Embedding theorems for quasitoric manifolds

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Abstract

The talk is based on joint work with Andrey Kustarev.

The central subject are theorems on equivariant embeddings of quasitoric manifolds in terms of combinatorial data.

One of our tasks is to improve the classical theorems in the case when combinatorial data defines the corresponding structure on the underlying quasitoric manifold.

Manifolds with an action of compact group

Theorem (Mostow-Palais, 1957)

Let M be a compact smooth manifold with a smooth action of a compact Lie group G.

Then there exists a smooth embedding $M \to \mathbb{R}^N$ equivariant with respect to a linear representation $G \to GL(N, \mathbb{R})$.

See:

Mostow, George D.,
Equivariant embeddings in Euclidean space,
Annals of Mathematics, Second Series 65: 432-446, 1957.

Palais, Richard S,
Imbedding of compact, differentiable transformation groups in orthogonal representations, J. Math. Mech. 6: 673-678, 1957.

Complex manifolds

Theorem (Kodaira, 1954)

Let M be a compact complex manifold with a positive holomorphic linear bundle
(for example, M possesses a rational Kähler form).

Then there exists a complex-analytic embedding $M \to \mathbb{C}P^N$.

See:
Kodaira, Kunihiko,
On Kahler varieties of restricted type
(an intrinsic characterization of algebraic varieties),
Annals of Mathematics. Second Series 60 (1): 28-48 (1954).

Compact symplectic manifolds

Theorem (Gromov-Tishler, 1970 - 1977)

Let M be a compact symplectic manifold with an integral symplectic form ω.

Then there exists a symplectic embedding of M to $\mathbb{C}P^N$ with the standard symplectic form.

See:

M. Gromov,
A topological technique for the construction of solutions of differential equations and inequalities,
Actes, Congres intern. Math., Tome 2, pages 221-225, 1970.

D. Tischler,
Closed 2-forms and an embedding theorem for symplectic manifolds,
Journal of Differential Geometry, (12):229-235, 1977.

Notation

$\mathbb{C}^m$ is the standard complex linear m-dimensional space endowed with the canonical basis $e_1 = (1, 0, \ldots, 0), \ldots, e_m = (0, \ldots, 0, 1)$;

$\mathbb{R}^m \subset \mathbb{C}^m$ is the standard real linear space generated by $e_1, \ldots, e_m$;

$\mathbb{R}^m_{\geqslant} \subset \mathbb{R}^m$ is the positive cone i.e. the area formed by all points in $\mathbb{R}^m$ with nonnegative coordinates;

$\mathbb{Z}^m \subset \mathbb{R}^m$ is the standard lattice generated by $e_1, \ldots, e_m$;
Set

$$D^2 = \{z \in \mathbb{C}; |z| \leq 1\},$$

$$S^1 = \{z \in D^2, |z| = 1\}.$$

Notation

$\mathbb{T}^m \subset \mathbb{C}^m$ is the standard compact torus
$\{(t_1, \dots, t_m) \in \mathbb{C}^m \colon |t_k| = 1,\ k \in [1, m]\}$;

the map $exp \colon \mathbb{R}^m \to \mathbb{T}^m$ given by the formula
$exp(x_1, \dots, x_m) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_m})$ induces
the canonical isomorphism of $\mathbb{T}^m$ and $\mathbb{R}^m/\mathbb{Z}^m$.

The standard k-dimensional compact torus is denoted by $\mathbb{T}^k$ and
an abstract k-dimensional toric subgroup in the standard torus $\mathbb{T}^m$ is
denoted by T^k;

$\mathbb{T}_\omega \subset \mathbb{T}^m$ is a toric subgroup corresponding to an index set
$\omega \subset [1, m]$;

the one point set $\omega = \{i\}$ defines a coordinate torus $\mathbb{T}_i \subset \mathbb{T}^m$;

Notation

$\rho\colon \mathbb{C}^m \to \mathbb{R}^m_{\geqslant}$ is the standard moment map given by the formula

$$\rho(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2); \tag{1}$$

$s\colon \mathbb{R}^m_{\geqslant} \to \mathbb{C}^m$ is the map given by the formula

$$s(x_1, \dots, x_m) = (\sqrt{x_1}, \dots, \sqrt{x_m}), \tag{2}$$

note that $\rho \circ s = id$.

