Self-Dual Binary Codes from Small Covers and Simple Polytopes

— A joint work with Bo Chen and Zhi Lü

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Binary Linear Codes

A binary linear code C of length l — a linear subspace of the l-dimensional linear space $\mathbb{F}_2^l$ over $\mathbb{F}_2$.

The Hamming weight of an element $u = (u_1, ..., u_l) \in \mathbb{F}_2^l$, denoted by $wt(u)$, is the number of nonzero components u_i in u. The Hamming distance $d(u, v)$ of any elements $u, v \in C$ is defined by

$$d(u, v) = wt(u - v).$$

The minimum of the distances $d(u, v)$ for all $u, v \in C$, $u \neq v$, is called the minimum distance of C. It is also equal to the minimal Hamming weight of all the nonzero elements in C.

A binary code $C \subset \mathbb{F}_2^l$ is called type $[l, k, d]$ if $\dim_{\mathbb{F}_2} C = k$ and the minimum distance of C is d.

The inner product $\langle\ ,\ \rangle$ on $\mathbb{F}_2^l$ is defined by:

$$\langle u, v\rangle := \sum_{i=1}^{l} u_i v_i,\ \ u = (u_1, ..., u_l), v = (v_1, ..., v_l) \in \mathbb{F}_2^l.$$

Note that

$$\langle u, u\rangle = \sum_{i=1}^{l} u_i,\ \ u = (u_1, ..., u_l) \in \mathbb{F}_2^l.$$

Self-dual Binary Code

Any binary linear code C in $\mathbb{F}_2^l$ has a dual code $C^\perp$ defined by

$$C^\perp := \{u \in \mathbb{F}_2^l \mid \langle u, c\rangle = 0 \text{ for all } c \in C\}$$

It is clear that $\dim_{\mathbb{F}_2} C + \dim_{\mathbb{F}_2} C^\perp = n$. We call C self-dual if

$$C = C^\perp.$$

If C is self-dual, we have:

- The code length $l = 2\dim_{\mathbb{F}_2} C$ must be even;
- For any $u \in C$, the Hamming weight $wt(u)$ is an even integer;
- The minimum distance of C is an even integer.

m-involutions on manifolds

An involution τ on a manifold M is called an m-involution if

- τ only has isolated fixed points, and
- the number of fixed points of τ is equal to $\sum_i b_i(M;\mathbb{F}_2)$.

Let $G_\tau = \langle \tau \rangle \cong \mathbb{Z}_2$. Then we can show that

(a) The number of fixed points $|M^{G_\tau}| = 2r$, $r \geq 1$.

(b) $H^*_{G_\tau}(M;\mathbb{F}_2)$ is a free $H^*(BG_\tau;\mathbb{F}_2)$-module, so

$$H^*_{G_\tau}(M;\mathbb{F}_2) = H^*(M;\mathbb{F}_2) \otimes H^*(BG_\tau;\mathbb{F}_2).$$

Localization of Equivariant Cohomology

(c) The inclusion of the fixed point set, $\iota : M^{G_\tau} \hookrightarrow M$, induces a monomorphism

$$\iota^* : H^*_{G_\tau}(M;\mathbb{F}_2) \to H^*_{G_\tau}(M^{G_\tau};\mathbb{F}_2) \cong \mathbb{F}_2^{2r} \otimes \mathbb{F}_2[t].$$

So the image of $H^*_{G_\tau}(M;\mathbb{F}_2)$ in $\mathbb{F}_2^{2r} \otimes \mathbb{F}_2[t]$ under the map ι^* is isomorphic to $H^*_{G_\tau}(M;\mathbb{F}_2)$ as graded algebras. Define

$$V_k^M = \{y \in \mathbb{F}_2^{2r} \mid y \otimes t^k \in \mathrm{Im}(\iota^*)\} \subset \mathbb{F}_2^{2r},\ \ k = 0, \cdots, n.$$

We have a filtration:

$$\mathbb{F}_2 \cong V_0^M \subset V_1^M \subset \cdots \subset V_{n-2}^M \subset V_{n-1}^M = \mathcal{V}_{2r} \subset V_n^M = \mathbb{F}_2^{2r}$$

where $\mathcal{V}_{2r} = \left\{x = (x_1, ..., x_{2r}) \in \mathbb{F}_2^{2r} \mid \langle x, x\rangle = 0\right\}$.

Binary Codes Constructed from m-involutions

By the localization theorem for equivariant cohomology,

$$H^k(M^n;\mathbb{F}_2) \cong V_k^M/V_{k-1}^M,\ 0 \le k \le n. \tag{1.1}$$

So we have: $\dim_{\mathbb{F}_2} V_k^M = \sum_{j=0}^{k} b_j(M;\mathbb{F}_2)$.

