Dual Polyhedral Products, Cocategory and Nilpotence

K - simplicial complex on vertex set $[m] = \{1, \dots, m\}$.

$(\underline{X}, \underline{A})$ = set of pairs of ptd, connected CW-complexes for $1 \le i \le m$.

If $\sigma \in K$, let $(\underline{X}, \underline{A})^{\sigma} = \prod_{i=1}^{m} Y_i$ $\qquad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$

Def: The polyhedral product is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^{\sigma}.$$

Ex: $K = m$ disjoint pts

$$(\underline{X}, \underline{*})^K \simeq \bigvee_{i=1}^{m} X_i.$$

Alternatively, for $I \subseteq [m]$ let K_I = full subcomplex of K on I.

$\exists$ inclusion of simplicial complexes $K_I \hookrightarrow K$. So if $K \neq \Delta^{m-1}$,

$$(\underline{X}, \underline{A})^K = \underset{\substack{\longrightarrow \\ I \subsetneq [m]}}{\mathrm{colim}} \, (\underline{X}, \underline{A})^{K_I}.$$

Dualize: $\nexists$ projection $K \to K_I$ but $\exists$ proj $(\underline{X}, \underline{A})^K \to (\underline{X}, \underline{A})^{K_I}$.

Def: The dual polyhedral product is

$$(\underline{X}, \underline{A})^K_D = \underset{\substack{\longleftarrow \\ I \subsetneq [m]}}{\mathrm{holim}} \, (\underline{X}, \underline{A})^{K_I}.$$

Note: Initial object $I = [m]$ is left out!

Special case: $(\underline{X}, \underline{*})^K$, $K = m$ disjoint pts.

$$(\underline{X}, \underline{*})^K \simeq \bigvee_{i=1}^{m} X_i \xrightarrow{\text{pinch map}} (\underline{X}, *)^{K_I} \simeq \bigvee_{i \in I} X_i$$

Def (Hovey) The thin product of $X_1, \dots, X_m$ is

$$P^m(\underline{X}) = (\underline{X}, \underline{*})^K_D.$$

- dual to the fat wedge
- used to define cocategory, dual to L-S category.

Def: Let $X_i = X$ $\forall i$. The cocategory of X is the least m s.t.

$\exists$ extension

$$\begin{tikzcd} \bigvee_{i=1}^{m+1} X \arrow[r] \arrow[d, "\text{fold}"] & P^{m+1}(X) \arrow[dl, dashed] \\ X & \end{tikzcd}$$

Write $\operatorname{cocat}(X) = m$.

Ex: $$P^2(X) = \underleftarrow{\operatorname{holim}}\left(\begin{tikzcd} & X \arrow[d] \\ X \arrow[r] & * \end{tikzcd} \right) = X \times X$$

$\operatorname{cocat}(X) = 1 \iff \exists$ extension

$$\begin{tikzcd} X \vee X \arrow[r] \arrow[d] & P^2(X) = X \times X \arrow[dl] \\ X & \end{tikzcd}$$

$\iff X$ is an H-space.

A weaker version: consider the htpy fib $F^{m+1} \to \bigvee_{i=1}^{m+1} X_i \to P^{m+1}(X)$

Def: The weak cocategory of X is the least m s.t.

$$\begin{tikzcd} F^{m+1} \arrow[r] & \bigvee_{i=1}^{m+1} X \arrow[d, "\text{fold}"] \\ & X \end{tikzcd}$$

is null htpic. Write $\operatorname{wcocat}(X) = m$.

Nilpotence: Let G be an H-gp. Let $c_m : G^{(m+1)} \to G$ be the $(m+1)$-fold Samelson product of the identity map on G with itself.

Def: The htpy nilpotence class of G is the least m s.t. $c_m \simeq *$ but $c_{m-1} \not\simeq *$. Write $\operatorname{nil}(G) = m$.

Ex: $\mathrm{nil}(G) = 1 \iff G \wedge G \xrightarrow{c_1} G \simeq * \iff G$ is htpy commutative.

<u>Thm (T)</u>: Let X be a 1-connected, ptd space. Then

$$\mathrm{nil}(\Omega X) = m \iff \mathrm{wcocat}(X) = m.$$

- $\exists$ several notions of cocategory, all attempt to obtain T.
- Can be used for explicit calculation.

Approach: Compare $\Omega(\underline{X}, \underline{A})^K$ and $\Omega(\underline{X}, \underline{A})^K_D$.

<u>Loop Space Decomposition</u>

Let $I_j = [m] \setminus \{j\}$.

$\exists$ idempotents $e_j : (\underline{X}, \underline{A})^K \longrightarrow (\underline{X}, \underline{A})^{K_{I_j}} \longrightarrow (\underline{X}, \underline{A})^K$.

Observe $e_i \circ e_j = e_j \circ e_i \Rightarrow e_i e_j$ is an idempotent.

Loop to add. Get 2^m idempotents

$$f_{(a_1, \dots, a_m)} = f_{a_1} \circ \dots \circ f_{a_m} \text{ where } f_{a_i} = \begin{cases} \Omega e_i & \text{if } a_i = 0 \\ 1 - \Omega e_i & \text{if } a_i = 1 \end{cases}$$

<u>Prop</u>: $\Omega(\underline{X}, \underline{A})^K \simeq \prod_{(a_1, \dots, a_m) \in \mathcal{J}} \mathrm{Tel}(f_{(a_1, \dots, a_m)})$.

