Higher Whitehead product: computations and applications

Marek Golasiński
Toruń (Poland)

Combinatorial and Toric Homotopy
August 23–31, 2015
Singapore

All the best to you Fred,
for many years of
mathematics to come.

J. H. C. Whitehead and a friend at his home.
Manor Farm, Noke.

Joint work:
Thiago de Melo, Rio Claro–SP (Brazil)

The aim of the talk is to present:

1. Properties and computations of higher order Whitehead products;

The aim of the talk is to present:

1. Properties and computations of higher order Whitehead products;
2. Applications of higher order Whitehead products.

Prerequisites

Recall that given $f \in \pi_k(X)$, $g \in \pi_l(X)$, the Whitehead product $[f, g] \in \pi_{k+l-1}(X)$ is defined as follows: the product $\mathbb{S}^k \times \mathbb{S}^l$ of spheres can be obtained by attaching a $(k+l)$-cell to the wedge sum $\mathbb{S}^k \vee \mathbb{S}^l$:

$$\begin{tikzcd}
\mathbb{S}^{k+l-1} \arrow[r, "\omega"] \arrow[d, hook] & \mathbb{S}^k \vee \mathbb{S}^l \arrow[d, hook] \\
\mathbb{D}^{k+l} \arrow[r] & \mathbb{S}^k \times \mathbb{S}^l.
\end{tikzcd}$$

Then, the compose

$$\mathbb{S}^{k+l-1} \xrightarrow{\omega} \mathbb{S}^k \vee \mathbb{S}^l \xrightarrow{f \vee g} X$$

represents $[f, g] \in \pi_{k+l-1}(X)$.

Given maps $f : \Sigma A \to X$ and $g : \Sigma B \to X$, Arkowitz defined the generalized Whitehead product $[f, g] \in \pi(\Sigma(A \wedge B), X)$ as the compose

$$\Sigma(A \wedge B) \xrightarrow{\omega} \Sigma A \vee \Sigma B \xrightarrow{f \vee g} X,$$

Given maps $f : \Sigma A \to X$ and $g : \Sigma B \to X$, Arkowitz defined the generalized Whitehead product $[f, g] \in \pi(\Sigma(A \wedge B), X)$ as the compose

$$\Sigma(A \wedge B) \xrightarrow{\omega} \Sigma A \vee \Sigma B \xrightarrow{f \vee g} X,$$

where $\omega : \Sigma(A \wedge B) \to \Sigma A \vee \Sigma B$ (the Whitehead map) is given by the pushout:

$$\begin{tikzcd}
\Sigma(A \wedge B) \arrow[r, "\omega"] \arrow[d, hook] & \Sigma A \vee \Sigma B \arrow[d, hook] \\
C\Sigma(A \wedge B) \arrow[r] & \Sigma A \times \Sigma B.
\end{tikzcd}$$

For the category $\mathcal{CO}$ of simply-connected co-H-spaces, Gray defined a functor (called the Theriault product):

$$\circ : \mathcal{CO} \times \mathcal{CO} \longrightarrow \mathcal{CO}$$

and a natural transformation $w : A \circ B \longrightarrow A \vee B$ with a pushout:

$$\begin{tikzcd}
A \circ B \arrow[r, "w"] \arrow[d, hook] & A \vee B \arrow[d, hook] \\
C(A \circ B) \arrow[r] & A \times B.
\end{tikzcd}$$

Given simply connected co-H-spaces A, B, by a result of Gray, there are maps

$$A \circ B \xrightarrow{\zeta} \Sigma(\Omega A \wedge \Omega B) \xrightarrow{\kappa} A \circ B$$

with $\kappa\zeta = \mathrm{id}_{A \circ B}$ and the homotopy fibration

$$\Sigma(\Omega A \wedge \Omega B) \xrightarrow{w'} A \vee B \to A \times B$$

determines a natural transformation

$$w : A \circ B \xrightarrow{\zeta} \Sigma(\Omega A \wedge \Omega B) \xrightarrow{w'} A \vee B$$

generalizing the Whitehead product map.

Armed with this construction, given maps $f : A \to X$ and $g : B \to X$, the generalized Whitehead product $[f, g] \in \pi(A \circ B, X)$ is the compose

$$A \circ B \xrightarrow{w} A \vee B \xrightarrow{f \vee g} X,$$

generalizing the Whitehead map.

