The topology and geometry of the moment-angle manifolds

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- Topology of moment-angle manifolds
- Positive Ricci curvature of moment-angle manifolds

Definition

K is a simplicial $n-1$ complex. Let $[m]=\{1,\ldots,m\}$ represent the m vertices of the simplicial complex, σ be a simplex in the complex K, $|\sigma|$ is the number of the vertices of σ. Define

$$D_{\sigma}^{2|\sigma|}\times T_{\widehat{\sigma}}=\{(z_1,z_2,\cdots,z_m)\in(D^2)^m:|z_j|=1 for j\notin\sigma\}.$$

and define Z_K corresponding to K as

$$Z_K=\bigcup_{\sigma\in K}D_{\sigma}^{2|\sigma|}\times T_{\widehat{\sigma}}\subset(D^2)^m.$$

1. A conjecture and the idea of the proof

We want to study the topology of the moment-angle manifold. Method: considering the change of $Z(K)$ after we do some operation on K. One operation is making connected sum of K with $\partial\triangle^n$.

For this case, S.Gitler and S.Lopez conjecture:

Conjecture

let K be a simplicial $n-1$ sphere with m vertices and $K_\sigma = K\#_\sigma \partial\triangle^n$. Let Z and Z_σ be the corresponding moment-angle manifolds, then Z_σ is homeomorphic to

$$\partial\left[\left(Z - \overset{\circ}{D}{}^{n+m}\right)\times D^2\right] \# \mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2}\times S^{m+n-j-1}).$$

If K is dual to the boundary of a simple polytope, the 'homeomorphism' will be 'diffeomorphism'.

By the definition, the moment-angle complex corresponding to K is:

$$Z = \bigcup_{\sigma \in K} D_{\sigma}^{2|\sigma|} \times T_{\widehat{\sigma}} \subset (D^2)^m.$$

Then we can express the moment-angle complex corresponding to K_σ as follows:

$$\begin{aligned} Z_\sigma &= \left(Z \times S^1 - T_{\widehat{\sigma}}^{m-n} \times \overset{\circ}{D}{}_{\sigma}^{2n} \times S^1 \right) \cup_{T_{\widehat{\sigma}}^{m-n} \times S_{\sigma}^{2n-1} \times S^1} T_{\widehat{\sigma}}^{m-n} \times S_{\sigma}^{2n-1} \times D^2 \\ &\simeq \partial \left[\left(Z - T_{\widehat{\sigma}}^{m-n} \times \overset{\circ}{D}{}_{\sigma}^{2n} \right) \times D^2 \right] \end{aligned}$$

In the case $m < 3n$, S. Gitler and S. López firstly proved that $T_{\hat{\sigma}}^{m-n} \times \{0\}$ can be contracted to a point in Z. Since $m < 3n$, it is isotopic to a $(m-n)$-torus inside an open disk in Z. therefore,

$$Z - T_{\widehat{\sigma}}^{m-n} \times \overset{\circ}{D}{}_{\sigma}^{2n} \cong Z - T^{m-n} \times \overset{\circ}{D}{}^{2n} \cong (Z - \overset{\circ}{D}{}^{n+m}) \cup (D^{n+m} - T^{m-n} \times \overset{\circ}{D}{}^{2n})$$

and

$$Z_{\sigma} \simeq \partial \left[\left(Z - \overset{\circ}{D}{}^{n+m} \right) \times D^2 \right] \# \partial \left[\left(S^{m+n} - T^{m-n} \times \overset{\circ}{D}{}^{2n} \right) \times D^2 \right].$$

Then they prove

$$\partial \left[\left(S^{m+n} - T^{m-n} \times \overset{\circ}{D}{}^{2n} \right) \times D^2 \right] \simeq \mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1}).$$

We prove the conjecture in general case, the main idea is :

- Firstly, we construct an isotopy of $T^{m-n}_{\hat{\sigma}}$ in Z to move it to the regular embedding $T^{m-n} \subseteq D^{m-n+1} \subseteq D^{m+n} \subseteq Z$, thus we prove the following:

Proposition

Z_σ is homeomorphic to

$$\partial\left[\left(Z - \overset{\circ}{D}{}^{n+m}\right) \times D^2\right] \# \, \partial\left[\left(S^{m+n} - T^{m-n} \times \overset{\circ}{D}{}^{2n}\right) \times D^2\right],$$

where $T^{m-n} \times \overset{\circ}{D}{}^{2n}$ is the regular embedding in S^{m+n}.

