# Regular Maps on Spheres and Projective Spaces

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Young Topologist Seminar
11 - 19 August 2015
IMS, NUS

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# Introduction

Let $X$ be a topological space and $k \geq 2$. Let $\mathbb{K}$ denote the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Let $S^m$ denote the $m$-sphere and $\mathbb{R}P^m$, $\mathbb{C}P^m$ denote the real and complex projective spaces respectively.

## Definition 1

A map $f : X \to \mathbb{K}^N$ is called (real or complex) $k$-regular if for any distinct $k$ points $x_1, \cdots, x_k \in X$, $f(x_1), \cdots, f(x_k)$ are linearly independent over $\mathbb{K}$. For simplicity, a real $k$-regular map is also called a $k$-regular map.

# Introduction

Let $\alpha(k)$ denote the number of ones in the dyadic expansion of $k$. Some lower bounds of $N$ for $k$-regular maps of $\mathbb{R}^2$ into $\mathbb{R}^N$ were given in the following theorem.

**Theorem 2 ([7], Example 1.2, Theorem 1.4)**

*If there exists a $k$-regular map of $\mathbb{R}^2$ into $\mathbb{R}^N$, then $N \geq 2k - \alpha(k)$. Moreover, when $k$ is a power of 2, there exists a $k$-regular map of $\mathbb{R}^2$ into $\mathbb{R}^N$ for $N = 2k - \alpha(k)$.*

The following theorem partially generalized Theorem 2, giving a lower bound of $N$ for $k$-regular maps of $\mathbb{R}^d$ into $\mathbb{R}^N$ for all $d \geq 1$.

**Theorem 3 ([1], Theorem 2.1)**

*Let $d \geq 1$. If there exists a $k$-regular map of $\mathbb{R}^d$ into $\mathbb{R}^N$, then $N \geq d(k - \alpha(k)) + \alpha(k)$.*

# Introduction

The lower bounds of $N$ for complex $k$-regular maps of Euclidean spaces into $\mathbb{C}^N$ were studied in the following two theorems.

### Theorem 4 ([2], Theorem 5.2)

*Let $p$ be an odd prime and $d \geq 1$. If there exists a complex $p$-regular map of $\mathbb{R}^d$ into $\mathbb{C}^N$, then $N \geq [(d+1)/2](p-1)+1$.*

### Theorem 5 ([2], Theorem 5.3)

*Let $p$ be an odd prime, $\alpha_p(k)$ the sum of coefficients in the $p$-adic expansion of $k$ and $d = p^t$ for some $t \geq 1$. If there exists a complex $k$-regular map of $\mathbb{C}^d$ into $\mathbb{C}^N$, then $N \geq d(k - \alpha_p(k)) + \alpha_p(k)$.*

# Introduction

Motivated by Theorem 2 - Theorem 5, lower bounds of $N$ for $k$-regular maps of non-Euclidean spaces into $\mathbb{R}^N$ are of interest. For example, some 3-regular maps of $S^m$ into $\mathbb{R}^{m+2}$ can be constructed.

## Example 6 ([1], Lemma 2.5, Example 2.6-(2))

*Let $m \geq 1$. Let $i : S^m \to \mathbb{R}^{m+1}$ be the standard embedding and $1 : S^m \to \mathbb{R}$ the constant map with image 1. There is a 3-regular map*

$$S^m \xrightarrow{(1,i)} (\mathbb{R}, \mathbb{R}^{m+1}) \cong \mathbb{R}^{m+2}.$$

# Introduction

From Example 6 and [8, Theorem 4.1], [15, Theorem 5.2, Theorem 5.7] and [18, Theorem 5] (resp. [22, Theorem 5.4]), we have the following corollary.