<u>Observe</u>: $\mathrm{Tel}(\Omega e_j) = \Omega(\underline{X}, \underline{A})^{K_{I_j}}$.

$\Rightarrow$ If $a_j = 0$ then $\mathrm{Tel}(f_{(a_1, \dots, a_m)})$ retracts off $\Omega(\underline{X}, \underline{A})^{K_{I_j}}$.

Also,
$$\begin{tikzcd} (\underline{X}, \underline{A})^{K_{I_j}} \arrow[r] \arrow[rrr, bend left, "="] & (\underline{X}, \underline{A})^K \arrow[r] & (\underline{X}, \underline{A})^K_D \arrow[r] & (\underline{X}, \underline{A})^{K_{I_j}} \end{tikzcd}$$

by def of holim

$\Rightarrow \mathrm{Tel}(f_{(a_1, \dots, a_m)})$ retracts off $\Omega(\underline{X}, \underline{A})^K_D$.

Only 1 case is left: $(a_1, \dots, a_m) = (1, \dots, 1)$

<u>Thm</u>: $\Omega(\underline{X}, \underline{A})^K_D \simeq \prod_{(a_1, \dots, a_m) \in \mathcal{J} \setminus (1, \dots, 1)} \mathrm{Tel}(f_{(a_1, \dots, a_m)}) \Rightarrow \Omega(\underline{X}, \underline{A})^K \simeq \Omega(\underline{X}, \underline{A})^K_D \times \mathrm{Tel}(f_{(1, \dots, 1)})$

More: Htpy fib $F_{[m]} \to (\underline{X},\underline{A})^K \to (\underline{X},\underline{A})^K_D$.

Prop: $\mathrm{Tel}(f_{(1,\dots,1)}) \simeq \Omega F_{[m]}$.

Even more: For example, $\mathrm{Tel}(f_{(1,\dots,1,0)})$ is a retract of $\Omega(\underline{X},\underline{A})^{K_{I_m}}$

$$\begin{aligned} &\mathrm{Tel}(f_{(1,\dots,1,0)}) \\ &\qquad \| \\ &\mathrm{Tel}(f_{\underbrace{(1,\dots,1)}_{m-1}}) \text{ for idemp constr on } \Omega(\underline{X},\underline{A})^L,\ L = K_{I_m} \end{aligned}$$

$\Rightarrow \mathrm{Tel}(f_{(1,\dots,1,0)}) \simeq \Omega F_{I_m}$.

$\Rightarrow$ Thm: $\Omega(\underline{X},\underline{A})^K \simeq \prod_{I \subseteq [m]} \Omega F_I$.

Next step: Identify F_I.

Specialize to $(\underline{CX},\underline{X})^K$ for K shifted, shellable, ...

$\Rightarrow (\underline{CX},\underline{X})^K \simeq$ wedge of spaces of the form $\Sigma^t X_{i_1} \wedge \dots \wedge X_{i_k}$.

Write $(\underline{CX},\underline{X})^K \in W$.

Careful analysis of the fibre of $(\underline{CX},\underline{X})^K \to (\underline{CX},\underline{X})^K_D$ shows:

Prop: Each $F_I \in W$.

Cor: $\Omega(\underline{CX},\underline{X})^K \simeq$ product of loop suspensions. (K shifted, etc).

& $\Omega(\underline{X},\underline{*})^K \simeq \prod_{i=1}^m \Omega X_i \times \Omega(\underline{C\Omega X},\underline{\Omega X})^K$

some control row.

Further: special case of $K = m$ disjoint pts

$$F_{[m]} \xrightarrow{f} \bigvee_{i=1}^m X_i \longrightarrow P^m(X)$$

satisfies: ① $F_{[m]} \in W$

② f is htpic to a sum of iterated Whitehead products of length $\geq m$ involving all ΩX_i for $1 \leq i \leq m$

③ Any length $\geq m$ Wh prod involving all ΩX_i factors through f.

Return to cocategory and nilpotence.

<u>Thm</u>: $\mathrm{nil}(\Omega X) = m-1 \iff \mathrm{wcocat}(X) = m-1$.

<u>Pf</u>: ($\Rightarrow$) $\mathrm{nil}(\Omega X) = m-1 \Rightarrow$ all length m Samelson products on ΩX vanish

$\Rightarrow$ " " Whitehead " X ".

$$\Rightarrow \begin{tikzcd} F[m] \arrow[r, "f"] & \bigvee_{i=1}^{m} X \arrow[d, "\mathrm{fold}"] \\ & X \end{tikzcd} \quad \text{is null htpic}$$

$\Rightarrow \mathrm{wcocat}(X) \leq m-1$.

($\Leftarrow$) $\mathrm{wcocat}(X) = m-1 \Rightarrow$

$$\begin{tikzcd} F[m] \arrow[r, "f"] & \bigvee_{i=1}^{m} X \arrow[d, "\nabla = \mathrm{fold}"] \\ & X \end{tikzcd} \quad \text{is null htpic.}$$

Given a Samelson prod of length m on ΩX.

Adjoint to get a Whitehead prod of length m on X.

Any such Wh prod lifts through ∇ to one on $\bigvee_{i=1}^{m} X$

involving all m variables. By ③ this lifts through f.

But $\nabla \circ f \simeq *$ so the original Sam. prod is trivial.

Hence $\mathrm{nil}(\Omega X) \leq m-1$.

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