Discrete Morse Theory and Classifying Spaces of 2-Categories

— Workshop on Applied Topology at IMS —

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August 21, 2015

What is Discrete Morse Theory?

- Robin Forman discovered a discrete analogue of Morse theory in 1995.
- The notion of discrete Morse function on a regular cell complex was introduced.
  - critical cells
  - gradient vector fields

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- Given a discrete Morse function f on a finite regular cell complex K, we can deform it to a cell complex $K(f)$ whose cells are in one-to-one correspondence with critical cells of f.
- We can also deform the cellular chain complex $C_*(X)$ to a chain complex $C(f)_*$ generated by critical cells whose boundary operators are described by gradient flows.

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- Given a discrete Morse function f on a finite regular cell complex K, we can deform it to a cell complex $K(f)$ whose cells are in one-to-one correspondence with critical cells of f.
- We can also deform the cellular chain complex $C_*(X)$ to a chain complex $C(f)_*$ generated by critical cells whose boundary operators are described by gradient flows.
- Discrete Morse theory is useful for reducing the number of cells or generators.

This Talk

Theorem

For a "good" discrete Morse function f on a finite regular cell complex K, there exists a small 2*-category* $C(f)$ *whose set of objects is* $\mathrm{Crit}(f)$ *and whose classifying space is homotopy equivalent to K.*

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Motivations?

- Explicit functorial construction of a cell complex $K(f)$ obtained from a discrete Morse function f.
- A discrete analogue of Cohen-Jones-Segal Morse theory.
- Correct notion of discrete gradient flows.
- Ghrist's question: discrete Morse theory for non-acyclic partial matchings?
- Appearance of higher categories in topological combinatorics.

Outline

Forman's Discrete Morse Theory

Flows

The 2-Category of Flows

Sketch of Proof

Forman's Discrete Morse Theory

Discrete Morse Function

Let K be a regular cell complex.

- The face poset of K is denoted by $F(K)$ with partial order $e \leq e' \Leftrightarrow e \subset \overline{e'}$.
- When $e < e'$ and $\dim e' = \dim e + 1$, we denote $e <_1 e'$.

Discrete Morse Function

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- When $e < e'$ and $\dim e' = \dim e + 1$, we denote $e <_1 e'$.
- A function $f\colon F(K) \to \mathbb{R}$ is called a *discrete Morse function* if there exists a decomposition

$$F(K) = D(f) \amalg \mathrm{Crit}(f) \amalg U(f)$$

  such that,

  1. for any $e \in D(f)$, there exists a unique $e' \in U(f)$ such that $e <_1 e'$ and $f(e) \geq f(e')$,
  2. for any $e \in U(f)$, there exists a unique $e' \in D(f)$ such that $e' <_1 e$ and $f(e') \geq f(e)$,
  3. for any $c \in \mathrm{Crit}(f)$ and $e' <_1 e$, we have $f(e') < f(e)$, and
  4. for any $c \in \mathrm{Crit}(f)$ and $e <_1 e'$, we have $f(e) < f(e')$.
- Cells in $\mathrm{Crit}(f)$ are called *critical*.

Discrete Morse Function

- The pairs of codimension 1 cells on which f is not order preserving are matched.

Definition

The one-to-one correspondence between $D(f)$ and $U(f)$ in the definition of discrete Morse function is denoted by

$$\mu_f : D(f) \longrightarrow U(f).$$

This is called the *(partial) matching* induced by f.

Discrete Morse Function: Examples

Example

The dimension function $\dim : F(K) \to \mathbb{Z} \subset \mathbb{R}$ is a discrete Morse function with $D(\dim) = \emptyset$, $U(\dim) = \emptyset$, and $\mathrm{Crit}(\dim) = F(K)$.

Example

Let $K = \partial[v_0, v_1, v_2]$. Define

$$\begin{aligned} f([v_0]) &= 0 \\ f([v_0, v_1]) = f([v_0, v_2]) &= 1 \\ f([v_1]) = f([v_2]) &= 2 \\ f([v_1, v_2]) &= 3. \end{aligned}$$

Then this is a discrete Morse function with

$$\mathrm{Crit}(f) = \{[v_0], [v_1, v_2]\}.$$

Discrete Morse Function: Examples

Example

Let $K = \partial[v_0, v_1, v_2, v_3]$. Define

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Then this is a discrete Morse function with

$$\mathrm{Crit}(f) = \{[v_0], [v_1, v_2, v_3]\}.$$

Acyclic Partial Matching

- The induced matching $\mu_f : D(f) \to U(f)$ indicates the "direction" we can collapse cells without changing the homotopy type.