Consider a simple polytope P.
Denote by f_k the number of k-dimensional faces of P. Set $m = f_{n-1}$.
We have the face lattice $L(P)$ of P.
Let $\{F_1, \dots, F_m\}$ be the set of facets and $\{v_1, \dots, v_{f_0}\}$ – the set of vertices. For any face $F \in L(P)$, $F \neq \varnothing$ set

$$\mathcal{Z}_{P,F} = \prod_{i:\ F_i \supset F} D_i^2 \times \prod_{j:\ F_j \not\supset F} S_j^1 \subset \mathbb{D}^{2m}.$$

The moment-angle complex of a simple polytope P is

$$\mathcal{Z}_P = \bigcup_{F \in L(P) \setminus \{\varnothing\}} \mathcal{Z}_{P,F}$$

Vertex partition of a moment-angle complex

Lemma

There is a homeomorphism:

$$\mathcal{Z}_P = \bigcup_{i=1}^{f_0} \mathcal{Z}_{P,v_i}$$

- Let $P = I = [0;1]$.
  Then $\mathcal{Z}_I = \mathcal{Z}_{I,v_0} \cup \mathcal{Z}_{I,v_1} = (\mathbb{D}^2 \times S^1) \cup (S^1 \times \mathbb{D}^2)$
- $P = \Delta^n \Longleftrightarrow \mathcal{Z}_P = S^{2n+1}$.

The space $\mathcal{Z}_P$ has the canonical structure of a multigraded subcomplex in $\mathbb{D}^{2m}$.
The embedding $\mathcal{Z}_P \subset \mathbb{D}^{2m}$ is $\mathbb{T}^m$ – equivariant.

Theorem

There is a homeomorphism $P \cong \mathcal{Z}_P/\mathbb{T}^m$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Z}_P & \longrightarrow & \mathbb{D}^{2m} \\ \downarrow & & \downarrow \\ P & \longrightarrow & \mathbb{I}^m \end{array}$$

Construction of a moment-angle manifold

Take a simple polytope

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{a}_i\boldsymbol{x} + b_i \geqslant 0, i = 1, \ldots, m\}$$

Using the embedding

$$j_P \colon P \longrightarrow \mathbb{R}^m_{\geq} \ : \ j_P(x) = (y_1, \ldots, y_m)$$

where $y_i = \boldsymbol{a}_i\boldsymbol{x} + b_i$, we will consider P as the subset in $\mathbb{R}^m_{\geqslant}$.
A moment-angle manifold $\hat{\mathcal{Z}}_P$ is the product
of $\mathbb{C}^m$ and P over $\mathbb{R}^m_{\geqslant}$ described by the pullback diagram:

$$\begin{array}{ccc} \hat{\mathcal{Z}}_P & \xrightarrow{j_{\mathcal{Z}}} & \mathbb{C}^m \\ {\scriptstyle \rho_P}\downarrow & & \downarrow{\scriptstyle \rho} \\ P & \xrightarrow{j_P} & \mathbb{R}^m_{\geqslant} \end{array}$$

where $\rho(z_1, \ldots, z_m) = \big(|z_1|^2, \ldots, |z_m|^2\big)$.

Moment-angle manifold

We have

$$\hat{\mathcal{Z}}_P = \{(x, z) \in P \times \mathbb{C}^m : j_P(x) = \rho(z)\}$$

Here $\rho(z) = (|z_1|^2, \ldots, |z_m|^2)$ and $j_P(x) = Ax + b$,
where A is a $(m \times n)$-matrix, rank$A = n$ and $b \in \mathbb{R}^m$.
Mapping j_P is an embedding because rank of A is n.
Thus mapping $(x, z) \to z$ gives the embedding $\hat{\mathcal{Z}}_P \subset \mathbb{C}^m$.
Denote by C such $((m - n) \times m)$-matrix that $CA = 0$ and rank of C is $m - n$.
Consider the map

$$\Phi : \mathbb{C}^m \to \mathbb{R}^{m-n} : \ \Phi(z) = C(\rho(z) - b).$$

It is a $\mathbb{T}^m$-equivariant quadratic map with respect to the trivial action of $\mathbb{T}^m$ on $\mathbb{R}^{m-n}$.

Set $\Phi(z) = (\Phi_1(z), \ldots, \Phi_{m-n}(z))$.

Theorem (Buchstaber-Panov-Ray, 2007)

For any simple polytope P there is a homeomorphism $\hat{\mathcal{Z}}_P \to \mathcal{Z}_P$, using it in what follows we identify $\hat{\mathcal{Z}}_P$ and $\mathcal{Z}_P$.

- *$\mathcal{Z}_P$ is a complete intersection of real quadratic hypersurfaces in $\mathbb{R}^{2m} \cong \mathbb{C}^m$.*

$$\mathcal{F}_k = \{z \in \mathbb{C}^m : \Phi_k(z) = 0\},\ k = 1, \ldots, m-n.$$

- *There is a trivialisation of the normal bundle of the $\mathbb{T}^m$-equivariant embedding $\mathcal{Z}_P \subset \mathbb{C}^m$* **that is**
- *$\mathcal{Z}_P$ has the structures of a framed manifold.*

Let $H \subset \mathbb{T}^m$ be a subgroup of dimension $r \leqslant m - n$.
Choosing a basis, we can write it in the form

$$H = \left\{ \left(e^{2\pi i (s_{11}\varphi_1 + \dots + s_{1r}\varphi_r)}, \dots, e^{2\pi i (s_{m1}\varphi_1 + \dots + s_{mr}\varphi_r)} \right) \in \mathbb{T}^m \right\},$$

where $\varphi_i \in \mathbb{R}$, $i = 1, \dots, r$ and $S = (s_{ij})$ is an integral $(m \times r)$-matrix which defines a monomorphism $\mathbb{Z}^r \to \mathbb{Z}^m$.
For any subset $\omega = \{i_1, \dots, i_n\} \subset [m]$ denote by $S_{\hat{\omega}}$ the $((m-n) \times r)$-submatrix of S obtained by deleting the rows $i_1, \dots, i_n$.