Moreover, we have

$$(V_k^M)^{\perp} = V_{n-1-k}^M. \tag{1.2}$$

This is because V_{n-1-k}^M is perpendicular to V_k^M with respect to $\langle\ ,\ \rangle$ and by the Poincaré duality of M, we have

$$\dim_{\mathbb{F}_2} V_k^M + \dim_{\mathbb{F}_2} V_{n-1-k}^M = \sum_{j=0}^{n} b_j(M;\mathbb{F}_2) = 2r.$$

Each V_k^M above can be thought of as a binary code in $\mathbb{F}_2^{2r}$. So

when n is odd, $V_{\frac{n-1}{2}}^M$ is a self-dual binary code in $\mathbb{F}_2^{2r}$.

Theorem [Puppe 2001]

For any m-involution τ on a closed manifold M^n where n is odd, we obtain a self-dual binary code $V_{\frac{n-1}{2}}^M$ from the localization of $H_{G_\tau}^*(M^n;\mathbb{F}_2)$ to the fixed point sets.

Theorem [Puppe-Kreck 2012]

Any self-dual binary code can be obtained from an m-involution on some closed 3-manifold in the above way.

Self-dual binary codes $\longleftrightarrow$ m-involutions on manifolds

Problem: Construct m-involutions on manifolds? (Not easy)

Small covers — closed n-manifold with locally standard $(\mathbb{Z}_2)^n$-actions whose orbit space is a simple convex polytope.

They are introduced by Davis-Januszkiewicz (1991 Duke. Math. J.) as an analogue of toric manifolds.

Small Covers

Suppose M^n is a small cover whose orbit space under the locally standard $(\mathbb{Z}_2)^n$-action is P^n (a simple n-polytope). Let

$$\pi : M^n \to P^n \text{ (the orbit map).}$$

For any facet F_i of P^n, the isotropy subgroup of $\pi^{-1}(F_i) \subset M^n$ under the $(\mathbb{Z}_2)^n$-action is a rank one subgroup of $(\mathbb{Z}_2)^n$ generated by a nonzero element, say $g_{F_i} \in (\mathbb{Z}_2)^n$. Then we obtain a map

$$\begin{aligned} \lambda_{M^n} : \mathcal{F}(P^n) &\longrightarrow (\mathbb{Z}_2)^n \\ F_i &\longmapsto g_{F_i} \end{aligned}$$

We call λ_{M^n} the <u>characteristic function</u> associated to M^n.

Conversely, Davis-Januszkiewicz showed that up to equivariant homeomorphism, M^n can be recovered from (P^n, λ_{M^n}) by

$$M^n = P^n \times (\mathbb{Z}_2)^n / \sim \tag{1.3}$$

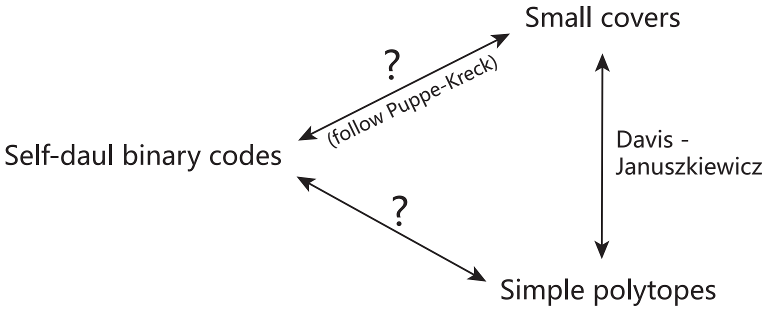
where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g' \in G_p$ where

$$G_p = \text{ the subgroup of } (\mathbb{Z}_2)^n \text{ generated by } \{\lambda_{M^n}(F) \,|\, p \in F\}$$

Many topological invairants (fundamental group, cohomology groups, characteristic classes etc.) can be explicitly computed from the combinatorics of P^n and λ. For example,

$$b_i(M; \mathbb{F}_2) = h_i(P^n),\ 0 \leq i \leq n$$

where $(h_0(P^n), h_1(P^n), ..., h_n(P^n))$ is the h-vector of P^n


Small covers
?
(follow Puppe-Kreck)
Self-daul binary codes
Davis -
Januszkiewicz
?
Simple polytopes

m-involutions on Small Covers

Let $\pi : M^n \to P^n$ be a small cover and $\lambda : \mathcal{F}(P^n) \to (\mathbb{Z}_2)^n$ be its characteristic function. Any $g \neq 0 \in (\mathbb{Z}_2)^n$ determines an involution τ_g on M^n, called a regular involution on M^n.

Theorem [Chen-Lü-Yu]

The following statements are equivalent.