## Corollary 7

*There exist 3-regular maps of $\mathbb{R}P^m$ into $\mathbb{R}^N$ (resp. 3-regular maps of $\mathbb{C}P^m$ into $\mathbb{R}^N$) in the cases listed in the following Table.*

| | | |
|---|---|---|
| $\mathbb{R}P^m$ | $m = 8q+3$ or $8q+5$, $q > 0$ | $N \geq 2m - \min\{5, \alpha(q)\}$ |
| | $m = 8q+1$, $q > 0$ | $N \geq 2m - \min\{7, \alpha(q)\} + 2$ |
| | $m = 32q+7$, $q > 0$ | $N \geq 2m - 6$ |
| | $m = 8q+7$, $q > 1$ | $N \geq 2m - 5$ |
| | $m \equiv 3 \pmod 8$, $m \geq 19$ | $N \geq 2m - 4$ |
| | $m \equiv 1 \pmod 4$, $m \neq 2^i + 1$ | $N \geq 2m - 2$ |
| | $m = 4q+i$, $i = 0$ or $2$, $q \neq 2^j$ or $0$ | $N \geq 2m - 1$ |
| | $m = 2^j + 1$, $j \geq 2$ | $N \geq 2m - 1$ |
| | $m = 2^j + 2$, $j \geq 3$ | $N \geq 2m$ |
| $\mathbb{C}P^m$ | $m \geq 5$, $m \neq 2^j$ | $N \geq 4m$ |
| | $m = 2^j$, $j \geq 2$ | $N \geq 4m + 1$ |

Our results are supplementary to [1, 2, 5, 7].

### Theorem 8

*Let $m \geq 2$. The following are equivalent*
*(a). there exists a 3-regular map of $S^m$ into $\mathbb{R}^N$,*
*(b). there exists a 2-regular map of $S^m$ into $\mathbb{R}^N$,*
*(c). $N \geq m + 2$.*

### Theorem 9

*Let $m \geq 2$. If there exists a complex 2-regular map of $S^m$ into $\mathbb{C}^N$, then $N \geq m/2 + 2$ if $m$ is even and $N \geq (m-1)/2 + 2$ if $m$ is odd.*

### Theorem 10

*Let $2^i \leq m < 2^{i+1}$, $i \geq 2$. If there exists a 2-regular map of $\mathbb{R}P^m$ into $\mathbb{R}^N$, then $N \geq 2^{i+1} + 1$.*

### Corollary 11

*Let $m = 2^i + 1$, $i \geq 2$. Then the following are equivalent*
*(a). there exists a 3-regular map of $\mathbb{R}P^m$ into $\mathbb{R}^N$,*
*(b). there exists a 2-regular map of $\mathbb{R}P^m$ into $\mathbb{R}^N$,*
*(c). $N \geq 2m - 1$.*

**Theorem 12**

*Let $2^i \leq m < 2^{i+1}$, $i \geq 2$. If there exists a 2-regular map of $\mathbb{C}P^m$ into $\mathbb{R}^N$, then $N \geq 2^{i+2}$.*

**Theorem 13**

*Let $m \geq 4$. If there exists a complex 2-regular map of $\mathbb{C}P^m$ into $\mathbb{C}^N$, then $N \geq 2m$.*

For positive integers $M \geq k$, let $G_k(\mathbb{K}^M)$ be the (real or complex) Grassmannian and $G_k(\mathbb{K}^\infty)$ the direct limit of $G_k(\mathbb{K}^M)$. Consider the inclusion $\mathbb{K}^N \to \mathbb{K}^\infty$ on the first $N$ coordinates of $\mathbb{K}^\infty$. Then there is an induced map $i : G_k(\mathbb{K}^N) \to G_k(\mathbb{K}^\infty)$.

- $\textsc{Case 1}$: $\mathbb{K} = \mathbb{R}$. It is known that

$$H^*(G_k(\mathbb{R}^\infty); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \cdots, w_k]$$

where $w_i$ is the $i$-th universal Stiefel-Whitney class with $|w_i| = i$. And

$$H^*(G_k(\mathbb{R}^M); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \cdots, w_k]/(\bar{w}_{M-k+1}, \bar{w}_{M-k+2}, \cdots, \bar{w}_M)$$

where $\bar{w}_j$ is defined as the $j$-th degree term in the expansion of $(1 + w_1 + \cdots + w_k)^{-1}$ and $(\bar{w}_{M-k+1}, \bar{w}_{M-k+2}, \cdots, \bar{w}_M)$ is the ideal generated by $\bar{w}_{M-k+1}$, $\bar{w}_{M-k+2}$, $\cdots$, $\bar{w}_N$. The canonical inclusion $i : G_k(\mathbb{R}^M) \to G_k(\mathbb{R}^\infty)$ induces an epimorphism on mod 2 cohomology.