Theorem

The matching μ_f induced by a discrete Morse matching is acyclic, i.e. there is no sequence of the form

$$e_1 <_1 \mu_f(e_1) >_1 e_2 <_1 \mu_f(e_2) >_1 \cdots >_1 e_n <_1 \mu_f(e_n) >_1 e_1$$

with $n \geq 2$ and all e_i's distinct.

- In other words, there is no oriented cycle in the Hasse diagram of the face poset $F(K)$ with arrows corresponding to matchted pairs inverted.

Acyclic Partial Matching

Theorem

For a partial matching $\mu : D \to U$ *on* $F(K)$,
$\mu = \mu_f$ *for a discrete Morse function* $f \iff \mu$ *is acyclic.*

- Forman regarded the acyclic partial matching μ_f as an analogue of gradient vector field.
- He introduced the notion of gradient flows to describe homology.

Flows

Forman Flows

- A sequence of cells

$$c >_1 e_1 <_1 \mu_f(e_1) >_1 \cdots <_1 e_n >_1 \mu_f(e_n) >_1 c'$$

gives rise to a decreasing sequence of real numbers

$$f(c) > f(e_1) \geq f(\mu_f(e_1)) > f(e_2) \geq f(\mu_f(e_2)) > \cdots > f(e_n) \geq f(\mu_f(e_n)) > f(c').$$

- This kind of sequence of cells can be regarded as a gradient flow.
- The set of gradient flows from a critical cell c to another c' is denoted by $\Gamma(c, c')$.

Algebraic Morse Theory

The notion of gradient flows plays an essential role in "algebraic Morse theory".

Theorem

For a discrete Morse function f on a finite regular cell complex K, let $C_n(f)$ be the free Abelian group generated by the critical cells of dimension n. Then there exist homomorphisms

$$\partial_n : C_n(f) \longrightarrow C_{n-1}(f)$$

such that

- $C_*(f) = \{C_n(f), \partial_n\}$ *is a chain complex.*
- $C_*(f)$ *is chain homotopy equivalent to* $C_*(K)$.
- $\partial_n(c) = \sum_{c' \in \mathrm{Crit}_{n-1}(f)} \sum_{\gamma \in \Gamma(c,c')} m(\gamma) c'$.

Flow Paths

- In order to obtain homotopy type instead of homology, we need relations among all cells.

Definition

A *flow path* γ with respect to a partial matching μ ending at a critical cell c is a sequence

$$\gamma = (e_1, u_1, \ldots, e_n, u_n; c = e_{n+1})$$

of distinct cells satisfying the following conditions:

1. $u_i \in U(\mu)$ for $1 \leq i \leq n$.
2. Either $e_i = u_i$ or $e_i = \mu^{-1}(u_i)$.
3. $u_i > e_{i+1}$ for $1 \leq i \leq n$.

We denote $\ell(\gamma) = n$.
The set of all flow paths is denoted by $\mathrm{FP}(\mu)$.

Flow Paths

A flow path $\gamma = (e_1, u_1, \ldots, e_n, u_n; c)$ can be regarded as a sequence

$$\begin{aligned} u_1 > \cdots > u_{i_1 - 1} \\ > \mu^{-1}(u_{i_1}) < u_{i_1} > \cdots > u_{i_2 - 1} \\ > \mu^{-1}(u_{i_2}) < u_{i_2} > \cdots > u_n > c \end{aligned}$$

or

$$\begin{aligned} \mu^{-1}(u_1) < u_1 > \cdots > u_{i_1 - 1} \\ > \mu^{-1}(u_{i_1}) < u_{i_1} > \cdots > u_{i_2 - 1} \\ > \mu^{-1}(u_{i_2}) < u_{i_2} > \cdots > u_n > c. \end{aligned}$$

Faithful Morse Function

Definition

A discrete Morse function $f\colon F(K) \to \mathbb{R}$ is said to be *faithful* if

1. f is injective.
2. If $e < e'$ and $e' \neq \mu_f(e)$, then $f(e) < f(e')$.

