Linearisation of algebraic structures via functor calculus

Combinatorial and toric homotopy

Singapore, august 2015

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Reminder: Relations between groups and Lie algebras

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2. The associated graded of arbitrary groups: For any group G and elements $x, y \in G$ let $[x,y] = (xy)(yx)^{-1}$. An N-series of G is a filtration

$$\mathcal{N}\colon\ G = N_1 \supset N_2 \supset \dots$$

of G by subgroups N_n such that $[N_i, N_j] \subset N_{i+j}$.

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of G by subgroups N_n such that $[N_i, N_j] \subset N_{i+j}$. Then $\mathrm{Gr}_n^{\mathcal{N}}(G) = N_n / N_{n+1}$ is an abelian group, and

$$\mathrm{Gr}^{\mathcal{N}}(G) = \sum_{k \geq 1} \mathrm{Gr}_n^{\mathcal{N}}(G)$$

is a graded Lie ring whose bracket is induced by the commutator of G.

Examples of N-series

1. The lower central series

$$\gamma\colon G = \gamma_1(G) \supset \gamma_2(G) \supset \dots$$

where $\gamma_n(G) = \langle [x_1, \dots, x_n] \mid x_1, \dots, x_n \in G \rangle$ with

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2. The dimension series: let $\mathbb{K}$ be a commutative ring. Then the subgroups

$$D_{n,\mathbb{K}}(G) = G \cap (1 + I^n_{\mathbb{K}}(G))$$

form an N-series where $I^n_{\mathbb{K}}(G)$ denotes the n-th power of the augmentation ideal of the group algebra $\mathbb{K}(G)$.

Relations between groups and Lie algebras - sequel

3. Mal'cev/Lazard equivalence: There is a canonical equivalence between the categories of radicable n-step nilpotent groups and n-step nilpotent Lie algebras over $\mathbb{Q}$, based on the Baker-Campbell-Hausdorff formula.

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4. Primitive operations on group algebras: Primitive elements of Hopf algebras (the bialgebra type of group algebras) form a Lie algebra under the usual ring commutator.

Relations between groups and Lie algebras - Summary

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2. The associated graded of arbitrary groups
3. Mal'cev/Lazard equivalence
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GOAL:

Given a suitable non-linear "algebraic" structure generalizing groups,

- exhibit a related linear structure = type of algebras (linear operad), generalizing Lie algebras

- generalize the relations 1. to 4. above to this situation.

Approach

- develop and use algebraic functor calculus to construct a suitable notion of commutators and a suitable operad in abelian groups, satisfying relation 2. (basically done)

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- try to generalize relation 1. (dream!)

Framework

Definition [Janelidze, Márki, Tholen 2002]: A category $\mathcal{C}$ is called semi-abelian if it is pointed, finitely complete and cocomplete, protomodular and Barr-exact.

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1. C is pointed and finitely complete and cocomplete.
2. For any morphism $p \colon X \to Y$ in $\mathcal{C}$ admitting a section $s \colon Y \to X$, X "is generated by the kernel of p and the image of s", that is the morphism $\mathrm{Ker}(p) + Y \to X$ given by the injection of $\mathrm{Ker}(p)$ and s is a cokernel.

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- the category of cocommutative Hopf algebras (Gadjo-Gran-Vercruyssen)

- ....

The idea of the categorical (Higgins) commutator calculus

(blackboard)

Basic (algebraic) functor calculus

In the sequel, $F\colon \mathcal{C} \to \mathcal{D}$ denotes a functor between categories satisfying
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$$\lhd \; F(X_1 + \ldots + X_n)$$

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In particular, $cr_1 F(X) = \mathrm{Ker}\Big(F(0)\colon F(X) \to F(0)\Big)$ and

$$cr_2 F(X, Y) = \mathrm{Ker}\Big(r_{12}\colon F(X+Y) \twoheadrightarrow F(X) \times F(Y)\Big).$$

Examples

- A functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is additive iff $cr_2 F = 0$.