Freely acting subgroups

Write each vertex $v \in P^n$ as v_ω if $v = F_{i_1} \cap \ldots \cap F_{i_n}$

Lemma

The subgroup H acts freely on $\mathcal{Z}_P$ if and only if for every vertex v_ω the $(m-n) \times r$ submatrix $S_{\hat{\omega}}$ defines a monomorphism $\mathbb{Z}^r \hookrightarrow \mathbb{Z}^{m-n}$ onto a direct summand.

Corollary

The subgroup H of rank $r = m - n$ acts freely on $\mathcal{Z}_P$ if and only if for any vertex $v_\omega = F_{i_1} \cap \ldots \cap F_{i_n}$ of P^n,

$$\det S_{\hat{\omega}} = \pm 1.$$

Characteristic map

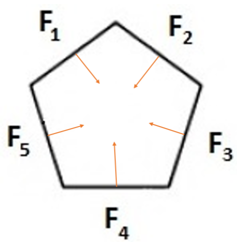
Definition

An $(n \times m)$-matrix Λ gives a characteristic map

$$\ell\colon \{F_1, \ldots, F_m\} \longrightarrow \mathbb{Z}^n$$

for a given simple polytope P^n with facets $\{F_1, \ldots, F_m\}$ if the column-vectors $\lambda_{j_1}, \ldots, \lambda_{j_n}$ of Λ corresponding to any vertex v_ω form a basis for $\mathbb{Z}^n$.

Example: P^2_5, $\Lambda = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$



The problem

By permuting the facets of P if necessary, we may assume that the intersection $F_1 \cap \cdots \cap F_n$ is the vertex v_*.
Associate to a simple n-dim polytope P an integral $(n \times m)$-matrix

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \dots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix},$$

in which the column $\lambda_j = (\lambda_{1j}, \dots, \lambda_{nj})$ corresponds to the facet F_j, $j = 1, \dots, m$, and the columns $\lambda_{j_1}, \dots, \lambda_{j_n}$ corresponding to any vertex $v_\omega = F_{j_1} \cap \cdots \cap F_{j_n}$ are required to form a basis for $\mathbb{Z}^n$.

Combinatorial data

Definition

The combinatorial quasitoric data (P, Λ) consists of an oriented combinatorial simple polytope P and an integral $(n \times m)$-matrix Λ which gives a characteristic map.

The matrix Λ defines an epimorphism

$$\ell \colon \mathbb{T}^m \to \mathbb{T}^n.$$

The kernel of ℓ (which we denote $K(\Lambda)$) is isomorphic to $\mathbb{T}^{m-n}$. The action of $K(\Lambda)$ on $\mathcal{Z}_P$ is free due to the condition on the minors of Λ.

Quasitoric manifold with structure

Construction

The quotient $M = \mathcal{Z}_P / K(\Lambda)$ is a $2n$-dimensional smooth manifold with an action of the n-dimensional torus $\mathbb{T}^m / K(\Lambda)$. We denote this action by α.
It satisfies the *Davis–Januszkiewicz' conditions*:

1. α is locally isomorphic to the standard coordinatewise representation of $\mathbb{T}^n$ in $\mathbb{C}^n$,
2. there is a projection $\pi\colon M \to P$ whose fibres are orbits of α.

We refer to $M = M(P, \Lambda)$ as the *quasitoric manifold associated with the combinatorial data* (P, Λ).

Let

$$P = \{x \in \mathbb{R}^n \colon Ax + b \geqslant 0\}.$$

Definition

The manifold $M = M(P, \Lambda)$ is called the quasitoric manifold with (A, Λ)-structure.

Consider the standard complex projective space

$$\mathbb{C}P^n = \{(z_1 : \cdots : z_{n+1}), z \in \mathbb{C}^{n+1}, |z| = 1\}$$

with the standard action of torus $\mathbb{T}^{n+1}$. Let $M(n+1, \mathbb{C})$ be the space of $(n+1) \times (n+1)$-matrices with the following action of $\mathbb{T}^{n+1}$

$$t \circ M = t^\top M \bar{t}.$$

The mapping $\mathbb{C}P^n \to M(n+1, \mathbb{C})$

$$(z_1 : \cdots : z_{n+1}) \to z\overline{z}^\top,$$

where z is a column vector, gives the $\mathbb{T}^{n+1}$-equivariant embedding $\mathbb{C}P^n \to \mathbb{R}^{n+1}_{\geqslant} \times \mathbb{C}^N$, $N = \frac{n(n+1)}{2}$.