- CASE 2: $\mathbb{K} = \mathbb{C}$. It is known that

$$H^*(G_k(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_k]$$

where $c_i$ is the $i$-th universal Chern class with $|c_i| = 2i$. And

$$H^*(G_k(\mathbb{C}^M); \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \cdots, c_k]/(\bar{c}_{M-k+1}, \bar{c}_{M-k+2}, \cdots, \bar{c}_M)$$

where $\bar{c}_j$ is defined as the $2j$-th degree term in the expansion of $(1 + c_1 + \cdots + c_k)^{-1}$ and $(\bar{c}_{M-k+1}, \bar{c}_{M-k+2}, \cdots, \bar{c}_M)$ is the ideal generated by $\bar{c}_{M-k+1}$, $\bar{c}_{M-k+2}$, $\cdots$, $\bar{c}_M$. The canonical inclusion $i : G_k(\mathbb{C}^M) \to G_k(\mathbb{C}^\infty)$ induces an epimorphism on integral cohomology.

# Preliminaries

## Cohomology of unordered configuration spaces

Let $\Sigma_k$ be the permutation group of order $k$ and the $k$-th configuration space of $X$ be

$$F(X,k) = \{(x_1,\cdots,x_k) \in X \times \cdots \times X \mid \text{for any } i \neq j, x_i \neq x_j\}.$$

For any $\sigma \in \Sigma_k$, let $\sigma$ act on $F(M,k)$ by

$$\sigma(x_1,\cdots,x_k) = (x_{\sigma(1)},\cdots,x_{\sigma(k)})$$

and act on $\mathbb{K}^k$ by

$$(r_1,\cdots,r_k)\sigma = (r_{\sigma^{-1}(1)},\cdots,r_{\sigma^{-1}(k)}).$$

Then we have a space $F(X,k)/\Sigma_k$, called the $k$-th unordered configuration space of $X$, and an $O(\mathbb{K}^k)$-bundle

$$\xi^{\mathbb{K}}_{X,k} : \mathbb{K}^k \to F(X,k) \times_{\Sigma_k} \mathbb{K}^k \to F(X,k)/\Sigma_k.$$

Let $h : F(X,k)/\Sigma_k \to G_k(\mathbb{K}^\infty)$ be the classifying map of $\xi^{\mathbb{K}}_{X,k}$.

# Preliminaries

Cohomology of unordered configuration spaces

For any $m \geq 1$, $F(S^m, 2)/\Sigma_2 \simeq \mathbb{R}P^m$. Consequently,

$$H^*(F(S^m,2)/\Sigma_2;\mathbb{Z}_2) = \mathbb{Z}_2[u]/(u^{m+1}),\ |u| = 1, \tag{1}$$

$$H^*(F(S^m,2)/\Sigma_2;\mathbb{Z}) = \begin{cases} \mathbb{Z}[x]/(2x, x^{\frac{m+2}{2}}),\ |x| = 2,\ \text{if } m \text{ is even}, \\ \mathbb{Z}[x]/(2x, x^{\frac{m+1}{2}}),\ |x| = 2,\ \text{if } m \text{ is odd}. \end{cases} \tag{2}$$

# Preliminaries

## Cohomology of unordered configuration spaces

### Theorem 14 ([8], Theorem 3.7)

*As $\mathbb{Z}_2$-algebras, $H^*(F(\mathbb{R}P^m, 2)/\Sigma_2; \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2[u, x_1, x_2]/(u^2 - ux_1, \tilde{\sigma}_m(x_1, x_2), \tilde{\sigma}_{m+1}(x_1, x_2))$. Here $u = w_1(\xi^{\mathbb{R}}_{\mathbb{R}P^m,2})$ and $|x_i| = i$, $i = 1, 2$.*