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- For $T^2\colon \mathrm{Ab} \to \mathrm{Ab}$, $T^2(A) = A \otimes A$, we have

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- Let Gr denote the category of groups. Then for groups $X_1, \ldots, X_n$ and elements $x_k \in X_k$, $k = 1, \ldots, n$, we have

$$[x_1, \ldots, x_n] \in Id_{Gr}(X_1|\ldots|X_n).$$

If $n = 2$ these elements generate $Id_{Gr}(X_1|X_2)$ (freely if one takes $x_1, x_2 \neq e$).

- Let Lp denote the category of loops. Then for loops X_1, X_2, X_3 and elements $x_k \in X_k$ the associator

$$A(x_1, x_2, x_3) = (x_1(x_2x_3))\backslash((x_1x_2)x_3) \in Id_{Lp}(X_1|X_2|X_3).$$

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- The functor $cr_n\colon Func(\mathcal{C}, \mathcal{D}) \to Func(\mathcal{C}^n, \mathcal{D})$ is exact.
- "Pseudo-right-exactness" [Van der Linden]:

If F preserves coequalizers of reflexive parallel pairs of morphisms (reflexive meaning that these morphisms admit a common section) then so does $cr_n F$ in all variables, for any n.

Preservation of coequalizers of reflexive parallel pairs

A functor $F\colon \mathcal{C} \to \mathcal{D}$ as before preserves coequalizers of reflexive parallel pairs iff for any right-exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

in $\mathcal{C}$ the sequence

$$F(A) + F(A|B) \xrightarrow{\left\langle \begin{smallmatrix} F(a) \\ \delta \end{smallmatrix} \right\rangle} F(B) \xrightarrow{F(b)} F(C) \longrightarrow 0$$

in $\mathcal{D}$ is exact, where

$$\delta\colon F(A|B) \xrightarrow{F(a|1_B)} F(B|B) \rightarrowtail F(B+B) \xrightarrow{F(\nabla^2)} F(B)\,.$$

Operadic structure of cross-effects

- Operadic structure: Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be reduced functors where the category $\mathcal{E}$ is semi-abelian, too.

Denote "multi-objects", i.e. sequences of objects in $\mathcal{C}$, by $\underline{X}_j = X_{j,1}, \ldots, X_{j,k_j}$ and concatenation of such by

$$\underline{X}_1 \cup \ldots \cup \underline{X}_n = X_{1,1}, \ldots, X_{1,k_1}, \ldots, X_{n,1}, \ldots, X_{n,k_n}.$$

Then there is a natural transformation

$$\begin{array}{c} cr_n G\Big(cr_{k_1} F(\underline{X}_1), \ldots, cr_{k_n} F(\underline{X}_n)\Big) \\ \downarrow \\ cr_{k_1+\ldots+k_n}(G \circ F)\big(\underline{X}_1 \cup \ldots \cup \underline{X}_n\big) \end{array}$$

rendering a certain canonical diagram commutative.

Polynomial functors

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Example 1: A reduced functor between abelian categories is linear (that is, polynomial of degree ≤ 1) iff it is additive.

Example 2: The n-th tensor power functor $T^n \colon \mathrm{Ab} \to \mathrm{Ab}, T^n(A) = A^{\otimes n}$, is polynomial of degree n.

Commutators via functor calculus

Let $F: \mathcal{C} \to \mathcal{D}$ be a reduced functor as before.

For subobjects $x_k : X_k \rightarrowtail X$, $k = 1, \ldots, n$, of an object X of $\mathcal{C}$ define the subobject $[X_1, \ldots, X_n]_F$ of $F(X)$ to be the image of the morphism

$$F(X_1|\ldots|X_n) \rightarrowtail F(X_1 + \ldots + X_n) \xrightarrow{F(x_1,\ldots,x_n)} F(X)$$

Note that

$$[X_1, \ldots, X_n]_{Id_{\mathcal{D}}} \leq X,$$

and that

$$[X_1]_F = \mathrm{Im}\Big(cr_1 F(X_1) \rightarrowtail F(X_1) \xrightarrow{F(x_1)} F(X)\Big).$$

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$$[X_1, [X_2, X_3]].[X_2, [X_3, X_1]].[X_3, [X_1, X_2]].$$