Example

$$\mathbb{C}P^1 \to \mathbb{R}^2_{\geqslant} \times \mathbb{C},$$

$$(z_1 : z_2) \to (|z_1|^2, |z_2|^2, z_1\overline{z_2}).$$

Monomial functions

Every vector $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ determines
a real-algebraic monomial function:

$$\varphi_a \colon \mathbb{C}^m \to \mathbb{C},$$
$$\varphi_a(z_1, \ldots, z_m) = \hat{z}_1^{a_1} \cdot \ldots \cdot \hat{z}_m^{a_m},$$

where

- $\hat{z}_i^{a_i} = 1$ if $a_i = 0$,
- $\hat{z}_i^{a_i} = z_i^{a_i}$ if $a_i > 0$,
- $\hat{z}_i^{a_i} = \overline{z}_i^{-a_i}$ if $a_i < 0$.

Example

The vector $a = (1, 0, -1) \in \mathbb{Z}^3$ gives a function

$$\varphi_a \colon \mathbb{C}^3 \to \mathbb{C},$$

$$\varphi_a(z_1, z_2, z_3) = z_1 \overline{z}_3$$

Equivariance of monomial functions

Let $t = (t_1, \ldots, t_r) \in \mathbb{T}^r$, $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ and A be an integer $(s \times r)$-matrix and A_k be the k-th row of the matrix A.

Notation:

$$t^a = t_1^{a_1} \cdot \ldots \cdot t_r^{a_r} \in \mathbb{T}^1 \textit{ and } t^A = (t^{A_1}, \ldots, t^{A_s}) \in \mathbb{T}^s.$$

Any $a \in \mathbb{Z}^m$ gives the character $\chi_a : \mathbb{T}^m \to \mathbb{T}^1 : t \to t^a$.

Any monomial function φ_a is equivariant.
That is if $a \in \mathbb{Z}^m$ and $t \in \mathbb{T}^m$, then

$$\varphi_a(tz) = t^a \varphi_a(z)$$

for all $z \in \mathbb{C}^m$.

Example of an equivariant map

Example

Let $a = (1, 0, -1) \in \mathbb{Z}^3$ and $t = (t_1, t_2, t_3) \in \mathbb{T}^3$.
Set

$$tz = (t_1 z_1, t_2 z_2, t_3 z_3).$$

We have:

$$t^a = t_1 t_3^{-1}$$

and

$$\varphi_a \colon \mathbb{C}^3 \to \mathbb{C},$$

$$\varphi_a(tz) = t_1 z_1 \overline{(t_3 z_3)} = (t_1 t_3^{-1}) z_1 \overline{z}_3 = t^a \varphi_a(z)$$

Characteristic map and free action

Lemma

Suppose $(n \times m)$*-matrix* $\Lambda = (I_n, \Lambda_*)$ *gives a characteristic map*

$$\ell\colon \{F_1, \ldots, F_m\} \longrightarrow \mathbb{Z}^n$$

Then the matrix $S = (-\Lambda_*, I_{m-n})$ *gives the* $(m-n)$*-dimensional subgroup*

$$K(\Lambda) = \left\{(e^{2\pi i\psi_1}, \ldots, e^{2\pi i\psi_m}) \in \mathbb{T}^m\right\},\ i = \sqrt{-1},$$

which acts freely on $(m+n)$*-dimensional manifold* $\mathcal{Z}_P$*.*
Here for $\xi = (\xi_1, \cdots, \xi_{m-n}) \in \mathbb{Z}^{m-n}$ *we take*

$$\psi_k = -\sum_{j=1}^{m-n} \lambda_{k+n,j}\xi_j,\ k = 1, \ldots, n;\ \psi_{n+k} = \xi_k,\ k = 1, \ldots, m-n.$$

The commutative diagram of a characteristic map

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{m-n} & \xrightarrow{S} & \mathbb{Z}^m & \xrightarrow{\Lambda} & \mathbb{Z}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{R}^{m-n} & \longrightarrow & \mathbb{R}^m & \longrightarrow & \mathbb{R}^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{T}^{m-n} & \xrightarrow{S_{\mathbb{T}}} & \mathbb{T}^m & \xrightarrow{\Lambda_{\mathbb{T}}} & \mathbb{T}^n & \longrightarrow & 0
\end{array}
$$

We have a real algebraic map

$$\varphi_\Lambda : \mathbb{C}^m \to \mathbb{C}^n : \varphi_\Lambda(z) = (\varphi_{\Lambda_1}(z), \cdots, \varphi_{\Lambda_n}(z)),$$

where Λ_k is a k-th row of Λ.

