact unitary manifold without boundary with trivial G action.

By using Segal's theorem:

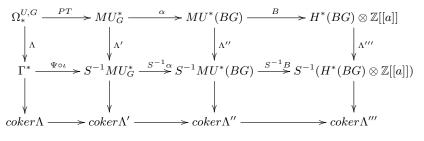
 $\Gamma(F,\eta) \in \Gamma_*.$

• For $[M]_G \in \Omega^{U,G}_*, \{\nu_{F,M} \to F\}$ is the fixed point data of M.

$$\Lambda([M]_G) := \sum_F \Gamma(F, \nu_{F,M}) \in \Gamma_*.$$

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Unitary $G = T^k$ -equivariant cobordism theory: Theorem



 $\Lambda, \Lambda', \Lambda'', \Lambda'''$ and all the horizontal maps are injective.

$$\Omega^{U,G}_* \xrightarrow{PT} MU^*_G \xrightarrow{\alpha} MU^*(BG) \xrightarrow{B} H^*(BG) \otimes \mathbb{Z}[[a]]$$

Proposition

Denote $c^G(M) = \prod (1+x_i)$ then

$$B \cdot \alpha \cdot PT([M]_G) = \sum_{\omega} S_{\omega}[M] b^{\omega},$$

where $(1 + b_1 t + b_2 t^2 + \cdots) \cdot (1 + a_1 t + a_2 t^2 + \cdots) = 1.$

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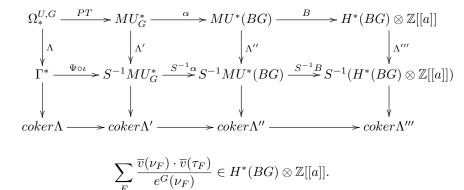
$\Gamma^* \xrightarrow{\Psi \circ \iota} S^{-1}MU^*_G \xrightarrow{S^{-1}\alpha} S^{-1}MU^*(BG) \xrightarrow{S^{-1}B} S^{-1}(H^*(BG) \otimes \mathbb{Z}[[a]])$

Theorem

 $\nu_F \to F$ as above,

$$S^{-1}(B \circ \alpha) \circ \Psi \circ \iota(\Gamma(F, \nu_F)) = \frac{\overline{v}(\nu_F) \cdot \overline{v}(\tau_F)}{e^G(\nu_F)}.$$

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Then there must be an unitary G-manifold M satisfying

$$\Lambda([M]_G) = \sum_F \Gamma(F, \nu_F).$$

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If $\nu_F \to F$ and $\nu_{F'} \to F'$ are r-dim complex G bundle,

$$\Gamma(F,\nu_F) = -\Gamma(F',\nu_{F'}) \in \Gamma_*$$

and $\dim F = \dim F' = s$.

Lemma

There must be a (s+2r)-dim unitary G-manifold M_F which $M_F \sim_G 0$, and $\{\nu_F \to F, \nu_{F'} \to F'\}$ are the fixed point data of M_F .

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Theorem

 $\{\nu_F \to F\}$ is the fixed point data of an unitary G-manifold if and only if

$$\sum_{F} \frac{\overline{v}(\nu_F) \cdot \overline{v}(\tau_F)}{e^G(\nu_F)} [F] \in H^*(BG) \otimes \mathbb{Z}[[a]].$$

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Thank You!

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