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2. If $\mathcal{D}$ is the category of loops Lp, then

- $[X_1, X_2]_{Id_{Lp}}$ is the normal subloop of $\langle X_1 \cup X_2 \rangle$ generated by the elements $[x_2, x_1]$, $A(x_1, y_1, y_2)$, $A(x_1, x_2, y_1)$, $A(x_1, x_2, y_2)$ $A(x_2, x_1, y_2)$, and $A_3(x_1, x_2, x_1, y_2)$ where $x_i, y_i \in X_i$ and

$$[a, b] = ba \backslash ab$$

$$A_3(a, b, c, d) = (A(a, b, c)A(a, b, d)) \backslash A(a, b, cd)\,.$$

Examples - sequel 1

3. If $\mathcal{D}$ is a category of ω-loops then $[X_1, X_2]_{Id_{\mathcal{D}}}$ is the normal subobject of $X_1 \vee X_2$ generated by the elements

$$[(x_1, \dots, x_n), (y_1, \dots, y_n)]_\theta =$$

$$\theta(x_1 y_1, \dots, x_n y_n)/(\theta(x_1, \dots, x_n)\theta(y_1, \dots, y_n))$$

where $x_1, \dots, x_n \in X_1$, $y_1, \dots, y_n \in X_2$ and θ is a generating operation of $\mathcal{D}$.

- 3a) If $\mathcal{D} = Groups$ then $[x, y]_i = y^{-1}x^{-1}yx$ and $[(x_1, x_2), (y_1, y_2)]_. = {}^{x_1}[y_1, x_2]$.

- 3a) If $\mathcal{D} = Loops$ then $[(x_1, x_2), (y_1, y_2)]_. = ((x_1 y_1)(x_2 y_2))/((x_1 x_2)(y_1 y_2))$. In particular, $[(e, x_2), (y_1, e)]_. = (y_1 x_2)/(x_2 y_1)$ and $[(x_1, e), (y_1, y_2)]_. = ((x_1 y_1) y_2)/(x_1 (y_1 y_2))$.

Examples - sequel

4. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras $\mathcal{P}\text{-}Alg$, then

$$[X_1,\ldots,X_n]_{\mathcal{P}\text{-}Alg} = \sum_{p_k\geq 1} \mu_p(X_1^{\otimes p_1}\otimes\ldots\otimes X_n^{\otimes p_n}\otimes\mathcal{P}(p)).$$

where $p = p_1 + \ldots + p_n$.

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- Preservation by morphisms: For $f : X \to Y$ in $\mathcal{C}$,

$$F(f)([X_1, \ldots, X_n]_F) = [f(X_1), \ldots, f(X_n)]_F.$$

Lower central series

For an object X of $\mathcal{D}$ let

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for $N_k = \gamma_k^F(X)$. In particular, taking $F = Id_{\mathcal{D}}$ we obtain the (categorically defined) lower central series (c.l.c.s.) of X,

$$X = \gamma_1^{Id_{\mathcal{D}}}(X) \geq \gamma_2^{Id_{\mathcal{D}}}(X) \geq \ldots$$

Examples

1. If $\mathcal{D}$ is the category of groups, then the categorically defined lower central series coincides with the classical l.c.s.

2. If $\mathcal{D}$ is the category of loops, then the categorically defined lower central series coincides with the commutator-associator filtration introduced by Mostovoy.

3. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras $\mathcal{P}\text{-}Alg$, then

$$\gamma_n^{Id_{\mathcal{P}-Alg}}(X) = \sum_{k \geq n} \mu_k(X^{\otimes k} \otimes \mathcal{P}(k)).$$

How to prove this?

Characterisation of the c.l.c.s.

Theorem. Let $\mathcal{X}(X)\colon X = X_1 \geq X_2 \geq \dots$ be a natural filtration of all objects X in $\mathcal{D}$ by normal subobjects X_n of X.

