Free Loop Cohomology of Complete Flag Manifolds

Matt Burfitt *University of Southampton*

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Lie Groups

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Three classes of classical compact, connected, non-abelian Lie groups are given by

$$SO(n) = \{A \in M_n(\mathbb{R}) \mid A^{\intercal}A = I_n,\ det(A) = 1\},$$

$$SU(n) = \{A \in M_n(\mathbb{C}) \mid \bar{A}^{\intercal}A = I_n,\ det(A) = 1\},$$

$$Sp(n-1) = \{A \in M_n(\mathbb{H}) \mid \bar{A}^{\intercal}A = I_{n-1}, \},$$

for each $n \geq 2$.

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Theorem (Classification of abelian Lie group)

A connected abelian Lie group is isomorphic to $T^\alpha \times \mathbb{R}^\beta$ for some α, β.

Homogeneous Spaces

Definition

A manifold M is called a homogeneous space if there is a Lie group G which acts transitively on it.

Under weak assumptions this means that M is diffeomorphic to G/H for some closed subgroup H of G.

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Examples

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2. *Projective space* $\mathbb{R}P^n$ *or* $\mathbb{C}P^n$,
3. *Grassmannians* $Gr(r, n)$.

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Proposition

When H is a closed connected subgroup of compact Lie group G we have a fibration

$$H \hookrightarrow G \to G/H.$$

Definition

Subgroup $T \subseteq G$ *with* $T \cong T^n$ *is a maximal torus, if any* $T' \supseteq T$ *with* $T' \cong T^m$ $\implies T' = T$.

Complete Flag Manifolds

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For each Lie group G and maximal torus T, G/T is a homogeneous space called a *complete flag manifold*.

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Maximal tori in G are conjugate.
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Examples

For $\alpha_j \in \mathbb{R}$ A maximal torus in $SU(n)$ is given by elements of the form

$$\begin{bmatrix} e^{2\pi\alpha_1 i} & & \mathit{0} \\ & \ddots & \\ \mathit{0} & & e^{2\pi\alpha_n i} \end{bmatrix},$$

such that $e^{2\pi\alpha_1 i} \cdots e^{2\pi\alpha_n i} = 1$.
Without this condition this is a maxima torus for $Sp(n)$.

Complete Flag Manifolds

Examples

Maximal tori in $SO(n)$ and $SO(n+1)$ are given by elements of the form

$$\begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & & & \\ sin(\theta_1) & cos(\theta_1) & & \mathit{0} & \\ & & \ddots & & \\ & \mathit{0} & & cos(\theta_{\frac{n}{2}}) & -sin(\theta_{\frac{n}{2}}) \\ & & & sin(\theta_{\frac{n}{2}}) & cos(\theta_{\frac{n}{2}}) \end{bmatrix},$$

$$\begin{bmatrix} cos(\theta_1) & -sin(\theta_1) & & & & \\ sin(\theta_1) & cos(\theta_1) & & & \mathit{0} & \\ & & \ddots & & & \\ & & & cos(\theta_{\frac{n}{2}}) & -sin(\theta_{\frac{n}{2}}) & \\ & \mathit{0} & & sin(\theta_{\frac{n}{2}}) & cos(\theta_{\frac{n}{2}}) & \\ & & & & & 1 \end{bmatrix},$$

for $\theta_1, \dots, \theta_{\frac{n}{2}} \in \mathbb{R}$

Cohomology of Lie Groups

Theorem (Hopf)

Any compact connected Lie group G has free commutative cohomology algebra on odd degree generators over a field.

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Examples

For $v \geq 2$ and $n \geq 1$ as a $\mathbb{Z}$-algebras

$$H^*(SU(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_3, x_5, \dots, x_{2n-1}),$$

$$H^*(Sp(n);\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_3, x_7, \dots, x_{4n-1}),$$

where $|x_i| = i$.

However the integral cohomology of $SO(n)$ contains 2-torsion.

Cohomology of Complete Flag Manifolds

Definition

A polynomial $p \in \mathbb{Z}[x_1, \ldots, x_n]$ is called symmetric if it is invariant under permutations of the indices $1, \ldots, n$.

