The fixed point data and equivariant Chern numbers

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- Preliminaries.
- Main Result.
- Proof.

G-manifold and G-map

NOTE : All the mfds and maps are smooth.

- G-manifold M:
$$\theta : G \times M \to M$$
satisfying:
  1. $e \in G$, $\theta(e, x) = x$,
  2. $g_1, g_2 \in G$, $\theta(g_1 g_2, x) = \theta(g_1, \theta(g_2, x))$.
- G-map f:
$$f : M \to N$$
satisfying the following square commute,

$$\begin{tikzcd} G \times M \arrow[r, "(id,f)"] \arrow[d, "\theta"'] & G \times N \arrow[d, "\theta"'] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

- G-(complex) bundle
  Bundle $\xi : E \to M$ satisfied:
  1. M is a G-manifold,
  2. G acts linearly on the fibres (i.e. $g \in G$, $g : E_x \to E_{gx}$ is a G-(complex) liner map)
- G-unitary manifold
  M is a G-manifold and

$$\tau(M) \oplus \underline{\mathbb{R}}^l \to M$$

is a G-complex bundle for some l. Where $\underline{\mathbb{R}}^l$ is trivial G-bundle with trivial G-action on the fibres $\mathbb{R}^l$.

Fixed Point Set

fixed points: $M^G = \{m \in M |\ gm = m,\ \forall g \in G\}$.

Lemma

M is a G-manifold and $M^G = \coprod_F F$, where F be the connected component of the fixed point set:

1. *F is a closed manifold with trivial G action,*
2. *$\nu_{F,M}$ is a G-bundle without trivial summand.*

Lemma

M is a G-unitary manifold and $M^G = \coprod_F F$, where F be the connected component of the fixed point set:

1. *F is an unitary manifold with trivial G action,*
2. *$\nu_{F,M}$ is a G-complex bundle without trivial summand.*

Fixed Point Data

We focus on the following two cases.

- $G = \mathbb{Z}_2^k$-manifold M and the fixed point set $M^G = \coprod_F F$. $\{\nu_{F,M} \to F\}$ is called the Fixed Point Data of M.
- $G = T^k$-unitary manifold M and the fixed point set $M^G = \coprod_F F$. $\{\nu_{F,M} \to F\}$ is called the Fixed Point Data of the unitary manifold M.

Realising fixed point data

Question

For given a family $\mathbb{Z}_2^k$-bundle (or T^k-complex bundle)

$$\{\nu_F \to F\},$$

find necessary and sufficient conditions for the existence of a $\mathbb{Z}_2^k$-manifold (T^k-unitary manifold) with the given fixed point data.

Bordism Ring

- Unoriented Bordism Ring: $\Omega_*^O = \sum \Omega_n^O$

$$\Omega_n^O = \{n\text{-dim closed mfds}\}/\sim .$$

- Unitary Bordism Ring: $\Omega_*^U = \sum \Omega_n^U$

$$\Omega_n^U = \{n\text{-dim closed unitary mfds}\}/\sim .$$

geometric G-equivariant Bordism Ring

For given G-mfd M_1, M_2, we can define $G \curvearrowright M_1 \times M_2$.

- geometric unoriented $\mathbb{Z}_2^k$-equivariant Bordism Ring:

$$\Omega_*^{O,\mathbb{Z}_2^k} = \sum \Omega_n^{O,\mathbb{Z}_2^k}$$

$$\Omega_n^{O,\mathbb{Z}_2^k} = \{n\text{-dim } \mathbb{Z}_2^k \text{ closed } \ \text{mfds}\}/\sim_{\mathbb{Z}_2^k} .$$

- geometric unitary T^k-equivariant Bordism Ring:

$$\Omega_*^{U,T^k} = \sum \Omega_n^{U,T^k}$$

$$\Omega_n^{U,T^k} = \{n\text{-dim } T^k \text{ closed unitary mfds}\}/\sim_{T^k} .$$

Cobordism theory and Characteristic numbers

- $\Omega_*^O \longleftrightarrow$ Stiefel-Whitney numbers.
- $\Omega_*^U \longleftrightarrow$ Chern numbers.

$\pi : EG \to BG$ is the universal principal G-bundles.

The Borel construction gives us $EG \times_G \tau_M$ over $EG \times_G M$.