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- multifunctors $M_n\colon \mathcal{D}^n \to \mathcal{D}$
- natural maps $m_n\colon M_n(X,\dots,X) \to X_n$

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such that the following two conditions are satisfied:

1. Factorisations $\overline{m_n}$ exist and are cokernels rendering the following diagrams commutative:

$$\begin{tikzcd}
M_n(X,\dots,X) \arrow[r, "m_n"] \arrow[d, two heads, "t_{\underline{1}}"] & X_n \arrow[d, two heads, "q_n"] \\
(T_{\underline{1}}M_n)(X,\dots,X) \arrow[r, dotted, two heads, "\overline{m_n}"] & X_n/X_{n+1}
\end{tikzcd}$$

2. The images of the maps

$$m_k\colon M_k(X,\dots,X) \to X_k \hookrightarrow X_n,$$

$k \geq n$, jointly generate X_n as a normal subobject of X.

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Application:

Proof of the identity $\gamma_n^{Id_{\mathcal{D}}}(X) = \gamma_n(X)$ in $\mathcal{D} = \mathit{Groups}$:

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Application:

Proof of the identity $\gamma_n^{Id_{\mathcal{D}}}(X) = \gamma_n(X)$ in $\mathcal{D} = Groups$: take

- $M_n(X_1,\dots,X_n)$ to be the free group generated by the set $X_1 \times \dots \times X_n$ modulo the normal subgroup generated by the tuples $(x_1,\dots,x_n)$ where one of the x_k's is trivial

- m_n to send a basis element $(x_1,\dots,x_n) \in X^n$ to $[x_1,\dots,x_n]$.

Lower central series of the group ring functor

Let $F\colon \mathit{Groups} \to \mathrm{Ab}$ be the functor sending a group G to its group ring $\mathbb{Z}[G]$. Then

$$\gamma_n^F(G) = I^n(G)$$

where $I^n(G)$ is the n-th power of the augmentation ideal of $\mathbb{Z}[G]$.

Polynomialization of functors

For any functor $F\colon \mathcal{C} \to \mathcal{D}$ as before let

$$T_nF = F/\gamma_n(F) \text{ and } t_n\colon F \longrightarrow\!\!\!\!\!\to T_nF\ .$$

Then the functor T_nF is polynomial of degree $\leq n$ and t_n is initial among all natural transformations from F to polynomial functors of degree $\leq n$.

Nilpotency

Define an object X of $\mathcal{D}$ to be *n*-step nilpotent if $\gamma_{n+1}^{Id_{\mathcal{D}}}(X) = 0$.

"Polynomiality subsumes nilpotency"

1. Global statement: All objects of $\mathcal{D}$ are n-step nilpotent iff the identity functor of $\mathcal{D}$ is polynomial of degree $\leq n$. For arbitrary $\mathcal{D}$, the "n-step nilization" functor $X \mapsto Nil_n(X) = X/\gamma_{n+1}^{Id_{\mathcal{D}}}(X)$ equals $T_n Id_{\mathcal{D}}$.

1. Local statement: A single object X of $\mathcal{D}$ is n-step nilpotent iff its "commutator map"

$$S_2^{Id_{\mathcal{D}}} \colon Id_{\mathcal{D}}(X|X) \rightarrowtail X + X \xrightarrow{\nabla^2} X$$

is polynomial of degree $\leq n-1$ in both (equivalently any of the two) variables, which by definition means that $S_2^{Id_{\mathcal{D}}}$ factors through the bi-polynomialization

$$t_{n-1,n-1} \colon cr_2 Id_{\mathcal{D}}(X, X) \twoheadrightarrow T_{n-1,n-1}(cr_2 Id_{\mathcal{D}})(X, X).$$

Example: 2-step nilpotency in groups

Let $\mathcal{D} = Gr$ and $n = 2$. Here

$$\begin{array}{c} X \\ \Big\uparrow {\scriptstyle S_2^{Id_{Gr}}\,:\,(x,y)\mapsto[x,y]} \\ cr_2 Id_{Gr}(X,X) = \mathrm{Free}(|X^*| \times |X^*|) \\ \Big\downarrow {\scriptstyle t_{1,1}\,:\,(x,y)\mapsto(xX')\otimes(yY')} \\ T_{1,1}(cr_2 Id_{Gr})(X,X) = X_{ab} \otimes_{\mathbb{Z}} X_{ab} \end{array}$$

Hence $S_2^{Id_{Gr}}$ is polynomial of degree ≤ 1 in both variables iff the classical commutator map $c\colon X \times X \to X$, $(x,y) \mapsto [x,y]$, is bi-additive, which indeed is a well-known characterization of the fact that X is 2-nilpotent.

