Free Loop Cohomology of Complete Flag Manifolds

Matt Burfitt University of Southampton

Young Topologist Seminar

August 13th, 2015

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Recall that a Lie group is a space with a group structure where inversion and group multiplication are smooth.

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Recall that a Lie group is a space with a group structure where inversion and group multiplication are smooth.

Three classes of classical compact, connected, non-abelian Lie groups are given by

$$SO(n) = \{A \in M_n(\mathbb{R}) \mid A^{\mathsf{T}}A = I_n, \ det(A) = 1\},\$$

$$SU(n) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^{\mathsf{T}}A = I_n, \ det(A) = 1\},\$$

$$Sp(n-1) = \{A \in M_n(\mathbb{H}) \mid \bar{A}^{\mathsf{T}}A = I_{n-1}, \},\$$

for each $n \ge 2$.

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for each $n \ge 2$.

Theorem (Classification of abelian Lie group)

A connected abelian Lie group is isomorphic to $T^{\alpha} \times \mathbb{R}^{\beta}$ for some α, β .

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Homogeneous Spaces

Definition

A manifold M is called a homogeneous space if there is a Lie group G which acts transitively on it.

Under weak assumptions this means that M is diffeomorphic to G/H for some closed subgroup H of G.



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Examples

Many common space are homogeneous spaces e.g.

- **1** Spheres S^n ,
- **2** Projective space $\mathbb{R}P^n$ or $\mathbb{C}P^n$,
- **3** Grassmannians Gr(r, n).



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Proposition

When H is a closed connected subgroup of compact Lie group G we have a fibration

$$H \hookrightarrow G \to G/H$$
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Definition

Subgroup $T \subseteq G$ with $T \cong T^n$ is a maximal torus, if any $T' \supseteq T$ with $T' \cong T^m$ $\implies T' = T$.

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Definition

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For each Lie group G and maximal torus T, G/T is a homogeneous space called a *complete flag manifold*.

Proposition

Maximal tori in G are conjugate. G is covered by maximal tori.



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Examples

For $\alpha_i \in \mathbb{R}$ A maximal torus in SU(n) is given by elements of the form

$$\begin{bmatrix} e^{2\pi\alpha_1 i} & 0 \\ \vdots \\ 0 & e^{2\pi\alpha_n i} \end{bmatrix}$$

such that $e^{2\pi\alpha_1 i} \cdots e^{2\pi\alpha_n i} = 1$. Without this condition this is a maxima torus for Sp(n).

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Examples

Maximal tori in SO(n) and SO(n+1) are given by elements of the form

 $cos(heta_1) - sin(heta_1) \ sin(heta_1) \ cos(heta_1)$ $\cos(heta_{rac{n}{2}}) - \sin(heta_{rac{n}{2}}) \ \sin(heta_{rac{n}{2}}) - \cos(heta_{rac{n}{2}})$ $cos(\theta_1) - sin(\theta_1)$ $sin(\theta_1) cos(\theta_1)$ $\begin{array}{c} \cos(\theta_{\frac{n}{2}}) & -\sin(\theta_{\frac{n}{2}}) \\ \sin(\theta_{\frac{n}{2}}) & \cos(\theta_{\frac{n}{2}}) \end{array}$

for
$$\theta_1, \ldots, \theta_{\frac{n}{2}} \in \mathbb{R}$$

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Theorem (Hopf)

Any compact connected Lie group G has free commutative cohomology algebra on odd degree generators over a field.

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Theorem (Hopf)

Any compact connected Lie group G has free commutative cohomology algebra on odd degree generators over a field.

Examples

For $v \ge 2$ and $n \ge 1$ as a \mathbb{Z} -algebras

$$H^*(SU(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_3, x_5, \ldots, x_{2n-1}),$$

$$H^*(Sp(n);\mathbb{Z})\cong \Lambda_{\mathbb{Z}}(x_3,x_7,\ldots,x_{4n-1}),$$

where $|x_i| = i$.

However the integral cohomology of SO(n) contains 2-torsion.

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Cohomology of Complete Flag Manifolds

Definition

A polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$ is called symmetric if it is invariant under permutations of the indices $1, \ldots, n$.

Proposition

For each $1 \leq i \leq n$ the elements

$$\sigma_i = \sum_{1 \le j_1 < \cdots < j_i \le n} x_{i_1} \cdots x_{j_i},$$

form an algebraically independent generating set for the ring of symmetric polynomials in x_1, \ldots, x_n .