Armed with this construction, given maps $f : A \to X$ and $g : B \to X$, the generalized Whitehead product $[f, g] \in \pi(A \circ B, X)$ is the compose

$$A \circ B \xrightarrow{w} A \vee B \xrightarrow{f \vee g} X,$$

generalizing the Whitehead map.
The existence of $A \circ B$ generalizes a result of Theriault who showed that the smash product of two simply-connected co-associative co-H-spaces is the suspension of a co-H-space.

Given maps maps $f_i : \mathbb{S}^{m_i} \to X$ for $i = 1, \ldots, n$ with $n \geq 2$, Hardie follows Zeeman for $n = 3$ and deals with the n^{th} order spherical Whitehead product

$$[f_1, \ldots, f_n] \subseteq \pi_{m_1+\cdots+m_n-1}(X)$$

for $n \geq 3$ as follows:

Given maps maps $f_i : \mathbb{S}^{m_i} \to X$ for $i = 1, \ldots, n$ with $n \geq 2$, Hardie follows Zeeman for $n = 3$ and deals with the n^{th} order spherical Whitehead product

$$[f_1, \ldots, f_n] \subseteq \pi_{m_1+\cdots+m_n-1}(X)$$

for $n \geq 3$ as follows: the characteristic map

$$\omega_n : \mathbb{S}^{m_1+\cdots+m_n-1} \to T_1(\mathbb{S}^{m_1}, \ldots, \mathbb{S}^{m_n})$$

for attaching the top cell to the fat wedge $T_1(\mathbb{S}^{m_1}, \ldots, \mathbb{S}^{m_n})$

$$\begin{tikzcd}
\mathbb{S}^{m_1+\cdots+m_n-1} \arrow[r, "\omega_n"] \arrow[d, hook] & T_1(\mathbb{S}^{m_1},\ldots,\mathbb{S}^{m_n}) \arrow[d, hook] \\
\mathbb{D}^{m_1+\cdots+m_n} \arrow[r] & \mathbb{S}^{m_1}\times\cdots\times\mathbb{S}^{m_n}
\end{tikzcd}$$

leads the compose

$$\mathbb{S}^{m_1+\cdots+m_n-1} \xrightarrow{\omega_n} T_1(\mathbb{S}^{m_1},\ldots,\mathbb{S}^{m_n}) \xrightarrow{F} X$$

which is an element of $[f_1,\ldots,f_n]$ provided

$$F : T_1(\mathbb{S}^{m_1},\ldots,\mathbb{S}^{m_n}) \longrightarrow X$$

is an extension of $f_1\vee\cdots\vee f_n : \mathbb{S}^{m_1}\vee\cdots\vee\mathbb{S}^{m_n}\to X$.

- Porter has generalized the Hardie's construction and introduced the notion of the n^{th} order generalized Whitehead product $[f_1, \ldots, f_n]$ of maps $f_i : \Sigma A_i \to X$ for $i = 1, \ldots, n$ with $n \geq 2$.

- Porter has generalized the Hardie's construction and introduced the notion of the n^{th} order generalized Whitehead product $[f_1, \ldots, f_n]$ of maps $f_i : \Sigma A_i \to X$ for $i = 1, \ldots, n$ with $n \geq 2$.
- If $n = 2$ and A_1, A_2 are spheres it coincides with the classical Whitehead product.

- Porter has generalized the Hardie's construction and introduced the notion of the n^{th} order generalized Whitehead product $[f_1, \ldots, f_n]$ of maps $f_i : \Sigma A_i \to X$ for $i = 1, \ldots, n$ with $n \geq 2$.
- If $n = 2$ and A_1, A_2 are spheres it coincides with the classical Whitehead product.
- If A_1, A_2 are any spaces then the 2^{nd} order Whitehead product coincides with the generalized Whitehead product studied by Arkowitz.

More precisely, the pushout (up to homotopy)

$$\begin{tikzcd}
\Sigma^{n-1}A_1 \wedge \cdots \wedge A_n \arrow[r, "\omega_n"] \arrow[d, hook] & T_1(\Sigma A_1, \ldots, \Sigma A_n) \arrow[d, hook] \\
C(\Sigma^{n-1}A_1 \wedge \cdots \wedge A_n) \arrow[r] & \Sigma A_1 \times \cdots \times \Sigma A_n
\end{tikzcd}$$

leads the compose

$$\Sigma^{n-1}A_1 \wedge \cdots A_n \xrightarrow{\omega_n} T_1(\Sigma A_1, \ldots, \Sigma A_n) \xrightarrow{F} X$$

which is an element of $[f_1, \ldots, f_n]$ provided

$$F : T_1(\Sigma A_1, \ldots, \Sigma A_n) \longrightarrow X$$

is an extension of $f_1 \vee \cdots \vee f_n : \Sigma A_1 \vee \cdots \vee \Sigma A_n \to X$.