Proposiiton

For any discrete Morse function $f\colon F(K) \to \mathbb{R}$, there exists a faithful discrete Morse function $g\colon F(K) \to \mathbb{R}$ with $\mathrm{Crit}(f) = \mathrm{Crit}(g)$, $D(f) = D(g)$, $U(f) = U(g)$ and $\mu_f = \mu_g$.

Gradient Flows

If $\mu = \mu_f$ for a faithful discrete Morse function f, for a flow path $\gamma = (e; e_1, u_1, \ldots, e_n, u_n; e')$, the sequence

$$\begin{aligned} & u_1 > \cdots > u_{i_1 - 1} \\ & \qquad > \mu^{-1}(u_{i_1}) < u_{i_1} > \cdots > u_{i_2 - 1} \\ & \qquad\qquad > \mu^{-1}(u_{i_2}) < u_{i_2} > \cdots > u_n > c \end{aligned}$$

implies

$$\begin{aligned} & f(u_1) > \cdots > f(u_{i_1 - 1}) \\ & \qquad > f(\mu^{-1}(u_{i_1})) \geq f(u_{i_1}) > \cdots > f(u_{i_2 - 1}) \\ & \qquad\qquad > f(\mu^{-1}(u_{i_2})) \geq f(u_{i_2}) > \cdots > f(u_n) > f(c) \end{aligned}$$

and a similar sequence for the other case.

The Category of Flows

Definition

For a flow path $\gamma = (e_1, u_1, \ldots, e_n, u_n; c)$, we denote

$$\begin{aligned} \iota(\gamma) &= e_1 \\ \tau(\gamma) &= c. \end{aligned}$$

Definition

Given a partial matching μ, define a category $C(\mu)$ as follows.

- Objects are critical cells: $C(\mu)_0 = \mathrm{Crit}(\mu)$.
- For $c, c' \in C(\mu)_0$,

$$C(\mu)(c, c') = \left\{ \gamma \in \mathrm{FP}(\mu) \,\middle|\, c' > \iota(\gamma), c = \tau(\gamma) \right\}.$$

- Compositions are given by concatenations.
- Identity morphisms are given by $1_c = (c; c)$.

Example

Let f be the discrete Morse function on $K = \partial[v_0, v_1, v_2]$ discussed previously. We have

$$\mathrm{Crit}(f) = \{[v_0], [v_1, v_2]\}.$$

There are two flow paths from $[v_1, v_2]$ to $[v_0]$:

$$\begin{aligned}\gamma_1 &= ([v_1], [v_0, v_1]; [v_0]) \\ \gamma_2 &= ([v_2], [v_0, v_2]; [v_0]).\end{aligned}$$

And

$$\begin{aligned}C(\mu_f)([v_1, v_2], [v_1, v_2]) &= \{([v_1, v_2]; [v_1, v_2])\} = \{\gamma_{12}\} \\ C(\mu_f)([v_0], [v_0]) &= \{([v_0]; [v_0])\} = \{\gamma_0\}.\end{aligned}$$

Thus

$$BC(\mu_f) \cong S^1 \cong \partial[v_0, v_1, v_2].$$

Example

There are two more flow paths

$$\gamma_{01} = ([v_0, v_1], [v_0, v_1]; [v_0])$$
$$\gamma_{02} = ([v_0, v_2], [v_0, v_2]; [v_0])$$

and

$$\mathrm{FP}(\mu_f) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_{01}, \gamma_{02}, \gamma_{12}\}.$$

Define a partial order by

$$\gamma_1, \gamma_2 < \gamma_{12}$$
$$\gamma_0, \gamma_1 < \gamma_{01}$$
$$\gamma_0, \gamma_2 < \gamma_{02}.$$

Then

$$B\mathrm{FP}(\mu_f) \cong \mathrm{Sd}(\partial[v_0, v_1, v_2]).$$

Example

The target map $\tau : \mathrm{FP}(\mu_f) \to \mathrm{Crit}(\mu_f)$ can be regarded as a functor

$$\tau : \mathrm{FP}(\mu_f) \longrightarrow C(\mu_f).$$

which induces a homotopy equivalence

$$B\tau : B\mathrm{FP}(\mu_f) \xrightarrow{\simeq} BC(\mu_f).$$

- In general, each morphism set $C(\mu)(c, c')$ should be regarded as a poset or a small category.
- Thus $C(\mu)$ should be defined as a 2-category.