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The cohomology of many homogeneous spaces in the case when the subgroup has maximal rank were given by Borel.

As a \mathbb{Z} -algebras

$$H^*(SU(n+1)/T^n;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,\ldots,x_{n+1}]}{[\sigma_1,\ldots,\sigma_{n+1}]}$$

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 $|x_i| = 2$

For $v \geq 2$ and $n \geq 1$ as a \mathbb{Z} -algebras

$$\begin{split} H^*(SO(2v)/T^v;\mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1,x_2,...,x_v]}{[\sigma_1^{(2)},\sigma_2^{(2)},...,\sigma_{v-1}^{(2)},\sigma_v]} \\ H^*(SO(2v+1)/T^v;\mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1,x_2,...,x_v]}{[\sigma_1^{(2)},\sigma_2^{(2)},...,\sigma_v^{(2)}]}, \\ H^*(Sp(n)/T^n;\mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1,x_2,...,x_n]}{[\sigma_1^{(2)},\sigma_2^{(2)},...,\sigma_n^{(2)}]}, \\ H^*(G_2/T^2;\mathbb{Z}) &\cong \frac{\mathbb{Z}[x_1,x_2]}{[\sigma_2,\sigma_3^{(2)}]}, \end{split}$$

with $|x_i| = 2$ and

$$\sigma_i^{(2)} = \sum_{1 \le j_1 < \dots < j_i \le n} x_{j_1}^2 \cdots x_{j_i}^2.$$

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Definition (free loop space)

For any space X,

 $L(X) = Map(S^1, X)$

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Definition (free loop space)

For any space X,

$$L(X) = Map(S^1, X)$$

For a manifold M of dimension d.

Theorem (Gromoll, Meyer)

There are infinitely many geometrical distinct periodic geodesics for any metric on M if the Betti numbers of L(M) are unbounded.

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Theorem (Chas, Sullivan)

There is a shifted product structure on the homology of M,

$$\circ : H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$$

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Examples

The Free loop cohomology is known for classes of spaces such as

- Spheres LSⁿ,
- Complex Projective Space LCPⁿ,
- **3** Simple Lie groups LG

The Chas-Sullivan product is also known for 1,2 and most of 3.

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Examples

The Free loop cohomology is known for classes of spaces such as

- Spheres LSⁿ,
- 2 Complex Projective Space LℂPⁿ,
- **3** Simple Lie groups LG

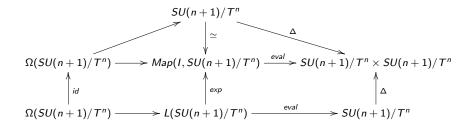
The Chas-Sullivan product is also known for 1,2 and most of 3.

I am interested in the free loop cohomology algebra of homogeneous spaces, in particular that of the complete flag manifolds. I am currently studying the easiest case

 $H^*(L(SU(n+1)/T^n);\mathbb{Z}).$

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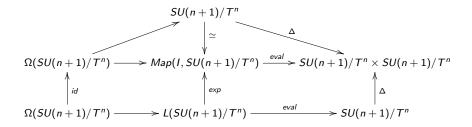




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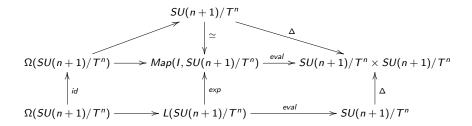
Proposition

As an algebra

 $H^*(\Omega(SU(n+1)/T^n);\mathbb{Z}) \cong H^*(\Omega(SU(n+1);\mathbb{Z}) \otimes H^*(T^n;\mathbb{Z}).$

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Proposition

As an algebra

 $H^*(\Omega(SU(n+1)/T^n);\mathbb{Z}) \cong H^*(\Omega(SU(n+1);\mathbb{Z}) \otimes H^*(T^n;\mathbb{Z}).$

$$H^*(L(SU(2)/T^1); \mathbb{Z}) \cong H^*(L(S^2); \mathbb{Z}) \cong \frac{\Lambda(\gamma, y, yx_k, \gamma x_k, y\gamma x_k)}{[\gamma^2, 2y\gamma, 2y\gamma x_k, yx_k\gamma x_j - \binom{k+j}{k}y\gamma x_{k+j}]},$$
where $|y| = 1, |\gamma| = 2, |x_i| = 2i$ and $k, j \ge 1$.

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Thank you for your attention

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