- G equivariant Stiefel-Whitney class

$$w^G(M) := w(EG \times_G \tau_M).$$

- G equivariant Stiefel-Whitney number
  The constant map gives $p_! : H^*_G(M, \mathbb{Z}_2) \to H^*(BG, \mathbb{Z}_2)$.
  Then
$$w^G_\omega[M] := p_!(w^G_\omega(M))$$

$\pi : EG \to BG$ is the universal principal G-bundles.

The Borel construction gives us $EG \times_G \tau_M$ over $EG \times_G M$.

- G equivariant Chern class

$$c^G(M) := c(EG \times_G \tau_M).$$

- G equivariant Chern number

  The constant map gives $p_! : H^*_G(M) \to H^*(BG)$. Then

$$c^G_\omega[M] := p_!(c^G_\omega(M))$$

Equivariant case

- $\Omega_*^{O,\mathbb{Z}_2^k} \longleftrightarrow \mathbb{Z}_2^k$-equivariant Stiefel-Whitney numbers. (tom Dieck in 1971 lnventiones math)
- $\Omega_*^{U,T^k} \longleftrightarrow T^k$-equivariant Chern number. (Guillemin-Ginzburg-Karshon's conjecture, answered by Lü-Wang)

Unoriented $\mathbb{Z}_2^k$-manifold with only isolated fixed-points:

Theorem (tom Dieck)

For given $G = \mathbb{Z}_2^k$ representation $W^1, \ldots, W^s$, they are the fixed point data of a closed G-manifold if and only if for any symmetric homogeneous polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_2$,

$$\sum_{i=1}^{s} \frac{f(x_1^r, \ldots, x_n^r)}{x_1^r \cdots x_n^r} \in H^*(BG, \mathbb{Z}_2).$$

where $W^r = \oplus_{i=1}^{n} W_i^r$ and $x_i^r = w_1^G(W_i^r)$.

Unoriented $\mathbb{Z}_2$-case:

Theorem (Stong and Kosniowski)

For given $\mathbb{Z}_2$-bundle $\{\nu_F^{n-r} \to F^r\}$, they are the fixed point data of a $\mathbb{Z}_2$-manifold, if and only if for any symmetric polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{Z}_2$ of degree at most n,

$$\sum_r \frac{f(1+y_1, \ldots, 1+y_{n-r}, z_1, \ldots, z_r)}{\prod(1+y_i)} [F] = 0,$$

where $w(F^r) = \prod(1+z_i)$ and $w(\nu_F) = \prod(1+y_i) \in H^*(F; \mathbb{Z}_2)$.

Remark

p is an isolated fixed point of the unitary T^k-manifold M. The normal bundle $\nu_{p,M}$ has two orientations:

1. induced by the orientation of M which comes from the unitary structure
2. induced by the orientation of $\nu_{p,M}$.

Then we can define the sign of the isolated fixed point p:

$$\zeta(p) := \begin{cases} +1, & \textit{two orientations are same,} \\ -1, & \textit{otherwise.} \end{cases}$$

Question (B, P and R in [Toric genera])

For any set of signs $\zeta(x)$ and complex representation W_x, and necessary and sufficient conditions for the existence of a tangentially stably complex T^k manifold with the given fixed point data.

- Preliminaries.
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Unitary $G = T^k$-manifold

n is even.

1. isolated fixed points x, $n/2$-dim complex representation W_x, and the sign $\zeta(x)$.
2. $G = T^k$ complex bundle $\nu_F \to F$ and $dimF = r > 0$, ν_F is $(n-r)/2$-dim G-complex bundle over F.
   (l is large enough to stablize all the τ_F.)

Theorem

They are the fixed point data of an unitary G-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}$ in $(n+l)/2$ variables,

$$\sum_F \frac{f(y,z)}{\prod y_i}\ [F] + \sum_x \zeta(x)\frac{f(u)}{\prod u_i} \in H^*(BG;\mathbb{Z}),$$

where $c^G(F) = \prod(1+z_i)$, $c^G(\nu_F) = \prod(1+y_j)$ and $c^G(W_x) = \prod(1+u_i)$.

n is even, $W^1, \ldots, W^s$ are the $n/2$-dim complex representations and $\zeta(r)$ is the sign of W^r.

Remark

They are the fixed point data of an unitary G-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}$ in $n/2$ variables,

$$\sum_{i=1}^{s} \zeta(r) \frac{f(x_1^r, \ldots, x_n^r)}{x_1^r \cdots x_n^r} \in H^*(BG),$$

where $W^r = \oplus_{i=1}^{n} W_i^r$, and $x_i^r = c_1^G(W_i^r)$.