(a) There exists a regular m-involution on M^n.

(b) There exists a regular involution on M^n with only isolated fixed points;

(c) The image $\mathrm{Im}(\lambda)$ of λ is a basis of $(\mathbb{Z}_2)^n$ (which implies that P^n is n-colorable).

Description of n-colorable simple n-polytopes

A simple polytope is n-colorable if we can color all the facets of the polytope by n different colors so that any neighboring facets are assigned different colors.

Theorem [Joswig 2002]

Let P^n be an n-dimensional simple polytope. The following statements are equivalent.

(a) P^n is n-colorable;

(b) Each 2-face of P^n has an even number of vertices.

(c) Each face of P^n with dimension greater than 0 (including P^n itself) has an even number of vertices.

(d) Each k-face of P^n is k-colorable.

Let $\pi : M^n \to P^n$ be an n-dimensional small cover which admits a regular m-involution. Then by our preceding discussions,

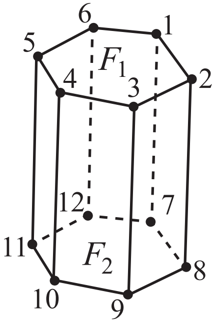
- P^n is an n-dimensional n-colorable simple polytope.
- The characteristic function λ of M^n satisfies: $\mathrm{Im}(\lambda) = \{e_1, \cdots, e_n\}$ is a basis of $(\mathbb{Z}_2)^n$.
- $\tau_{e_1+\cdots+e_n}$ is an m-involution on M^n.
- Suppose P^n has $2r$ vertices. There is a filtration

$$\mathbb{F}_2 \cong V_0^M \subset V_1^M \subset \cdots \subset V_{n-2}^M \subset V_{n-1}^M = \mathcal{V}_{2r} \subset V_n^M = \mathbb{F}_2^{2r}.$$

In particular, when n is odd, $C_{M^n} := V^M_{\frac{n-1}{2}} \subset \mathbb{F}_2^{2r}$ is a self-dual binary code determined by $(M^n, \tau_{e_1+\cdots+e_n})$.

Let $\{v_1, \cdots, v_{2r}\}$ be all the vertices of P^n. Any face f of P^n determines an element $\xi_f \in \mathbb{F}_2^{2r}$ where the i-th entry of ξ_f is 1 if and only if v_i is a vertex of f.

For example, $\xi_{v_i} = (0, \cdots, \overset{i}{1}, \cdots, 0)$, $\xi_{P^n} = \underline{1} = (1, \cdots, 1) \in \mathbb{F}_2^{2r}$.



$$\begin{pmatrix}
1&0&1&0&0&0&0&1\\
1&0&1&1&0&0&0&0\\
1&0&0&1&1&0&0&0\\
1&0&0&0&1&1&0&0\\
1&0&0&0&0&1&1&0\\
1&0&0&0&0&0&1&1\\
0&1&1&0&0&0&0&1\\
0&1&1&1&0&0&0&0\\
0&1&0&1&1&0&0&0\\
0&1&0&0&1&1&0&0\\
0&1&0&0&0&1&1&0\\
0&1&0&0&0&0&1&1
\end{pmatrix}$$

Main Theorem [Chen-Lü-Yu]

Let $\pi : M^n \to P^n$ be an n-dimensional small cover which admits a regular m-involution where n is odd. For any $0 \leq k \leq n$,

$$V_k^M = \mathrm{Span}_{\mathbb{F}_2}\{\xi_f \,;\, f \text{ is a codimension-}k \text{ face of } P^n\}$$

- The self-dual binary code $C_{M^n} = V^M_{\frac{n-1}{2}}$ is spanned by

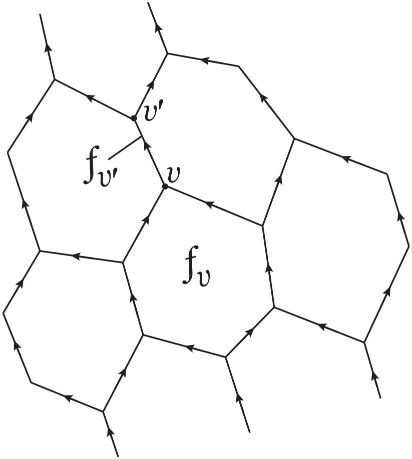
$$\{\xi_f \,;\, f \text{ is any face of } P^n \text{ with } \dim(f) = \tfrac{n+1}{2}\}.$$

- So the minimum distance of C_{M^n} is less or equal to

$$\min\{\#(\text{vertices of } f) \,;\, f \text{ is a } \tfrac{n+1}{2}\text{-dimensional face of } P^n\}.$$