In virtue of Porter, it is non-empty if and only if all the lower products $[f_{k_1}, \dots, f_{k_m}]$ for $1 \leq k_1 \leq \cdots \leq k_m \leq n$ with $m = 2, \dots, n-1$ contain the zero element 0.

In virtue of Porter, it is non-empty if and only if all the lower products $[f_{k_1}, \dots, f_{k_m}]$ for $1 \leq k_1 \leq \cdots \leq k_m \leq n$ with $m = 2, \dots, n-1$ contain the zero element 0.

We note that the set $[f_1, \dots, f_n]$ can be empty even if $f_i = 0$ for some i.

This is the case for $[0, \iota_2, \iota_2]$, since the classical Whitehead product $[\iota_2, \iota_2] = 2\eta_2 \neq 0$.

Theorem (Porter)

Let $f_i : \Sigma A_i \to \Sigma B_i$, $g_i : \Sigma B_i \to X$ *and* $k : X \to Y$ *be maps for* $i = 1, \ldots, n$. *Then:*

Theorem (Porter)

Let $f_i : \Sigma A_i \to \Sigma B_i$, $g_i : \Sigma B_i \to X$ *and* $k : X \to Y$ *be maps for* $i = 1, \ldots, n$. *Then:*

1. $k_*[g_1, \ldots, g_n] \subseteq [kg_1, \ldots, kg_n]$;

Theorem (Porter)

Let $f_i : \Sigma A_i \to \Sigma B_i$, $g_i : \Sigma B_i \to X$ and $k : X \to Y$ be maps for $i = 1, \ldots, n$. Then:

1. $k_*[g_1, \ldots, g_n] \subseteq [kg_1, \ldots, kg_n]$;
2. *If X is an H-space then* $[f_1, \ldots, f_n] = 0$;

Theorem (Porter)

Let $f_i : \Sigma A_i \to \Sigma B_i$, $g_i : \Sigma B_i \to X$ and $k : X \to Y$ be maps for $i = 1, \ldots, n$. Then:

1. $k_*[g_1, \ldots, g_n] \subseteq [kg_1, \ldots, kg_n]$;
2. *If X is an H-space then* $[f_1, \ldots, f_n] = 0$;
3. $0 \in [f_1, \ldots, f_r]$ *if and only if* $f_1 \vee \cdots \vee f_r : \Sigma A_1 \vee \cdots \vee \Sigma A_n \to X$ *has an extension to* $\Sigma A_1 \times \cdots \times \Sigma A_n$.

Theorem (Porter)

Let $f_i : \Sigma A_i \to \Sigma B_i$, $g_i : \Sigma B_i \to X$ and $k : X \to Y$ be maps for $i = 1, \ldots, n$. Then:

1. $k_*[g_1, \ldots, g_n] \subseteq [kg_1, \ldots, kg_n]$;
2. *If X is an H-space then* $[f_1, \ldots, f_n] = 0$;
3. $0 \in [f_1, \ldots, f_r]$ *if and only if* $f_1 \vee \cdots \vee f_r : \Sigma A_1 \vee \cdots \vee \Sigma A_n \to X$ *has an extension to* $\Sigma A_1 \times \cdots \times \Sigma A_n$.

Corollary.

If $[f_1, \ldots, f_r] \neq \emptyset$ *and* $f_{i_0} = 0_{i_0}$ *for some* $1 \leq i_0 \leq n$ *then* $0 \in [f_1, \ldots, f_n]$.

Referring to Williams, we say that a space X has property P_n if for every $f_i : \Sigma A_i \to X$ with $i = 1, \ldots, n$, we have $0 \in [f_1, \ldots, f_n]$.

Certainly, in view of the theorem above, any H-space posses not only property P_n for all $r \geq 2$ but 0 is the only element of $[f_1, \ldots, f_n]$.

(Williams has asked: *We note at this point that it is unresolved conjecture as to whether X has property P_n implies that 0 is the only element of $[f_1, \ldots, f_n]$.*)

Remark. The higher Whitehead product $[f_1, \dots, f_n]$ might be a singleton and non-zero.