### Theorem 15 ([22], Theorem 4.9)

*As a $H^*(G_2(\mathbb{C}^{m+1}); \mathbb{Z}_2)$-module, the cohomology $H^*(F(\mathbb{C}P^m, 2)/\Sigma_2; \mathbb{Z}_2)$ has $\{1, v, v^2\}$ as a basis. Moreover, the ring structure of $H^*(F(\mathbb{C}P^m, 2)/\Sigma_2; \mathbb{Z}_2)$ is given by $v^3 = e_1 v$. Here $v = w_1(\xi^{\mathbb{R}}_{\mathbb{C}P^m,2})$.*

### Theorem 16 ([22], Theorem 4.10)

*As a $H^*(G_2(\mathbb{C}^{m+1}); \mathbb{Z})$-module, the cohomology $H^*(F(\mathbb{C}P^m, 2)/\Sigma_2; \mathbb{Z})$ has $\{1, u\}$ as generators with $|u| = 2$. Moreover, the ring structure of $H^*(F(\mathbb{C}P^m, 2)/\Sigma_2; \mathbb{Z})$ satisfies $2u = 0$ and $u^2 = c_1 u$. Here $c_1 = c_1(\xi^{\mathbb{C}}_{\mathbb{C}P^m,2})$.*

## $k$-regular maps on topological spaces

Suppose $f : X \to \mathbb{K}^N$ is a (real or complex) $k$-regular map. We have a well-defined map

$$g : F(X,k)/\Sigma_k \quad \to \quad G_k(\mathbb{K}^N).$$

### Proposition 17

*The following diagram commutes*

$$\begin{tikzcd} F(X,k)/\Sigma_k \arrow[r, "g"] \arrow[dr, "h"'] & G_k(\mathbb{K}^N) \arrow[d, "i"] \\ & G_k(\mathbb{K}^\infty). \end{tikzcd}$$

Let $R$ be a ring. From Proposition 17, there is an induced commutative diagram on cohomology

$$\begin{tikzcd} H^*(F(X,k)/\Sigma_k; R) & H^*(G_k(\mathbb{K}^N); R) \arrow[l, "g^*"'] \\ & H^*(G_k(\mathbb{K}^\infty); R). \arrow[ul, "h^*"] \arrow[u, "i^*"'] \end{tikzcd} \tag{3}$$

# $k$-regular maps on topological spaces

CASE 1: $\mathbb{K} = \mathbb{R}$.

## Lemma 18 ([7])

*Lex $X$ be a topological space and $f : X \to \mathbb{R}^N$ a $k$-regular map. If $\bar{w}_t(\xi^{\mathbb{R}}_{X,k}) \neq 0$, then $N \geq t + k$.*

CASE 2: $\mathbb{K} = \mathbb{C}$.

## Lemma 19 ([2])

*Let $X$ be a topological space and $f : X \to \mathbb{C}^N$ be a complex $k$-regular map. If $\bar{c}_t(\xi^{\mathbb{C}}_{X,k}) \neq 0$, then $N \geq t + k$.*

Let $m \geq 2$. We first prove

$$\bar{w}_t(\xi^{\mathbb{R}}_{S^m,2}) \neq 0 \ \text{ for } t \leq m. \tag{4}$$

Then by applying (4) to Lemma 18 and with the help of Example 6, we obtain Theorem 8.

We first prove

$$\bar{c}_t(\xi^{\mathbb{C}}_{S^m,2}) \quad \neq \quad 0 \ \text{ for } t \leq \frac{m}{2} \tag{5}$$

if $m$ is even and

$$\bar{c}_t(\xi^{\mathbb{C}}_{S^m,2}) \quad \neq \quad 0 \ \text{ for } t \leq \frac{m-1}{2} \tag{6}$$

if $m$ is odd. Then by applying (5) and (6) to Lemma 19, we obtain Theorem 9.

We first prove that the smallest positive integer $\tau(m)$ such that for all $t \geq \tau(m)$, $\bar{w}_t(\xi^{\mathbb{R}}_{\mathbb{R}P^m,2}) = 0$ is

$$\tau(m) = 2^{i+1} \tag{7}$$

for $2^i \leq m < 2^{i+1}$, $i \geq 2$. Then by applying (7) to Lemma 18, we obtain Theorem 10.