For given $\{\nu_F \to F\}$,

Theorem

They are the fixed point data of a G-manifold if and only if for any symmetric homogeneous polynomial $f(x)$ over $\mathbb{Z}_2$

$$\sum_F \frac{f(y,z)}{\prod y_i} [F] \in H^*(BG;\mathbb{Z}_2),$$

where

$$w^G(F) = \prod(1+z_i) \in H^*_G(F;\mathbb{Z}_2),$$

$$w^G(\nu_F) = \prod(1+y_i) \in H^*_G(F;\mathbb{Z}_2).$$

- Preliminaries.
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Recall $G = \mathbb{Z}_2^k$ case

Theorem (tom Dieck)

$$\Omega_*^{O,G} \xrightarrow{PT} MO_G^* \xrightarrow{\alpha} MO^*(BG) \xrightarrow{B} H_G^*(BG) \otimes \mathbb{Z}_2[[a]].$$

All the maps are injective.

Thom class and Euler class

$\xi : E \to X$ is G-vector bundle. There is the G-equivariant classifying map:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\xi"'] & EO_G(n) \arrow[d] \\ X \arrow[r, "f"] & BO_G(n) \end{tikzcd}$$

- Thom class:

$$t(\xi) := \lim\{Th(f) : Th(\xi) \to MO_G(n)\} \in MO_G^n(Th(\xi))$$

-
$$s : X \hookrightarrow Th(\xi)$$

  Euler class:

$$e(\xi) := s^*(t(\xi)) \in MU_G^n(X)$$

Theorem (tom Dieck)

$G = \mathbb{Z}_2^k$ *equivariant cobordism theory:*

$$\begin{tikzcd}
\Omega_*^{O,G} \arrow[r, "PT"] \arrow[d, "\Lambda"] & MO_G^* \arrow[r, "\alpha"] \arrow[d, "\Lambda'"] & MO^*(BG) \arrow[r, "B"] \arrow[d, "\Lambda''"] & H^*(BG) \otimes \mathbb{Z}_2[[a]] \arrow[d, "\Lambda'''"] \\
\Gamma^* \arrow[r, "\Psi \circ \iota"] \arrow[d] & S^{-1}MO_G^* \arrow[r, "S^{-1}\alpha"] \arrow[d] & S^{-1}MO^*(BG) \arrow[r, "S^{-1}B"] \arrow[d] & S^{-1}(H^*(BG) \otimes \mathbb{Z}_2[[a]]) \arrow[d] \\
coker\Lambda \arrow[r] & coker\Lambda' \arrow[r] & coker\Lambda'' \arrow[r] & coker\Lambda'''
\end{tikzcd}$$

$\Lambda,\ \Lambda',\ \Lambda'',\ \Lambda'''$ *and all the horizontal maps are injective.*

Theorem (tom Dieck, Sinha, Hanke)

$$\begin{tikzcd}
\Omega_*^{U,G} \arrow[r, "PT"] \arrow[d, "\Lambda"] & MU_G^* \arrow[r, "\alpha"] \arrow[d, "\Lambda'"] & MU^*(BG) \arrow[d, "\Lambda''"] \\
\Gamma^* \arrow[r, "\Psi\circ\iota"] & S^{-1}MU_G^* \arrow[r, "S^{-1}\alpha"] & S^{-1}MU^*(BG)
\end{tikzcd}$$

The diagram is a pull back square. And

$$S^{-1}MU_*^G \cong MU_*(B) \otimes \mathbb{Z}[e_1(V)^{-1}, e_1(V)]$$

$$\Gamma = MU_*(B) \otimes \mathbb{Z}[e_1(V)^{-1}] \subset S^{-1}MU_*^G.$$

Λ

- $\eta : E \to F$ is a G equivariant complex bundle where F is compact unitary manifold without boundary with trivial G action.