Namely, consider the inclusion map $J_n : \mathbb{S}^2 \hookrightarrow \mathbb{C}P^n$ into the complex projective n-space for $n \geq 1$. Then, (Arkowitz+Porter)

$$[j_n, \overset{\times n+1}{\dots}, j_n] = (n+1)!\gamma_n$$

for the projection $\gamma_n : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$.

Remark. The higher Whitehead product $[f_1, \ldots, f_n]$ might be a singleton and non-zero.

Namely, consider the inclusion map $J_n : \mathbb{S}^2 \hookrightarrow \mathbb{C}P^n$ into the complex projective n-space for $n \geq 1$. Then, (Arkowitz+Porter)

$$[j_n, \overset{\times n+1}{\cdots}, j_n] = (n+1)!\gamma_n$$

for the projection $\gamma_n : \mathbb{S}^{2n+1} \to \mathbb{C}P^n$.

In particular, we get the classical homotopy theory result $[\iota_2, \iota_2] = 2\eta_2$ for $\iota_2 = \mathrm{id}_{\mathbb{S}^2}$ and $\eta_2 : \mathbb{S}^3 \to \mathbb{S}^2$ the Hopf fibration.

We make use of the Theriault product to define the higher order Gray–Whitehead product for co-H-spaces $(A_1, \ldots, A_n) = \underline{A}$.

First, applying $\Sigma(A_1 \circ A_2) = A_1 \wedge A_2$ (Gray), the inductive procedure shows that

$$(A_1 \circ A_2) \wedge A_3 \wedge \cdots \wedge A_n = \Sigma^{n-2}(A_1 \circ \cdots \circ A_n).$$

The Porter's homotopy fibration

$$\Sigma^{n-1}\Lambda\Omega(\underline{A}) \to T_1(\underline{A}) \to A_1 \times \cdots \times A_n$$

and coretractions $\nu_i : A_i \to \Sigma\Omega A_i$ for $i = 1, \ldots, n$ yields the generalized Gray–Whitehead map

$$w_n : \Sigma^{n-2}(A_1 \circ \cdots \circ A_n) \longrightarrow T_1(A_1, \ldots, A_n)$$

with a pushout (up to homotopy)

$$\begin{tikzcd}
\Sigma^{n-2}(A_1 \circ \cdots \circ A_n) \arrow[r, "w_n"] \arrow[d, hook] & T_1(A_1, \ldots, A_n) \arrow[d, hook] \\
C\Sigma^{n-1}(A_1 \circ \cdots \circ A_n) \arrow[r] & A_1 \times \cdots \times A_n.
\end{tikzcd}$$

Computations

Given $f_i : \mathbb{S}^{n_i} \to X$ for $i = 1, 2, 3$, Hardie has proved that:

Computations

Given $f_i : \mathbb{S}^{n_i} \to X$ for $i = 1, 2, 3$, Hardie has proved that:

Theorem

If the triple Whitehead product $[f_1, f_2, f_3]$ is non-empty then it is a coset of the subgroup

$$J(f_1, f_2, f_3) = [\pi_{n_2+n_3}(X), f_1] + [\pi_{n_1+n_3}(X), f_2] + [\pi_{n_1+n_2}(X), f_3]$$

of $\pi_{n_1+n_2+n_3-1}(X)$.

Computations

Given $f_i : \mathbb{S}^{n_i} \to X$ for $i = 1, 2, 3$, Hardie has proved that:

Theorem

If the triple Whitehead product $[f_1, f_2, f_3]$ is non-empty then it is a coset of the subgroup

$$J(f_1, f_2, f_3) = [\pi_{n_2+n_3}(X), f_1] + [\pi_{n_1+n_3}(X), f_2] + [\pi_{n_1+n_2}(X), f_3]$$

of $\pi_{n_1+n_2+n_3-1}(X)$.

Notice that this result is "in the spirit" of the Toda brackets.

A *trivial* triple spherical Whitehead product means that $[f_1, f_2, f_3] = J(f_1, f_2, f_3)$ or equivalently, $0 \in [f_0, f_2, f_3]$.

Hardie has stated:

A *trivial* triple spherical Whitehead product means that $[f_1, f_2, f_3] = J(f_1, f_2, f_3)$ or equivalently, $0 \in [f_0, f_2, f_3]$.

Hardie has stated:

Problem.

We do not know of a case when the triple (spherical) product $[f_1, f_2, f_3]$ is non-trivial for a sphere, and for $X = \mathbb{S}^4$, the triple product $[\eta_4, \eta_4^2, 2\iota_4] \subseteq \pi_{14}(\mathbb{S}^4)$ is possibly non-trivial.