We first prove that the smallest positive integer $\lambda(m)$ such that for all $t \geq \lambda(m)$, $\bar{w}_t(\xi^{\mathbb{R}}_{\mathbb{C}P^m,2}) = 0$ is

$$\lambda(m) = 2^{i+2} - 1 \text{ for } 2^i \leq m < 2^{i+1}, i \geq 2. \tag{8}$$

Then by applying (8) to Lemma 18, we obtain Theorem 12.

Let $\kappa(m)$ be the smallest positive integer such that for all $t \geq \kappa(m)$, $\bar{c}_t(\xi^{\mathbb{C}}_{\mathbb{C}P^m,2}) = 0$. We first prove that

$$\kappa(m) \geq 2m - 1. \tag{9}$$

Then by applying (9) to Lemma 19, we obtain Theorem 13.

Let $M$ be a manifold of dimension $m$ and $X$ be a topological space with non-degenerate base-point. The configuration space $C(M;X)$ is defined by

$$C(M;X) = \coprod_{k=0}^{\infty} F(M,k) \times_{\Sigma_k} X^k / \approx$$

where $F(M,0) \times_{\Sigma_0} X^0$ is defined to be the base-point $*$ and $\approx$ is generated by

$$(a_1, \cdots, a_k; x_1, \cdots, x_k) \approx (a_1, \cdots, a_{k-1}; x_1, \cdots, x_{k-1})$$

if $x_k = *$. The length of a configuration induces a natural filtration of $C(M;X)$ by the subspaces

$$C_k(M;X) = \coprod_{j=0}^{k} F(M,j) \times_{\Sigma_j} X^j / \approx .$$

For $k \geq 1$, define the quotient spaces

$$D_k(M;X) = C_k(M;X)/C_{k-1}(M;X).$$

# Plans

Configuration spaces

- My aim is to obtain cup-lengths of elements in the cohomology ring

$$H^*(D_k(M;S^0);\mathbb{Z}_2)$$

and use this to study the lower bounds of $N$ for $k$-regular maps of $M$ into $\mathbb{R}^N$.

- I want to study the cohomology ring

$$H^*(C(M;S^0);\mathbb{Z}_2).$$

then find ways to derive the cohomology ring

$$H^*(D_k(M;S^0);\mathbb{Z}_2).$$

- For primes $p \geq 2$, let $\beta_q = \dim_{\mathbb{Z}_p} H_q(M; \mathbb{Z}_p)$ and a tensor product of finite number of Hopf algebras $H_*(\Omega^{t-q}\Sigma^{t-q}S^{r+q}; \mathbb{Z}_p)$ (cf. [6]) as

$$\mathcal{C}^t(H_*(M; \mathbb{Z}_p); S^r) = \bigotimes_{q=0}^{t} \bigotimes^{\beta_q} H_*(\Omega^{t-q}S^{t+r}; \mathbb{Z}_p).$$

For $r \geq 2$ and $n \geq 2$, we have that as a $\mathbb{Z}_p$-filtered algebra (cf. [21]),

$$H_*(C(M \times \mathbb{R}^n; S^r); \mathbb{Z}_p) \cong \mathcal{C}^{m+n}(H_*(M; \mathbb{Z}_p); S^r). \tag{10}$$

I want to consider the case $n = 1$ and $r = 0$ in (10) and study the algebra structure of

$$H_*(C(M \times \mathbb{R}; S^0); \mathbb{Z}_2).$$

# Plans

The group-completion theorem

Let $\mathcal{M}$ be a topological monoid up to homotopy. Let the homology be taken with integral coefficients. Then the $H$-space structure on $\mathcal{M}$ implies that $H_*(\mathcal{M})$ is a Pontrjagin ring. It is known that $H_0(\mathcal{M}) = \mathbb{Z}[\pi_0\mathcal{M}]$, hence $\pi_0\mathcal{M}$ can be regarded as a multiplicative subset of the Pontrjagin ring $H_*(\mathcal{M})$.