Polynomial functors and nilpotency

Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is polynomial of degree $\leq n$. Then:

1. F takes values in the full subcategory $\mathrm{Nil}_n(\mathcal{D})$ of n-step nilpotent objects in $\mathcal{D}$.

2. If $\mathcal{C}$ is semi-abelian and F preserves coequalizers of reflexive parallel pairs then F factors through the n-step nilization functor $Nil_n\colon \mathcal{C} \longrightarrow \mathrm{Nil}_n(\mathcal{C})$. Hence F factors as

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Nil_n"] & \mathcal{D} \\ \mathrm{Nil}_n(\mathcal{C}) \arrow[r, dotted, "\overline{F}"] & \mathrm{Nil}_n(\mathcal{D}) \arrow[u, tail] \end{tikzcd}$$

Abbreviating "pre" for "pseudo-right exact" we obtain an equivalence of functor categories

$$Pol_{\leq n}(\mathcal{C},\mathcal{D})_{pre} \simeq Pol_{\leq n}(\mathrm{Nil}_n(\mathcal{C}),\mathrm{Nil}_n(\mathcal{D}))_{pre} .$$

Associated graded object of an N-series

Let $\mathcal{N}\colon X = N_1 \geq N_2 \geq \dots$ be an N-series of an object X of $\mathcal{D}$. Then each N_n is normal in X, and the quotient

$$\mathrm{Gr}_n^{\mathcal{N}}(X) = N_n/N_{n+1} \text{ is abelian },$$

so

$$\mathrm{Gr}^{\mathcal{N}}(X) = \bigoplus_{n\geq 1} \mathrm{Gr}_n^{\mathcal{N}}(X)$$

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QUESTION: Does $\mathrm{Gr}^{\mathcal{N}}(X)$ carry a natural global multilinear “multiplicative” structure relating its various components?

YES!!!!

MAIN THEOREM. Let $\mathcal{N}\colon X = N_1 \geq N_2 \geq \ldots$ be an N-series of an object X of $\mathcal{D}$. Then $\mathrm{Gr}^{\mathcal{N}}(X)$ has a natural structure of graded algebra over

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- an operad in abelian groups $\mathrm{AbOp}(\mathcal{D})$ if $\mathcal{D}$ is a (semi-abelian) algebraic category, that is the category of models of an algebraic theory in the sense of Lawvere; here

$$\mathrm{AbOp}(\mathcal{D})(n) = cr_n(U_{Ab} \circ \mathrm{Gr}_n^{\gamma} \circ L)(\underline{1}, \dots, \underline{1})$$

where $U_{Ab}\colon \mathrm{Ab}(\mathcal{D}) \to Ab$ is the forgetful functor and $L\colon FinSet \to \mathcal{D}$ is the functor assigning to the finite set $\underline{k} = \{1, \dots, k\} \cong \underline{1}^{+k}$ the canonical free object of rank k.

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3. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras then $\mathrm{AbOp}(\mathcal{D}) = \mathcal{P}$.

Work in progress

Let $\mathcal{D}$ be a semi-abelian algebraic category. A set operad $\mathcal{P}_{\mathcal{D}}$ is defined by $\mathcal{P}_{\mathcal{D}}(n) = \mathcal{D}(L(\underline{1}), L(\underline{n}))$ and operadic composition induced by composition in the category $\mathcal{D}$.

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For $\mathbb{K}$ a field of characteristic 0, the set operad $\mathcal{P}_{\mathcal{D}}$ gives rise to an operad in $\mathbb{K}$-vector spaces $\mathbb{K}[\mathcal{P}_{\mathcal{D}}]$, by taking $\mathbb{K}[\mathcal{P}_{\mathcal{D}}](n)$ to be the vector space with basis $\mathcal{P}_{\mathcal{D}}(n)$.