We construct the regular embedding of T^k into $\mathbb{R}^{k+1}$ as follows:
$S^1 \subseteq D^2 \subseteq \mathbb{R}^2$
assume that
$T^{i-1} \subseteq D^i \subseteq \mathbb{R}^i$
$T^{i-1} \times S^1 \subseteq D^i \times S^1 \subseteq D^i \times S^1 \cup S^{i-1} \times D^2 - \{*\} \cong \mathbb{R}^{i+1}$
The key lemma in the construction of the isotopy is:

Lemma

There are two embedding of the torus T^k into D^{k+2}:
1. $T^k \subseteq D^{k+1} \subseteq D^{k+2}$
2. $T^k = T^{k-1} \times S^1 \subseteq D^k \times D^2 = D^{k+2}$
The two embeddings are isotopic to each other in D^{k+2}.

- Then we prove the following by induction:

Proposition

$\partial\left[\left(S^{m+n} - T^{m-n} \times \overset{\circ}{D}{}^{2n}\right) \times D^2\right]$ is diffeomorphic to

$$\mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{j+2} \times S^{m+n-j-1}),$$

where $T^{m-n} \times \overset{\circ}{D}{}^{2n}$ is the regular embedding in S^{m+n}.

Combining these two propositions, the conjecture is proved.

Further study

Until now, we have proved the conjecture. However, we can prove the following:

Theorem

Let K_1, K_2 are two simplicial $(n-1)$-spheres. Z_{K_1} and Z_{K_2} are the moment-angle manifolds corresponding to K_1 and K_2. Let $K = K_1 \# K_2$ be their connected sum at some maximal simplex σ_1 and σ_2. The moment-angle manifold corresponding to K is homeomorphic to

$$\mathcal{G}^{m_2-n}(Z_{K_1}) \# \mathcal{G}^{m_1-n}(Z_{K_2}) \# M$$

$$M \cong \mathop{\#}_{j=1}^{m_1+m_2-2n-1} [\binom{m_1+m_2-2n}{j+1} - \binom{m_1-n}{j+1} - \binom{m_2-n}{j+1}] S^{j+2} \times S^{m_1+m_2-j-2}.$$

The operation $\mathcal{G}$ on an manifold M^k is defined as:

$$\mathcal{G}(M^k) = \partial[(M^k - \overset{\circ}{D^k}) \times D^2].$$

$\mathcal{G}^p(M^k)$ means doing the operation $\mathcal{G}$ on M^k p times.
If K_1, K_2 are dual to the boundary of simple polytopes P_1^n, P_2^n, the 'homeomorphism' will be 'diffeomorphism'.

By the definition, the moment-angle complex corresponding to K is:

$$Z_K = \bigcup_{\substack{\tau \in K_1 \quad or \quad \tau \in K_2 \\ \tau \neq \sigma}} D_\tau^{2|\tau|} \times T^{m_1+m_2-n-|\tau|}.$$

which can be expressed as

$$(Z_{K_1} - T^{m_1-n} \times \overset{\circ}{D}{}_{\sigma_1}^{2n}) \times T^{m_2-n} \cup (Z_{K_2} - T^{m_2-n} \times \overset{\circ}{D}{}_{\sigma_1}^{2n}) \times T^{m_1-n}$$

where the two pieces are glued along

$$T^{m_1-n} \times S_{\sigma_1}^{2n-1} \times T^{m_2-n} \simeq T^{m_1-n} \times S_{\sigma_2}^{2n-1} \times T^{m_2-n},$$

using the identification of σ_1 and σ_2.

Idea of the proof:
Firstly, using the method above, we prove that

$$(Z_{K_1} - T^{m_1-n} \times D^{2n}_{\sigma_1}) \times T^{m_2-n} \cup (Z_{K_2} - T^{m_2-n} \times D^{2n}_{\sigma_2}) \times T^{m_1-n}$$

is homeomorphic to

$$[Z_{K_1}\#(S^{m_1+n} - T^{m_1-n} \times \overset{\circ}{D}{}^{2n})] \times T^{m_2-n} \cup [Z_{K_2}\#(S^{m_2+n} - T^{m_2-n} \times \overset{\circ}{D}{}^{2n})] \times T^{m_1-n},$$

where the two pieces are glued along

$$T^{m_1-n} \times S^{2n-1} \times T^{m_2-n} \overset{id}{\simeq} T^{m_1-n} \times S^{2n-1} \times T^{m_2-n}$$

.