### Theorem 20 ([16])

*Suppose*

*(1).* $H_*(\mathcal{M})[\pi_0\mathcal{M}^{-1}]$*, the localization of* $H_*(\mathcal{M})$ *with respect to* $\pi_0\mathcal{M}$*, admits calculation by right fractions;*

*(2).* $\pi_0\mathcal{M}$ *is finitely generated.*

*Then the canonical map* $\psi : \mathcal{M} \to \Omega B\mathcal{M}$ *induces an isomorphism of Pontrjagin rings*

$$\begin{tikzcd} H_*(\mathcal{M})[\pi_0\mathcal{M}^{-1}] \arrow[rr, "\cong"] & & H_*(\Omega B\mathcal{M}) \\ H_*(\mathcal{M}) \arrow[u, "\gamma"] \arrow[urr, "\psi_*"'] & & \end{tikzcd}$$

*where* $\gamma : H_*(\mathcal{M}) \to H_*(\mathcal{M})[\pi_0\mathcal{M}^{-1}]$ *is the canonical ring homomorphism of the localization of the ring* $H_*(\mathcal{M})$ *with respect to* $\pi_0\mathcal{M}$*.*

## Proposition 21

*$C(M \times \mathbb{R}; X)$ is a monoid up to homotopy.*

## Problem 22

*The canonical map $\psi : C(M \times \mathbb{R}; X) \to \Omega BC(M \times \mathbb{R}; X)$ induces a ring isomorphism on homology*

$$\begin{tikzcd}
H_*(C(M \times \mathbb{R}; X))[\pi_0 C(M \times \mathbb{R}; X)^{-1}] \arrow[rr, "\cong"] & & H_*(\Omega BC(M \times \mathbb{R}; X)) \\
H_*(C(M \times \mathbb{R}; X)) \arrow[u, "\gamma"] \arrow[urr, "\psi_*"'] & &
\end{tikzcd}$$

*where $\gamma : H_*(C(M \times \mathbb{R}; X)) \to H_*(C(M \times \mathbb{R}; X))[\pi_0 C(M \times \mathbb{R}; X)^{-1}]$ is the canonical ring homomorphism of the localization of the ring $H_*(C(M \times \mathbb{R}; X))$ with respect to $\pi_0 C(M \times \mathbb{R}; X)$.*

# Plans

## Section spaces

Let $W$ be a manifold of dimension $m$ without boundary which contains $M$, for example, $W = M$ if $M$ is closed, or $W = M \cup \partial M \times [0,1)$ if $M$ has boundary. Let $\xi$ be the principal $O(\mathbb{R}^m)$-bundle of the tangent bundle of $W$. Let $\xi[S^m \wedge X]$ be the associated bundle and $O(\mathbb{R}^m)$ acts diagonally on $S^m \wedge X$, trivially on $X$ and canonically on $S^m \cong \mathbb{R}^m \cup \{\infty\}$. For each subspace pair $(B, B_0)$ in $W$, let $\Gamma_{\xi[S^m \wedge X]}(B, B_0)$ be the space of cross sections of $\xi[S^m \wedge X]$ which are defined on $B$ and take values at $\infty \wedge X$ on $B_0$. We will consider the section space $\Gamma_{\xi[S^m \wedge X]}(W, W - M)$ (cf. [3, 21]). For the manifold $M \times [0,1]$, the manifold $W$ is chosen as

$$W = M \times [0,1] \cup \partial(M \times [0,1]) \times [0,1) \cong M \times [0,1] \cup (M \times (-1,0] \cup M \times [1,2)) = M \times (-1,2).$$

### Problem 23

*As Hopf algebras,*

$$H_*(\Omega BC(M \times \mathbb{R}; X); \mathbb{Z}_2) \cong H_*(\Gamma_{\xi[\Sigma^{m+1} X]}(M \times (-1,2), M \times (-1,0) \cup M \times (1,2)); \mathbb{Z}_2).$$

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Thank You

Thank Professor Wu Jie for his instruction and guidance!