Proposition

For each $1 \leq i \leq n$ the elements

$$\sigma_i = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{i_1} \cdots x_{j_i},$$

form an algebraically independent generating set for the ring of symmetric polynomials in $x_1, \ldots, x_n$.

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The cohomology of many homogeneous spaces in the case when the subgroup has maximal rank were given by Borel.

As a $\mathbb{Z}$-algebras

$$H^*(SU(n+1)/T^n; \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, \ldots, x_{n+1}]}{[\sigma_1, \ldots, \sigma_{n+1}]}$$

$|x_i| = 2$

Cohomology of Complete Flag Manifolds

For $v \geq 2$ and $n \geq 1$ as a $\mathbb{Z}$-algebras

$$H^*(SO(2v)/T^v;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,x_2,\dots,x_v]}{[\sigma_1^{(2)},\sigma_2^{(2)},\dots,\sigma_{v-1}^{(2)},\sigma_v]}$$

$$H^*(SO(2v+1)/T^v;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,x_2,\dots,x_v]}{[\sigma_1^{(2)},\sigma_2^{(2)},\dots,\sigma_v^{(2)}]},$$

$$H^*(Sp(n)/T^n;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,x_2,\dots,x_n]}{[\sigma_1^{(2)},\sigma_2^{(2)},\dots,\sigma_n^{(2)}]},$$

$$H^*(G_2/T^2;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1,x_2]}{[\sigma_2,\sigma_3^{(2)}]},$$

with $|x_i| = 2$ and

$$\sigma_i^{(2)} = \sum_{1 \leq j_1 < \cdots < j_i \leq n} x_{i_1}^2 \cdots x_{j_i}^2.$$

Free Loop Cohomology

Definition (free loop space)

For any space X,

$$L(X) = Map(S^1, X)$$

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There are infinitely many geometrical distinct periodic geodesics for any metric on M if the Betti numbers of $L(M)$ are unbounded.

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Theorem (Chas, Sullivan)

There is a shifted product structure on the homology of M,

$$\circ : H_p(LM) \otimes H_q(LM) \to H_{p+q-d}(LM)$$

Free Loop Cohomology

Examples

The Free loop cohomology is known for classes of spaces such as

1. *Spheres* LS^n,
2. *Complex Projective Space* $L\mathbb{C}P^n$,
3. *Simple Lie groups* LG

The Chas-Sullivan product is also known for 1,2 and most of 3.

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I am interested in the free loop cohomology algebra of homogeneous spaces, in particular that of the complete flag manifolds. I am currently studying the easiest case

$$H^*(L(SU(n+1)/T^n);\mathbb{Z}).$$

Free Loop Cohomology

$$\begin{tikzcd}
 & SU(n+1)/T^n \arrow[d, "\simeq"] \arrow[dr, "\Delta"] & \\
\Omega(SU(n+1)/T^n) \arrow[ur] \arrow[r] & Map(I, SU(n+1)/T^n) \arrow[r, "eval"] & SU(n+1)/T^n \times SU(n+1)/T^n \\
\Omega(SU(n+1)/T^n) \arrow[u, "id"] \arrow[r] & L(SU(n+1)/T^n) \arrow[u, "exp"] \arrow[r, "eval"] & SU(n+1)/T^n \arrow[u, "\Delta"]
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Proposition

As an algebra

$$H^*(\Omega(SU(n+1)/T^n);\mathbb{Z}) \cong H^*(\Omega(SU(n+1);\mathbb{Z}) \otimes H^*(T^n;\mathbb{Z}).$$

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Proposition

As an algebra

$$H^*(\Omega(SU(n+1)/T^n);\mathbb{Z}) \cong H^*(\Omega(SU(n+1);\mathbb{Z}) \otimes H^*(T^n;\mathbb{Z}).$$

$$H^*(L(SU(2)/T^1);\mathbb{Z}) \cong H^*(L(S^2);\mathbb{Z}) \cong \frac{\Lambda(\gamma,\ y,\ yx_k,\ \gamma x_k,\ y\gamma x_k)}{[\gamma^2,\ 2y\gamma,\ 2y\gamma x_k,\ yx_k\gamma x_j - \binom{k+j}{k} y\gamma x_{k+j}]},$$

where $|y| = 1$, $|\gamma| = 2$, $|x_i| = 2i$ and $k, j \geq 1$.

Thank you for your attention