The 2-Category of Flows

Morphisms between Flows

Definition

For flow paths

$$\begin{aligned}\gamma &= (e_1, u_1, \ldots, e_n, u_n; c)\\ \gamma' &= (e'_1, u'_1, \ldots, e'_{n'}, u'_{n'}; c'),\end{aligned}$$

a morphism from γ to γ' is a strictly increasing function

$$\varphi : \{1, \ldots, k\} \longrightarrow \{1, \ldots, \ell(\gamma') + 1\}$$

satisfying the following conditions

1. $u_i = u'_{\varphi(i)}$ for $1 \leq i < k$.
2. $\varphi(k) = \ell(\gamma') + 1$.
3. For each $1 \leq j \leq k$, $e_j \leq e'_p$ for all $\varphi(j-1) < p \leq \varphi(j)$, where $\varphi(0) = 0$.

We denote $|\varphi| = k$.

Morphisms between Flows

For flow paths $\gamma, \gamma', \gamma''$ and morphisms $\varphi : \gamma \to \gamma'$ and $\varphi' : \gamma' \to \gamma''$, the composition $\varphi' \bullet \varphi : \{1, \dots, r\} \to \{1, \dots, \ell(\gamma'') + 1\}$ is defined by

$$(\varphi' \bullet \varphi)(i) = \begin{cases} \varphi'(\varphi(i)), & i < r \\ \varphi'(|\varphi|), & i = r, \end{cases}$$

where r is the unique number with $\varphi(r-1) < |\varphi'| < \varphi(r)$.

Proposiiton

For any partial matching μ, we obtain a category of flow paths $\mathbf{FP}(\mu)$.

Proposiiton

If $\mu = \mu_f$ for a faithful discrete Morse function f, then $\mathbf{FP}(\mu)$ is a poset.

The Flow Category

For $c, c' \in \mathrm{Crit}(f)$, regard $C(f)(c, c')$ as a full subcategory of $\mathbf{FP}(\mu)$. Then the concatenation of flow paths induces a functor

$$\circ : C(\mu)(c', c'') \times C(\mu)(c, c') \longrightarrow C(\mu)(c, c'')$$

and we obtain a 2-category $C(\mu)$

Definition

This 2-category $C(\mu)$ is called the *flow category* of μ.

The classifying space of $C(\mu)$

Definition

Define a topological category $BC(\mu)$ by

- $BC(\mu)_0 = C(\mu)_0 = \mathrm{Crit}(\mu)$
- $BC(\mu)(c, c') = B(C(\mu)(c, c'))$.

The composition of morphisms

$$\circ : BC(\mu)(c', c'') \times BC(\mu)(c, c') \longrightarrow BC(\mu)(c, c'')$$

induced by

$$\circ : C(\mu)(c', c'') \times C(\mu)(c, c') \longrightarrow C(\mu)(c, c'').$$

The classifying space of this topological category is denoted by $B^2C(\mu)$.

Cohen-Jones-Segal Morse Theory

- Ralph Cohen, John Jones, and Graeme Segal wrote a paper "Morse theory and classifying spaces" in early 90s.
- Given a Morse function $f : M \to \mathbb{R}$ on a closed manifold M, they constructed a topological category $C(f)$ whose objects are $\mathrm{Crit}(f)$. A morphism from c to c' is a gradient flow from c' to c.
- They claimed that, if f is Morse-Smale, $BC(f)$ is homeomorphic to M.

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- They claimed that, if f is Morse-Smale, $BC(f)$ is homeomorphic to M.
- But the paper has not been published yet.

Sketch of Proof

Outline of Proof

Theorem

For a faithful discrete Morse function f on a finite regular cell complex K, we have $B^2 C(f) \simeq K$.

1. Define a normal colax functor
$$\tau : \mathbf{FP}(f) \longrightarrow C(f)$$
by $(e_1, u_1, \dots, e_n, u_n; c) \mapsto c$.
2. Show that the "homotopy fibers" $c \downarrow \tau$ of τ are contractible.
3. Use Quillen's theorem A for colax functors between 2-categories to obtain a homotopy equivalence
$$B\tau : B^{ncl}\mathbf{FP}(f) = B\mathbf{FP}(f) \xrightarrow{\simeq} B^{ncl}C(f).$$
4. In general, $B^{ncl}C(f) \simeq B^2 C(f)$.
5. Show that
$$B\mathbf{FP}(f) \simeq K.$$

Theorem A for Colax Functors

Theorem (Bullejos-Cegarra 2003, del Hoyo 2012)

Let $f\colon C \to D$ be a normal colax functor between 2-categories. Suppose that f is prefibered in the sense that the canonical functor $i_y : f^{-1}(y) \to y \downarrow f$ has a right adjoint for each $y \in D_0$. Suppose further that $Bf^{-1}(f)$ is contractible for all $y \in D_0$. Then

$$B^{ncl}f\colon B^{ncl}C \longrightarrow B^{ncl}D$$

is a homotopy equivalence.