- $\varphi_\Lambda(tz) = (\Lambda_{\mathbb{T}} t)\varphi_\Lambda(z)$ for any $t \in \mathbb{T}^m$.
- $\varphi_\Lambda(tz) = \varphi_\Lambda(z) \Leftrightarrow t = S_{\mathbb{T}}\tau$ for any $\tau \in \mathbb{T}^{m-n}$.

Local embedding

Let $\hat{\varphi}_\Lambda : \mathcal{Z}_P \to \mathbb{C}^n$ be the restriction of the map $\varphi_\Lambda : \mathbb{C}^m \to \mathbb{C}^n$ to the moment-angle manifold $\mathcal{Z}_P \subset \mathbb{C}^m$.
The map $\hat{\varphi}_\Lambda$ induces the smooth map $\tilde{\varphi}_\Lambda : M(P, \Lambda) \to \mathbb{C}^n$ equivariant with respect to the representation $\mathbb{T}^m/\mathbb{T}^{m-n} \to \mathbb{T}^n$.
Let $v_* = F_1 \cap \cdots \cap F_n$. Then $\mathcal{Z}_{P,v_*} = (\mathbb{D}^2)^n \times (S^1)^{m-n} \subset \mathbb{C}^m$.

Lemma

There is the $\mathbb{T}^n$-equivariant embedding:

$$\tilde{\varphi}_\Lambda : \mathcal{Z}_{P,v_*}/K(\Lambda) \simeq (\mathbb{D}^2)^n \times (1)^{m-n} \subset \mathbb{C}^n \times (1)^{m-n},$$

$$\tilde{\varphi}_\Lambda(z_1, \cdots, z_n, 1, \cdots, 1) = (z_1, \cdots, z_n, 1, \cdots, 1).$$

Let $P = \Delta^2$. Then

$$\mathcal{Z}_{\Delta^2} = S^5 = (\mathbb{D}^2 \times \mathbb{D}^2 \times S^1) \cup (\mathbb{D}^2 \times S^1 \times \mathbb{D}^2) \cup (S^1 \times \mathbb{D}^2 \times \mathbb{D}^2),$$

$$\Lambda = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{S} \mathbb{Z}^3 \xrightarrow{\Lambda} \mathbb{Z}^2 \longrightarrow 0$$

Complex projective plane

$$\varphi_\Lambda : \mathbb{C}^3 \to \mathbb{C}^2 : \varphi_\Lambda(z) = (z_1\overline{z}_3, z_2\overline{z}_3)$$

$$\hat{\varphi}_\Lambda : \mathcal{Z}_{\Delta^2} \to \mathbb{C}^2$$

$$\tilde{\varphi}_\Lambda : \mathbb{C}P^2 \to \mathbb{C}^2 : (z_1 : z_2 : z_3) \to (z_1\overline{z_3}, z_2\overline{z_3})$$

Restrictions of the map $\tilde{\varphi}_\Lambda$ on the parts give:

$$(z_1 : z_2 : 1) \to (z_1, z_2)$$

$$(z_1 : 1 : z_3) \to (z_1\overline{z_3}, \overline{z_3})$$

$$(1 : z_2 : z_3) \to (\overline{z_3}, z_2\overline{z_3})$$

Thus we describe the embedding $\tilde{\varphi}_{\Lambda,v_1}$ for the $\mathcal{Z}_{\Delta^2,v_1}/\mathbb{T}$.
Using the similar maps $\tilde{\varphi}_{\Lambda,v_2}$ and $\tilde{\varphi}_{\Lambda,v_3}$, we obtain:

$$\begin{array}{ccc} \mathbb{C}P^2 & \longrightarrow & \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 \\ & \searrow \qquad \swarrow & \\ & \mathbb{R}^3_{\geqslant} \times \mathbb{C}^3 & \end{array}$$

Embedding theorem

Let P be a simple n-dimensional polytope.
For any vertex $v \in P$ we have the $\mathbb{T}^n$-equivariant map $\tilde{\varphi}_{\Lambda,v_k} : M(P,\Lambda) \to \mathbb{C}^n$, such that the restriction $\tilde{\varphi}_{\Lambda,v_k} : \mathcal{Z}_{P,v_k}/K(\Lambda) \to \mathbb{C}^n$ is a $\mathbb{T}^n$-equivariant embedding.

Theorem

For any combinatorial data (P,Λ) there is an equivariant real-algebraic embedding

$$\tilde{\varphi}_{\Lambda} : M(P,\Lambda) \to \mathbb{C}^n \times \cdots \times \mathbb{C}^n \cong \mathbb{C}^N,$$

where $N = f_0 n$ and $\tilde{\varphi}_{\Lambda} = \tilde{\varphi}_{\Lambda,v_1} \times \cdots \times \tilde{\varphi}_{\Lambda,v_{f_0}}$.