We have shown:

We have shown:

Proposition.

The groups $[\pi_9(\mathbb{S}^4), \eta_4^2]$ *and* $[\pi_{10}(\mathbb{S}^4), \eta_4]$ *are trivial. In particular,*

$$J(\eta_4, \eta_4^2, 2\iota_4) = [\pi_{11}(\mathbb{S}^4), 2\iota_4]$$

and it is a subgroup of $\pi_{14}(\mathbb{S}^4)$ *with order fifteen. In addition, the triple product* $[\eta_4, \eta_4^2, \iota_4]$ *is trivial.*

It follows from the Hardie's theorem above that the triple Whitehead product $[f_1, f_2, f_3]$ is an empty or a single set element provided $f_i : \mathbb{S}^{n_i} \to X$ for $i = 1, 2, 3$ with $X = \mathbb{S}^2$ or $\mathbb{R}P^2$, the real projective plane. More, generally:

It follows from the Hardie's theorem above that the triple Whitehead product $[f_1, f_2, f_3]$ is an empty or a single set element provided $f_i : \mathbb{S}^{n_i} \to X$ for $i = 1, 2, 3$ with $X = \mathbb{S}^2$ or $\mathbb{R}P^2$, the real projective plane. More, generally:

Theorem (Baues)

If $f_i : \mathbb{S}^{m_i} \to \mathbb{S}^2$ for $i = 1, \dots, n$ with $n \geq 2$ then the higher Whitehead product $[f_1, \dots, f_n] = 0$ provided $m_1 + \cdots + m_n \neq 4$.

It follows from the Hardie's theorem above that the triple Whitehead product $[f_1, f_2, f_3]$ is an empty or a single set element provided $f_i : \mathbb{S}^{n_i} \to X$ for $i = 1, 2, 3$ with $X = \mathbb{S}^2$ or $\mathbb{R}P^2$, the real projective plane. More, generally:

Theorem (Baues)

If $f_i : \mathbb{S}^{m_i} \to \mathbb{S}^2$ for $i = 1, \ldots, n$ with $n \geq 2$ then the higher Whitehead product $[f_1, \ldots, f_n] = 0$ provided $m_1 + \cdots + m_n \neq 4$.

Notice that a similar result might be stated for $\mathbb{R}P^2$ as well.

Let now $f_i : \Sigma A_i \to X$ for $i = 1, 2, 3$. Then, the Hardie's result above we have generalized as follows:

Let now $f_i : \Sigma A_i \to X$ for $i = 1, 2, 3$. Then, the Hardie's result above we have generalized as follows:

Theorem

If the triple Whitehead product $[f_1, f_2, f_3]$ is non-empty then it is a coset of the subgroup

$$J(f_1, f_2, f_3) = [\pi(\Sigma^2(A_2 \wedge A_3), X), f_1] + [\pi(\Sigma^2(A_1 \wedge A_3), X), f_2] + [\pi(\Sigma^2(A_1 \wedge A_2), X), f_3]$$

of $\pi(\Sigma^2(A_1 \wedge A_2 \wedge A_3), X)$.

Let now $f_i : \Sigma A_i \to X$ for $i = 1, 2, 3$. Then, the Hardie's result above we have generalized as follows:

Theorem

If the triple Whitehead product $[f_1, f_2, f_3]$ is non-empty then it is a coset of the subgroup

$$J(f_1, f_2, f_3) = [\pi(\Sigma^2(A_2 \wedge A_3), X), f_1] + [\pi(\Sigma^2(A_1 \wedge A_3), X), f_2] + [\pi(\Sigma^2(A_1 \wedge A_2), X), f_3]$$

of $\pi(\Sigma^2(A_1 \wedge A_2 \wedge A_3), X)$.

It follows $[0_1, 0_2, 0_3] = 0$ for the trivial maps $0_i : \Sigma A_i \to X$ with $i = 1, 2, 3$.

Question. What about $[0_1, \dots, 0_n] = 0$ for $n \geq 4$?
We have shown:

Question. What about $[0_1, \ldots, 0_n] = 0$ for $n \geq 4$?
We have shown:

Proposition

If $n \geq 4$ and $\underline{\mathbb{S}} = (\mathbb{S}^{m_1}, \ldots, \mathbb{S}^{m_n})$ with $m_i \geq 1$ for $i = 1, \ldots, n$ then the quotient map

$$T_1\Sigma(\underline{\mathbb{S}}) \to T_1\Sigma(\underline{\mathbb{S}}) \,/\, \mathbb{S}^{m_1} \vee \cdots \vee \mathbb{S}^{m_n}$$

leads to a non-trivial element of the n^{th} order generalized Whitehead product $[0_1, \ldots, 0_n]$.