A linear basis of V_k^M

- Choose a generic height function ϕ on P^n. Using ϕ, one makes the 1-skeleton of P^n into a directed graph by orienting each edge so that ϕ increases along it.
- For any face f of P^n with dimension > 0, $\phi|_f$ assumes its maximum (or minimun) at a vertex. Since ϕ is generic, each face f of P^n of a unique "top" and a unique "bottom" vertex.
- For any vertex v, let $m(v)$ denote the number of incident edges which point toward v, and let f_v be the smallest face of P^n which contains all the inward pointing edges incident to v. It is clear that $\dim(f_v) = m(v)$.


v'
$f_{v'}$
v
f_v

Fact

The number of vertices v of P^n with $m(v) = k$ is equal to $h_k(P^n)$.

Proposition

Let $\pi : M^n \to P^n$ be an n-dimensional small cover which admits a regular m-involution where n is odd. For any $0 \le k \le n$, the linear space V_k^M has a basis defined by

$$\mathcal{A}_k = \{\xi_{f_v}\,;\, v \text{ is any vertex of } P^n \text{ with } n-k \le m(v) \le n,\} \subset (\mathbb{F}_2)^{2r}.$$

So in particular, $\mathcal{A}_{\frac{n-1}{2}}$ is a basis of $C_{M^n} = V^M_{\frac{n-1}{2}}$.

Binary Codes from General Simple Polytopes

Given an arbitrary n-dimensional simple polytope P^n, let the vertices of P^n be $v_1, \cdots, v_l$. Then for any $0 \leq k \leq n$, the following definition still makes sense.

$$\mathfrak{B}_k(P^n) := \text{Span}_{\mathbb{F}_2}\{\xi_f \,;\, f \text{ is a codimension-}k \text{ face of } P\} \subset \mathbb{F}_2^l.$$

Question:

For what simple polytope P^n and what $0 \leq k \leq n$, is the $\mathfrak{B}_k(P^n)$ a binary self-dual code?

Theorem [Chen-Lü-Yu]

Let P be an n-dimensional simple polytope. Then $\mathfrak{B}_k(P)$ is a self-dual code if and only if P is n-colorable, n is odd and $k = \frac{n-1}{2}$.

Therefore,the set of self-dual binary codes we can obtain from simple polytopes agree with those obtained from small covers!

Properties of n-colorable simple n-polytopes

Proposition [Chen-Lü-Yu]

Let P^n be an n-dimensional simple polytope with m facets. Then the following statements are equivalent.

(1) P^n is n-colorable.

(2) There exists a partition $\mathcal{F}_1, ..., \mathcal{F}_n$ of the set $\mathcal{F}(P^n)$ of all facets, such that for each $1 \leq i \leq n$, all the facets in $\mathcal{F}_i$ are pairwise disjoint and $\sum_{F\in\mathcal{F}_i} \xi_F = \underline{1}$ (i.e., each vertex of P^n is incident to exactly one facet from every $\mathcal{F}_i$).

(3) $\mathfrak{B}_0(P^n) \subset \mathfrak{B}_1(P^n) \subset \cdots \subset \mathfrak{B}_{n-1}(P^n) \subset \mathfrak{B}_n(P^n) \cong \mathbb{F}_2^{|V(P^n)|}$.

(4) $\mathfrak{B}_{n-2}(P^n) \subset \mathfrak{B}_{n-1}(P^n)$.

(5) $\dim_{\mathbb{F}_2} \mathfrak{B}_1(P^n) = m - n + 1$.

Proposition [Chen-Lü-Yu]

Let P^n be an n-colorable simple n-polytope. For any codimension-k face f of P^n. Then $|V(P^n)| \geq 2^k|V(f)|$. Moreover, $|V(P^n)| = 2^k|V(f)|$ if and only if $P = f \times [0,1]^k$.

Corollary

For any n-colorable simple n-polytope P^n, we must have $|V(P^n)| \geq 2^n$. In particular, $|V(P^n)| = 2^n$ if and only if $P^n = [0,1]^n$ (the n-dimensional cube).

Minimum Distance of Self-Dual Codes from Simple Polytopes

Proposition [Chen-Lü-Yu]

For a 3-dimensional 3-colorable simple polytope P^3, the minimum distance of the self-dual code $\mathfrak{B}_1(P^3)$ is always equal to 4.

Conjecture: For an n-colorable simple n-polytope P^n where n is odd, the minimum distance of the self-dual binary code $\mathfrak{B}_{\frac{n-1}{2}}(P^n)$ is equal to

$$\min\{\#(\text{vertices of } f)\,;\ f \text{ is a } \tfrac{n+1}{2}\text{-dimensional face of } P^n\}.$$

End of Talk

August 24, 2015

Singapore National University