Monomial maps in the general case

Suppose that vectors $a_1, \ldots, a_q \in \mathbb{Z}^m$ define the characters

$$\chi_{a_1}, \cdots, \chi_{a_q},$$

such that

$$\chi_{a_1}(t) = \cdots = \chi_{a_q}(t),$$

if $t = S_{\mathbb{T}}\tau$ for any $\tau \in \mathbb{T}^{m-n}$. Thus we have the induced character

$$\hat{\chi} : \mathbb{T}^{m-n} \to \mathbb{T}^1 : \hat{\chi}(\tau) = \chi_{a_1}(S_\Lambda \tau) = \cdots = \chi_{a_q}(S_\Lambda \tau).$$

Let $\varphi\colon\ \mathcal{Z}_P \to \mathbb{C}^q$ be a restriction of the monomial maps $\varphi(z) = (\varphi_{a_1}(z), \ldots, \varphi_{a_q}(z))$ to the moment-angle manifold $\mathcal{Z}_P \subset \mathbb{C}^m$.

Monomial maps of the quasitoric manifold

- If the character $\hat{\chi}$ is trivial, then φ is constant on orbits of the action of $K(\Lambda)$ on $\mathcal{Z}_P$.
  The map φ induces a smooth map $\tilde{\varphi}\colon M(P, \Lambda) \to \mathbb{C}^q$ equivariant with respect to some representation $\mathbb{T}^n \to \mathbb{T}^q$, where action of $\mathbb{T}^q$ on $\mathbb{C}^q$ is supposed to be standard.
- Let $\sum\limits_{j=1}^{q} |\varphi_{a_j}(z)|^2 \neq 0$, if $z \in \mathcal{Z}_P \subset \mathbb{C}^m$ then the map φ induces a smooth map $\tilde{\varphi}_{\mathbb{P}} : M \to \mathbb{C}P^{q-1}$ equivariant with respect to some representation $\mathbb{T}^n \to \mathbb{T}^q$, where the action of $\mathbb{T}^q$ on $\mathbb{C}P^{q-1}$ is supposed to be standard.

Maps of the form $\tilde{\varphi}\colon M \to \mathbb{C}^q$ and $\tilde{\varphi}_{\mathbb{P}}\colon M \to \mathbb{C}P^{q-1}$ constructed by a family of vectors $a_1, \dots, a_q$ are called monomial maps of the quasitoric manifold M.

$P = \Delta^2$,

$$S = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $a_k = e_k \in \mathbb{Z}^3, k = 1, 2, 3$. Then

$$S^{\Lambda} g = \begin{pmatrix} g \\ g \\ g \end{pmatrix},$$

where $g = e^{2\pi i \xi}$ and $\chi_{a_k}(S_{\Lambda} g) = g$. Thus

$$\varphi : \mathbb{C}^3 \to \mathbb{C}^3$$

is the identify map. If $z \in \mathcal{Z}_{\Delta^2}$, then $z \neq 0$ and we obtain the map

$$\tilde{\varphi_{\Delta_2}} : \mathbb{C}P^2 \to \mathbb{C}P^2,$$

which is the identify map.

Edge-characters

Let $r \subset P$ be an edge of the polytope P. Since the action of $\mathbb{T}^n$ on M is locally standard, the isotropy subgroup $G(r)$ of the submanifold $\pi^{-1}(r)$ is an $(n-1)$-dimensional toric subgroup in $\mathbb{T}^n$.

Denote by χ_r the character $\mathbb{T}^n \to \mathbb{T}^1 = \mathbb{T}/G(r)$.

Definition

The edge-character μ_r of edge r is the composition

$$\mu_r \colon \mathbb{T}^m \xrightarrow{\Lambda} \mathbb{T}^n \xrightarrow{\chi_r} \mathbb{T}^1.$$

Let $\chi\colon K \to \mathbb{T}^1$ be a character of the group $K = K(\Lambda)$ and $v_\omega = F_{i_1} \cap \cdots \cap F_{i_n}$ be a vertex of P. The composition $p_\omega\colon K \hookrightarrow \mathbb{T}^m \to \mathbb{T}_{\hat{\omega}}$ is an isomorphism. Here $\hat{\omega} = [1, m] \setminus \omega$.

Definition

The χ-vertex-character χ_{v_ω} of v_ω is the character

$$\chi_{v_\omega} : \mathbb{T}^m \to \mathbb{T}_{\hat{\omega}} \xrightarrow{p_\omega^{-1}} K \xrightarrow{\chi} \mathbb{T}^1.$$

Vertex-edge-characters

Let $\chi\colon K \to \mathbb{T}^1$ be a character of the group K,
$r \subset P$ an edge of P and $v \in r$ a vertex lying on the edge r.

Definition

The vertex-edge-character of the pair (v, r) is a character $\chi_{v,r}\colon \mathbb{T}^m \to \mathbb{T}^1$ defined as the sum of the characters $\chi_v\colon \mathbb{T}^m \to \mathbb{T}^1$ and $\mu_r\colon \mathbb{T}^m \to \mathbb{T}^1$.