By using Segal's theorem:

$$\Gamma(F, \eta) \in \Gamma_*.$$

- For $[M]_G \in \Omega_*^{U,G}$, $\{\nu_{F,M} \to F\}$ is the fixed point data of M.

$$\Lambda([M]_G) := \sum_F \Gamma(F, \nu_{F,M}) \in \Gamma_*.$$

Unitary $G = T^k$-equivariant cobordism theory:

Theorem

$$\begin{tikzcd}
\Omega_*^{U,G} \arrow[r, "PT"] \arrow[d, "\Lambda"] & MU_G^* \arrow[r, "\alpha"] \arrow[d, "\Lambda'"] & MU^*(BG) \arrow[r, "B"] \arrow[d, "\Lambda''"] & H^*(BG) \otimes \mathbb{Z}[[a]] \arrow[d, "\Lambda'''"] \\
\Gamma^* \arrow[r, "\Psi \circ \iota"] \arrow[d] & S^{-1}MU_G^* \arrow[r, "S^{-1}\alpha"] \arrow[d] & S^{-1}MU^*(BG) \arrow[r, "S^{-1}B"] \arrow[d] & S^{-1}(H^*(BG) \otimes \mathbb{Z}[[a]]) \arrow[d] \\
coker \Lambda \arrow[r] & coker \Lambda' \arrow[r] & coker \Lambda'' \arrow[r] & coker \Lambda'''
\end{tikzcd}$$

$\Lambda,\ \Lambda',\ \Lambda'',\ \Lambda'''$ *and all the horizontal maps are injective.*

T^k equivariant Chern numbers

$$\Omega_*^{U,G} \xrightarrow{PT} MU_G^* \xrightarrow{\alpha} MU^*(BG) \xrightarrow{B} H^*(BG) \otimes \mathbb{Z}[[a]]$$

Proposition

Denote $c^G(M) = \prod(1 + x_i)$ *then*

$$B \cdot \alpha \cdot PT([M]_G) = \sum_{\omega} S_\omega[M] b^\omega,$$

where $(1 + b_1 t + b_2 t^2 + \cdots) \cdot (1 + a_1 t + a_2 t^2 + \cdots) = 1.$

$$\Gamma^* \xrightarrow{\Psi\circ\iota} S^{-1}MU_G^* \xrightarrow{S^{-1}\alpha} S^{-1}MU^*(BG) \xrightarrow{S^{-1}B} S^{-1}(H^*(BG)\otimes\mathbb{Z}[[a]])$$

Theorem

$\nu_F \to F$ as above,

$$S^{-1}(B\circ\alpha)\circ\Psi\circ\iota(\Gamma(F,\nu_F)) = \frac{\overline{v}(\nu_F)\cdot\overline{v}(\tau_F)}{e^G(\nu_F)}.$$

$$
\begin{tikzcd}
\Omega_*^{U,G} \arrow[r, "PT"] \arrow[d, "\Lambda"] & MU_G^* \arrow[r, "\alpha"] \arrow[d, "\Lambda'"] & MU^*(BG) \arrow[r, "B"] \arrow[d, "\Lambda''"] & H^*(BG) \otimes \mathbb{Z}[[a]] \arrow[d, "\Lambda'''"] \\
\Gamma^* \arrow[r, "\Psi\circ\iota"] \arrow[d] & S^{-1}MU_G^* \arrow[r, "S^{-1}\alpha"] \arrow[d] & S^{-1}MU^*(BG) \arrow[r, "S^{-1}B"] \arrow[d] & S^{-1}(H^*(BG) \otimes \mathbb{Z}[[a]]) \arrow[d] \\
coker\Lambda \arrow[r] & coker\Lambda' \arrow[r] & coker\Lambda'' \arrow[r] & coker\Lambda'''
\end{tikzcd}
$$

$$\sum_F \frac{\overline{v}(\nu_F) \cdot \overline{v}(\tau_F)}{e^G(\nu_F)} \in H^*(BG) \otimes \mathbb{Z}[[a]].$$

Then there must be an unitary G-manifold M satisfying

$$\Lambda([M]_G) = \sum_F \Gamma(F, \nu_F).$$

If $\nu_F \to F$ and $\nu_{F'} \to F'$ are r-dim complex G bundle,

$$\Gamma(F, \nu_F) = -\Gamma(F', \nu_{F'}) \in \Gamma_*$$

and $\dim F = \dim F' = s$.

Lemma

There must be a $(s+2r)$-dim unitary G-manifold M_F which $M_F \sim_G 0$, and $\{\nu_F \to F,\ \nu_{F'} \to F'\}$ are the fixed point data of M_F.

Theorem

$\{\nu_F \to F\}$ is the fixed point data of an unitary G-manifold if and only if

$$\sum_F \frac{\overline{v}(\nu_F)\cdot\overline{v}(\tau_F)}{e^G(\nu_F)}[F] \in H^*(BG)\otimes\mathbb{Z}[[a]].$$

Thank You!