Remark. We are deeply grateful to Jie Wu for his idea on the proof.

Applications

(I) Given a simplicial complex K on n vertices, Davis and Januszkiewicz associated two fundamental objects of toric topology: the moment–angle complex $\mathcal{Z}_K$ and the Davis–Januszkiewicz space DJ_K.

Applications

(I) Given a simplicial complex K on n vertices, Davis and Januszkiewicz associated two fundamental objects of toric topology: the moment–angle complex $\mathcal{Z}_K$ and the Davis–Januszkiewicz space DJ_K.

The homotopy fibration sequence

$$\mathcal{Z}_K \xrightarrow{\tilde{\omega}} DJ_K \to \prod_{i=1}^{n} \mathbb{C}P^\infty$$

and its generalization has been studied by J. Grbić and S. Theriault, and K. Iriye and D. Kishimoto, respectively. It was shown that $\tilde{\omega} : \mathcal{Z}_K \to DJ_K$ is a sum of higher and iterated Whitehead products for appropriate complexes K.

Applications

(II) The homotopy type of the Euclidean ordered configuration space $\mathcal{F}(\mathbb{R}^{n+1}, m)$, for $n \geq 2$, admits a minimal cellular model

$$* = X_0 \subseteq X_n \subseteq X_{2n} \subseteq \cdots \subseteq X_{mn}$$

whose cells are attached via higher order Whitehead products.

Applications

(II) The homotopy type of the Euclidean ordered configuration space $\mathcal{F}(\mathbb{R}^{n+1}, m)$, for $n \geq 2$, admits a minimal cellular model

$$* = X_0 \subseteq X_n \subseteq X_{2n} \subseteq \cdots \subseteq X_{mn}$$

whose cells are attached via higher order Whitehead products.

More precisely, let $\underline{\mathbb{S}}^n_k = (\mathbb{S}^n, \overset{\times k}{\ldots}, \mathbb{S}^n)$, for $k = 1, \ldots, m$.

Theorem (P. Salvatore)

The pushout of the diagram

$$\begin{tikzcd} T_1(\underline{\mathbb{S}}^n_k) \arrow[r, "\nu_k"] \arrow[d, hook] & X_{(k-1)n} \\ \mathbb{S}^n \times \overset{k}{\cdots} \times \mathbb{S}^n & \end{tikzcd}$$

for some map $\nu_k : T_1(\underline{\mathbb{S}}^n_k) \to X_{(k-1)n}$ *is obtained by attaching a kn-cell to* $X_{(k-1)n}$.

Consequently, *kn*-cells are attached to $X_{(k-1)n}$ via elements of higher order Whitehead products.

Applications

(III) Recall that the *exterior Whitehead product* $\{\alpha_1, \ldots, \alpha_n\}$ of maps $\alpha_i : \Sigma A_i \to X_i$ for $i = 1, \ldots, n$ with $n \geq 2$ is the composition

$$T_1(\underline{\alpha})\omega_n : \Sigma^{n-1}(A_1 \wedge \cdots \wedge A_n) \to T_1(X_1, \ldots, X_n).$$

Applications

(III) Recall that the *exterior Whitehead product* $\{\alpha_1, \ldots, \alpha_n\}$ of maps $\alpha_i : \Sigma A_i \to X_i$ for $i = 1, \ldots, n$ with $n \geq 2$ is the composition

$$T_1(\underline{\alpha})\omega_n : \Sigma^{n-1}(A_1 \wedge \cdots \wedge A_n) \to T_1(X_1, \ldots, X_n).$$

If $A_i = \mathbb{S}^{m_i}$ is the m_i-sphere with $m_i \geq 1$ for $i = 1, \ldots, n$ then $\{\alpha_1, \ldots, \alpha_n\}$ has been defined by Hardie.

We refer to such a product as the *spherical* one.

Now, let $J_n(X)$ be the n^{th}-stage of the James construction. Given $m_i \geq 1$ for $i = 1, \ldots, n$ with $n \geq 2$, write $m' = m_1 + \cdots + m_n$ and $m'' = m' - \min\limits_{1 \leq i \leq n} \{m_i\}$.