Characters of $\mathbb{T}^m$ associated with characters of K

Let $ch(\mathbb{T}^m)$ be the group of all characters of $\mathbb{T}^m$.
The set of pairs

$$\{\text{ character } \chi\colon K \to \mathbb{T}^1,\ \text{vertex } v \in P\}$$

determines the set of characters $\{\chi_v | v \in P\} \subset \mathrm{ch}(\mathbb{T}^{\mathrm{m}})$.
The set of triples

$$\{\text{ character } \chi\colon K \to \mathbb{T}^1,\ \text{edge } r \subset P,\ \text{vertex } v \in r\}$$

determines the set $\{\chi_{v,r} | v \in r \subset P\} \subset \mathrm{ch}(\mathbb{T}^{\mathrm{m}})$,
Therefore,

Every character $\chi\colon K \to \mathbb{T}^1$ gives the set of characters

$$X_\chi = (\{\chi_v\} \cup \{\chi_{v,r}\}) \subset \mathrm{ch}(\mathbb{T}^m).$$

The properties of constructed characters

- There are at most $f_0(P)$ characters $\{\chi_v\}$ and at most $nf_0(P) = 2f_1(P)$ characters of the form $\{\chi_{v,r}\}$. The number of different characters in X_χ does not exceed $f_0(P)(n+1)$.

  The cardinality $|X_\chi|$ is often much less than this upper bound, since characters χ_v and $\chi_{v,r}$ may be equal for different vertices $v \in P$ and pairs (v, r).
- The restriction of any of characters χ_v and $\chi_{v,r}$ to the subgroup $K \subset \mathbb{T}^m$ is equal to $\chi\colon K \to \mathbb{T}^1$.
- For any vertex $v \in P$ the character χ_v is trivial if and only if the character χ is trivial.
- The character μ_r does not depend on χ and is nontrivial for every $r \subset P$. The restriction of μ_r to the subgroup K is trivial.

The properties of constructed characters

- Every character χ_v, $v \in P$, is well-defined, but characters $\Phi_r \colon \mathbb{T}^n \to \mathbb{T}^1$ and $\mu_r \colon \mathbb{T}^m \to \mathbb{T}^1$ are defined only up to multiplication by ± 1, so the definition of $\chi_{v,r}$ is still ambiguous.
  If v_0, v_1 are vertices lying on an edge $r \subset P$, then, the character $\chi_{v_1} - \chi_{v_0}$ is a multiple of μ_r.
  We set $\chi_{v_0,r} = \chi_{v_0} + \mu_r$, where μ_r has the same direction as $\chi_{v_1} - \chi_{v_0}$. If $\chi_{v_1} = \chi_{v_0}$, then we assume that the first nonzero coordinate of m-vector defining $\mu_r \colon \mathbb{T}^m \to \mathbb{T}^1$ is positive.
- If the character χ is trivial, the cardinality of X_χ does not exceed $f_1(P) + 1$, because $\chi_v \equiv 1$ for all $v \in P$ and if v_0, v_1 are vertices of an edge $r \subset P$, then $\chi_{v_0,r} = \chi_{v_1,r} = \mu_r$.

Equivariant embeddings in linear spaces

Vectors w_r that define characters χ_r, $r \subset P$, form an integral $(n \times q)$-matrix W. Consider the corresponding linear representation W of the torus $\mathbb{T}^n$ in $\mathbb{C}^q$.

The moment map $\pi\colon M \to P \subset \mathbb{R}^n$ is equivariant with respect to the trivial torus action on $\mathbb{R}^n$.

Theorem

The moment map $\pi\colon M \to P$ can be extended to a real-algebraic embedding

$$\pi \times \tilde{\varphi}\colon M \to \mathbb{R}^n \times \mathbb{C}^q$$

equivariant with respect to the representation $W\colon \mathbb{T}^n \to \mathbb{T}^q$.

Here the number q does not exceed the number of edges of P, as follows from the construction of the representation W.

Theorem

Let $\chi\colon K \to \mathbb{T}^1$ *is an arbitrary character. The set of characters* X_χ *determines a monomial map* $\tilde{\varphi}_{\mathbb{P},\chi}\colon M \to \mathbb{C}P^{q-1}$*, which can be extended to an embedding* $\pi \times \tilde{\varphi}_{\mathbb{P},\chi}\colon M \to P \times \mathbb{C}P^{q-1}$*.*

The induced cohomology pullback

$$\tilde{\varphi}^*_{\mathbb{P},\chi}\colon H^2(\mathbb{C}P^{q-1},\mathbb{Z}) \to H^2(M,\mathbb{Z})$$

coincides with the classifying map

$$H^2(\mathbb{C}P^{\infty},\mathbb{Z}) \xrightarrow{\simeq} H^2(\mathbb{C}P^{q-1},\mathbb{Z}) \to H^2(M,\mathbb{Z})$$

of the bundle $\xi_\chi = \mathcal{Z}_P \times_K \mathbb{C} \to M = \mathcal{Z}_P \times_K (pt)$
If the character χ is trivial, then the image of the map $\tilde{\varphi}_{\mathbb{P},\chi}$ lies entirely in some affine chart $\mathbb{C}^{q-1} \subset \mathbb{C}P^{q-1}$.