Secondly, we construct an isotopy of $\{*\}\times T^{m_2-n}(\{*\}\times T^{m_1-n})$ in

$$(S^{m_1+n}-T^{m_1-n}\times \mathring{D}^{2n})\times T^{m_2-n}\cup(S^{m_2+n}-T^{m_2-n}\times \mathring{D}^{2n})\times T^{m_1-n}$$

to move it to the regular embedding $T^{m_2-n}\subseteq D^{m_1+m_2}(T^{m_1-n}\subseteq D^{m_1+m_2})$. Thus we prove that

$$[Z_{K_1}\#(S^{m_1+n}-T^{m_1-n}\times \mathring{D}^{2n})]\times T^{m_2-n}\cup[Z_{K_2}\#(S^{m_2+n}-T^{m_2-n}\times \mathring{D}^{2n})]\times T^{m_1-n}$$

is homeomorphic to

$$(Z_{K_1}\times T^{m_2-n}\#_{T^{m_2-n}}S^{m_1+m_2})\#(Z_{K_2}\times T^{m_1-n}\#_{T^{m_1-n}}S^{m_1+m_2})\#M.$$

$$M=(S^{m_1+n}\times T^{m_2-n}\#_{T^{m_1-n}\times T^{m_2-n}}S^{m_2+n}\times T^{m_1-n})$$

Finally, we inductively prove that

1. $Z_{K_1} \times T^{m_2-n} \#_{T^{m_2-n}} S^{m_1+m_2} \simeq \mathcal{G}^{m_2-n}(Z_{K_1})$.
2. $Z_{K_2} \times T^{m_1-n} \#_{T^{m_1-n}} S^{m_1+m_2} \simeq \mathcal{G}^{m_1-n}(Z_{K_2})$.
3. $S^{m_1+n} \times T^{m_2-n} \#_{T^{m_1-n} \times T^{m_2-n}} S^{m_2+n} \times T^{m_1-n}$

is homeomorphic to

$$\# \mathop{\#}_{j=1}^{m_1+m_2-2n-1} \left(\binom{m_1+m_2-2n}{j+1} - \binom{m_1-n}{j+1} - \binom{m_2-n}{j+1}\right) S^{j+2} \times S^{m_1+m_2-j-2}.$$

2.Positive Ricci curvature of moment-angle manifolds

(D^2, S^1) : moment-angle manifolds.
$(D^{k+1}, S^k), k \geq 2$:generalised moment-angle manifolds.
Similarly,

$$\overline{Z}_\sigma \approx \partial\left[\left(\overline{Z} - D^{\overset{\circ}{n+km}}\right) \times D^{k+1}\right] \# \mathop{\#}_{j=1}^{m-n} \binom{m-n}{j} (S^{k(j+1)+1} \times S^{k(m-j)+n-1}).$$

From this, we know that if $\overline{Z}$ has a metrics of positive Ricci curvature, then $\overline{Z}_\sigma$ also has a metrics of positive Ricci curvature. We ask: When $Z(P)$ has a metrics of positive Ricci curvature?

Conjecture

For a simple polytope,
1. If $Z(D^k, S^{k-1})$ *has a metrics of positive Ricci curvature, then* $Z(D^{k+1}, S^k)$ *also has a metrics of positive Ricci curvature.*
2. $Z(D^{k+1}, S^k)$ *has a metrics of positive Ricci curvature, for* $k \geq 2$*.*

Definition

Let Q be a simplicial convex polytope in $\mathbb{R}^n$ whose vertices are primitive lattice vectors($\in \mathbb{Z}^n$), and which contains 0 in the interior. If $a_1, \ldots, a_n$ are the vertices of a facet of Q, we suppose $det(a_1, \ldots, a_n) = \pm 1$ for every facet. Then we call Q a Fano polytope.

The dual of Q:

$$P = \{u \in \mathbb{R}^n | \langle u, v \rangle \leq 1, \forall v \in Q\}$$

is a simple polytope. We claim that the moment-angle manifold corresponding to P has a metrics of positive Ricci curvature.

In fact, we can construct a smooth toric Fano variety X_P from the polytope P. According to the Calabi-Yau's theorem, it has a metrics of positive Ricci curvature. However, the moment-angle manifold $Z(P)$ is a principal T^{m-n} bundle of X_P, by a theorem of Gilkey, we know that $Z(P)$ has a metrics of positive Ricci curvature.

Thank you very much !