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For an object X of $\mathcal{C}$ the vector space $\mathbb{K}[|X|]$ with basis $|X|$ has the structure of an algebra over $\mathbb{K}[\mathcal{P}_\mathcal{D}]$; e.g. if $\mathcal{D} = Gr$, this is the structure of group algebra (including the antipode). Let $\mathbb{K}[X]$ be this "object algebra" of X.

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$\mathbb{K}[|X|]$ also is a cocommutative and coassociative coalgebra defined by putting $\Delta(x) = x \otimes x$ for $x \in |X|$.

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$\mathbb{K}[|X|]$ also is a cocommutative and coassociative coalgebra defined by putting $\Delta(x) = x \otimes x$ for $x \in |X|$.

Thus $\mathbb{K}[X]$ is a (generalized) bialgebra which we call the "object bialgebra" of X.

Conjectures

1. Primitives conjecture. The triple

$$\left(\mathbb{K}[\mathcal{P}_{\mathbf{D}}], \mathcal{C}om_{\mathbb{K}}, \mathsf{AbOp}(\mathcal{D}) \otimes \mathbb{K}\right)$$

is a good triple of operads in the sense of Loday.

This in particular means that the subspace of primitive elements of the graded object bialgebras $\mathsf{Gr}(\mathbb{K}[X])$ has a canonical structure of an algebra over $\mathsf{AbOp}(\mathcal{D}) \otimes \mathbb{K}$.

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2. Conjectural generalized Jennings theorem. The n-th dimension subobject $D_{n,\mathbb{K}}(X) = X \cap (1_X + I^n_{\mathbb{K}}(X))$ is

$$D_{n,\mathbb{K}}(X) = \sqrt{\gamma_n(X)},$$

for suitably (already) defined notions of augmentation filtration $I^n_{\mathbb{K}}(X)$ of $\mathbb{K}[X]$ and of isolator $\sqrt{S}$ of $S \leq X$.

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3. Conjectural generalized Quillen theorem. There is a natural isomorphism of graded $\mathbb{K}[\mathcal{P}_{\mathcal{D}}]$-algebras

$$\mathrm{Gr}(\mathbb{K}[X]) \;\cong\; \mathrm{U}(\mathrm{Gr}^{\gamma}(X) \otimes \mathbb{K}).$$

Conjectures-II

3. Conjectural generalized Lazard theorem.

Suppose that $\mathcal{D}$ is

- n-step nilpotent, which by definition means that the identity functor of $\mathcal{D}$ is polynomial of degree $\leq n$, or equivalently, that all objects of $\mathcal{D}$ are n-step nilpotent;

- n-radicable, which by definition means that the abelian group $\mathrm{End}_{\mathcal{D}}(L(\underline{1})^{ab})$ is a $\mathbb{Z}[\frac{1}{2}, \ldots, \frac{1}{n}]$-module.

Then there is a (canonical?) equivalence of categories

$$\mathcal{D} \;\simeq\; \mathrm{Alg}(\mathrm{AbOp}(\mathcal{D}))$$

This equivalence would also induce a generalized Baker-Campbell-Hausdorff formula (actually, one for each $n \geq 1$), expressing operations of arity n in $\mathcal{D}$ in terms of the operations given by the operad $\mathrm{AbOp}(\mathcal{D})$.

APPROACH

Use the theory of polynomial functors from $\mathcal{D}$ to Ab which encodes them by kind of a "DNA"; the latter involves intricate both algebraic and combinatorical structures (e.g. non-linear pseudo-Mackey functors).

So far, this program is completely achieved only for $n = 2$ and all $\mathcal{D}$ [H., Vespa; Defourneau];

for all $n \geq 2$, the necessary polynomial functor theory is achieved only for $\mathcal{D} = \mathit{Groups}$ and $\mathcal{D} = \mathit{Loops}$ (actually, also for $\mathcal{D} =$ *free finitely generated free algebras over a set-operad*) [H., Pirashvili, Vespa].