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$$B^{ncl} f\colon B^{ncl} C \longrightarrow B^{ncl} D$$

is a homotopy equivalence.

Theorem

The collapsing functor $\tau : \mathbf{FP}(f) \to C(f)$ is a prefibered normal colax functor with $B\tau^{-1}(c)$ contractible for all $c \in \mathrm{Crit}(f)$. Thus it induces a homotopy equivalence

$$B^{ncl}\tau : B^{ncl}\mathbf{FP}(f) \xrightarrow{\simeq} B^{ncl} C(f).$$

Classifying Spaces of 2-Categories

- There are many ways to take a classifying space of a 2-category.
- Carrasco, Cegarra, and Garzón [Algebr. Geom. Topol. 2010] compared 10 definitions of classifying spaces of 2-categories and showed that they are all homotopy equivalent.
- In particular,

$$B^{ncl} C(\vec{f}) \simeq B^2 C(\vec{f}).$$

Proof of Step 5

1. Define a flow path $(e_1, u_1, \ldots, e_n, u_n; c)$ to be *reduced* iff $e_{i+1} \not< \mu^{-1}(u_i)$.
   The subposet of reduced flow paths is denoted by $\overline{\mathbf{FP}}(f)$.
2. There is a reduction map
   $$r : \mathbf{FP}(f) \longrightarrow \overline{\mathbf{FP}}(f),$$
   whose composition with the inclusion
   $$\mathbf{FP}(f) \longrightarrow \overline{\mathbf{FP}}(f) \longrightarrow \mathbf{FP}(f)$$
   is a descending closure operator. In particular
   $$B\mathbf{FP}(f) \simeq B\overline{\mathbf{FP}}(f).$$

Proof of Step 5

3. We also have a subcategory $\overline{C}(f)$ of $C(f)$ consisting of reduced flow paths with the reduction functor

$$r : C(f) \longrightarrow \overline{C}(f).$$

4. For each pair $c, c' \, \mathrm{Crit}(f)$,

$$Br : BC(f)(c, c') \longrightarrow B\overline{C}(f)(c, c')$$

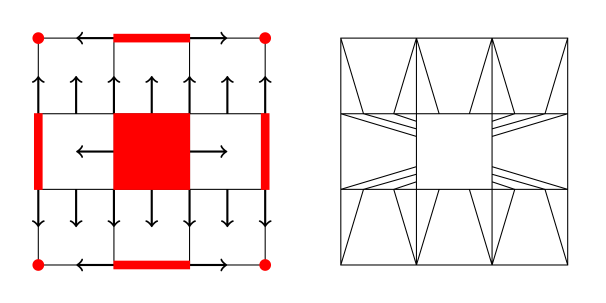
is a deformation retraction. Thus it induces a homotopy equivalence

$$B^2 r : B^2 C(f) \xrightarrow{\simeq} B^2 \overline{C}(f).$$

Proof of Step 5

5. Construct a subdivision $\mathrm{Sd}_f K$ of K whose face poset $F(\mathrm{Sd}_f K)$ is isomorphic to $\overline{\mathbf{FP}}(f)$. Then

$$B\overline{\mathbf{FP}}(f) \simeq BF(\mathrm{Sd}_f K) = \mathrm{Sd}(\mathrm{Sd}_f(K)) \cong K.$$



Work in Progress

- Cell decomposition?
  1. In Forman's discrete Morse theory, a cell complex $K(f)$ whose cells are in one-to-one correspondence to critical cells of f was constructed.
  2. We do have a decomposition of $B^{ncl}C(f)$ into contractible spaces indexed by $\mathrm{Crit}(f)$.

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  2. We do have a decomposition of $B^{ncl}C(f)$ into contractible spaces indexed by $\mathrm{Crit}(f)$.
- Hidetaka Tokuno is trying to remove the acyclicity condition.
- Vidit Nanda is trying to find a more abstract proof by using localizations of 2-categories.