Now, let $J_n(X)$ be the n^{th}-stage of the James construction. Given $m_i \geq 1$ for $i = 1, \ldots, n$ with $n \geq 2$, write $m' = m_1 + \cdots + m_n$ and $m'' = m' - \min_{1 \leq i \leq n} \{m_i\}$.

Next, consider the restriction

$$\mu_{\mathbf{m}}(X)_| : T_1(J_{m_1}(X), \ldots, J_{m_n}(X)) \to J_{m''}(X)$$

of the canonical multiplication

$$\mu_{\mathbf{m}}(X) : J_{m_1}(X) \times \cdots \times J_{m_n}(X) \to J_{m'}(X).$$

The *interior Whitehead product* $\langle \alpha_1, \ldots, \alpha_n \rangle$ of maps $\alpha_i : \Sigma A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$ is the composition

$$\{\alpha_1, \ldots, \alpha_n\}\mu_{\mathbf{m}}(X)_|$$

and $\langle \alpha_1, \ldots, \alpha_n \rangle \in \pi(\Sigma^{n-1}(A_1 \wedge \cdots \wedge A_n), J_{m''}(X))$.

In particular, for $\alpha_i : \Sigma A_i \to X$ with $i = 1, \ldots, n$, we have $\langle \alpha_1, \ldots, \alpha_n \rangle \in \pi(\Sigma^{n-1}(A_1 \wedge \cdots \wedge A_n), J_{n-1}(X))$.

In particular, for $\alpha_i : \Sigma A_i \to X$ with $i = 1, \ldots, n$, we have $\langle \alpha_1, \ldots, \alpha_n \rangle \in \pi(\Sigma^{n-1}(A_1 \wedge \cdots \wedge A_n), J_{n-1}(X))$.

Denote by $\mathrm{id}_{\Sigma A} : \Sigma A \to J_1(\Sigma A)$ the identity map. Then, we get (as it was shown by Jie Wu) a pushout (up to homotopy)

$$\begin{tikzcd}
\Sigma^{n-1}(A \wedge \cdots \wedge A) \arrow[r, "\langle \mathrm{id}_{\Sigma A}, \ldots, \mathrm{id}_{\Sigma A} \rangle"] \arrow[d, hook] & J_{n-1}(\Sigma A) \arrow[d, hook] \\
C\Sigma^{n-1}(A \wedge \cdots \wedge A) \arrow[r] & J_n(\Sigma A).
\end{tikzcd}$$

Let $SP_n(X)$ be the n^{th} symmetric power on X, write $q_n : J_n(X) \to SP^n(X)$ for the quotient map and

$$\langle \mathrm{id}_{\Sigma A}, \ldots, \mathrm{id}_{\Sigma A} \rangle' = q_n \langle \mathrm{id}_{\Sigma A}, \ldots, \mathrm{id}_{\Sigma A} \rangle.$$

Let $SP_n(X)$ be the n^{th} symmetric power on X, write $q_n : J_n(X) \to SP^n(X)$ for the quotient map and

$$\langle \mathrm{id}_{\Sigma A}, \ldots, \mathrm{id}_{\Sigma A} \rangle' = q_n \langle \mathrm{id}_{\Sigma A}, \ldots, \mathrm{id}_{\Sigma A} \rangle.$$

Thus, the diagram

$$\begin{tikzcd}
\Sigma^{n-1}(A \wedge \cdots \wedge A) \arrow[r, "\langle \mathrm{id}_{\Sigma A}, \ldots, \mathrm{id}_{\Sigma A} \rangle'"] \arrow[d, hook] & SP_{n-1}(\Sigma A) \arrow[d, hook] \\
C\Sigma^{n-1}(A \wedge \cdots \wedge A) \arrow[r] & SP_n(\Sigma A)
\end{tikzcd}$$

is a pushout (up to homotopy).

Next, write $\iota_n = \mathrm{id}_{\mathbb{S}^n}$, $\iota_{m,n} : \mathbb{S}^n \hookrightarrow J_m(\mathbb{S}^n)$ for the canonical inclusion map and $\eta_n \in \pi_{n+1}(\mathbb{S}^n)$ for generators with $n \geq 2$.

Next, write $\iota_n = \mathrm{id}_{\mathbb{S}^n}$, $\iota_{m,n} : \mathbb{S}^n \hookrightarrow J_m(\mathbb{S}^n)$ for the canonical inclusion map and $\eta_n \in \pi_{n+1}(\mathbb{S}^n)$ for generators with $n \geq 2$.

Proposition. (Hardie+Shar)

1. The element $\langle \iota_n, \overset{\times m}{\ldots}, \iota_n \rangle$ is of infinite order provided n is odd and $m \neq 2$ or n is even;

Next, write $\iota_n = \mathrm{id}_{\mathbb{S}^n}$, $\iota_{m,n} : \mathbb{S}^n \hookrightarrow J_m(\mathbb{S}^n)$ for the canonical inclusion map and $\eta_n \in \pi_{n+1}(\mathbb{S}^n)$ for generators with $n \geq 2$.

Proposition. (Hardie+Shar)

1. The element $\langle \iota_n, \overset{\times m}{\ldots}, \iota_n \rangle$ is of infinite order provided n is odd and $m \neq 2$ or n is even;
2. $\pi_{mn-1}(J_{m-1}(\mathbb{S}^n)) \approx \mathbb{Z} \oplus \pi_{mn}(\mathbb{S}^{n+1})$ and $\langle \iota_n, \overset{\times m}{\ldots}, \iota_n \rangle$ is a generator of the infinite cyclic group;

Next, write $\iota_n = \mathrm{id}_{\mathbb{S}^n}$, $\iota_{m,n} : \mathbb{S}^n \hookrightarrow J_m(\mathbb{S}^n)$ for the canonical inclusion map and $\eta_n \in \pi_{n+1}(\mathbb{S}^n)$ for generators with $n \geq 2$.

Proposition. (Hardie+Shar)

1. The element $\langle \iota_n, \overset{\times m}{\ldots}, \iota_n \rangle$ is of infinite order provided n is odd and $m \neq 2$ or n is even;
2. $\pi_{mn-1}(J_{m-1}(\mathbb{S}^n)) \approx \mathbb{Z} \oplus \pi_{mn}(\mathbb{S}^{n+1})$ and $\langle \iota_n, \overset{\times m}{\ldots}, \iota_n \rangle$ is a generator of the infinite cyclic group;
3. $[\iota_{m-2,n}, \langle \iota_n, \overset{\times (m-1)}{\ldots}, \iota_n \rangle] = 0$ if and only if $n = 2$ and m is an odd prime; this element has order m otherwise.

Given $\alpha_i : \Sigma A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$, we say that $F : \Sigma A_1 \times \cdots \times \Sigma A_n \to J(X)$ is *strongly of type* $(\alpha_1, \ldots, \alpha_n)^k$ if its image is contained in $J_k(X)$ and coincides on $T_1(\Sigma A_1, \ldots, \Sigma A_n)$ with $F' = \mu_{\mathbf{m}}(X)(\alpha_1 \times \cdots \times \alpha_n)$.

Given $\alpha_i : \Sigma A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$, we say that $F : \Sigma A_1 \times \cdots \times \Sigma A_n \to J(X)$ is *strongly of type* $(\alpha_1, \ldots, \alpha_n)^k$ if its image is contained in $J_k(X)$ and coincides on $T_1(\Sigma A_1, \ldots, \Sigma A_n)$ with $F' = \mu_{\mathbf{m}}(X)(\alpha_1 \times \cdots \times \alpha_n)$.

For $F : \Sigma A_1 \times \cdots \times \Sigma A_n \to J(X)$, the generalized Hopf construction leads to a map $c(F) : \Sigma^{n+1}(A_1 \wedge \cdots \wedge A_n) \to \Sigma X$.

Given $\alpha_i : \Sigma A_i \to J_{m_i}(X)$ for $i = 1, \ldots, n$, we say that $F : \Sigma A_1 \times \cdots \times \Sigma A_n \to J(X)$ is *strongly of type* $(\alpha_1, \ldots, \alpha_n)^k$ if its image is contained in $J_k(X)$ and coincides on $T_1(\Sigma A_1, \ldots, \Sigma A_n)$ with $F' = \mu_{\mathbf{m}}(X)(\alpha_1 \times \cdots \times \alpha_n)$.

For $F : \Sigma A_1 \times \cdots \times \Sigma A_n \to J(X)$, the generalized Hopf construction leads to a map $c(F) : \Sigma^{n+1}(A_1 \wedge \cdots \wedge A_n) \to \Sigma X$.

The proposition above implies the existence of a map F strongly of type

$$\left(\iota_{m-2,2}, \langle \iota_2, \overset{\times(m-1)}{\ldots\ldots}, \iota_2 \rangle\right)^{m-2}$$

for an odd prime m which yields in view of Hardie an element $c(F)$ of order m in $\pi_{2m}(\mathbb{S}^3)$.

Great thanks for your kind attention!