Equivariant embeddings in projective spaces

Let

$$P = \{x \in \mathbb{R}^n \colon Ax + b \geqslant 0\}.$$

Suppose that (A, Λ)-structure of a quasitoric manifold M satisfies 2 the following conditions:

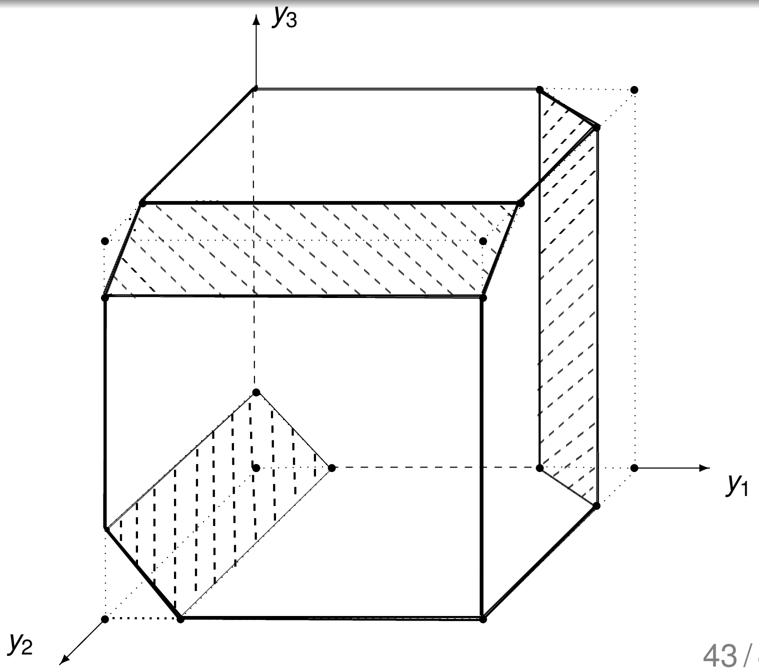
- $b \in \mathbb{Z}^m$,
- $A^\top = B\Lambda D$, where B is an integral $(n \times n)$-matrix with $|B| \neq 0$ and D is a diagonal $(m \times m)$-matrix with all nonzero elements equal to ± 1.

Consider the character $\chi_B \colon \mathbb{T}^m \to \mathbb{T}^1$ determined by vector $b \in \mathbb{Z}^m$ and denote by χ_P the character $K \subset \mathbb{T}^m \to \mathbb{T}^1$.

Theorem

The projective map $\tilde{\varphi}_{\mathbb{P},\chi_P} \colon M \to \mathbb{C}P^{q-1}$ constructed using the set X_{χ_P} is a smooth equivariant embedding.

3-dimensional Stasheff polytope



Combinatorial data

Let us consider an example of toric variety of complex dimension three over Stasheff polytope K_5.

The matrix $\Lambda = A_P^\top$ has the form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix},$$

and the vector b_P is equal to $(0, 0, 0, 3, 3, 3, 5, -1, 2)^T$.

Moment-angle manifold of Stasheff polytope

The manifold $\mathcal{Z}_P \subset \mathbb{C}^9$ is given by the following equations:

$$\begin{aligned}
|z_1|^2 + |z_4|^2 &= 3, \\
|z_2|^2 + |z_5|^2 &= 3, \\
|z_3|^2 + |z_6|^2 &= 3, \\
|z_1|^2 + |z_3|^2 + |z_7|^2 + |z_9|^2 &= 7, \\
|z_7|^2 + |z_8|^2 + |z_9|^2 &= 6, \\
|z_2|^2 + |z_3|^2 + |z_7|^2 &= 5.
\end{aligned}$$

The moment-angle manifold $\mathcal{Z}_P \subset \mathbb{C}^9$ has a real dimension 12.
It is endowed with the action of the compact torus $\mathbb{T}^9$,
with 6-dimensional subgroup $K \subset T^9$ acting freely on $\mathcal{Z}_P$.

Embedding of quasitoric manifold of Stasheff polytope

In the case of Stasheff polytope we have $q = 6$ and $f_1(P) = 21$. These monomial functions embed the quasitoric manifold corresponding to Stasheff polytope to Euclidean space $\mathbb{R}^3 \times \mathbb{C}^6$.

The following monomial functions define a projective embedding of the manifold corresponding to Stasheff polytope K_5:

$$\varphi_{b_1} = z_1 z_4^2 z_5^3 z_6^3 z_7^5 z_9, \qquad \varphi_{b_2} = z_3 z_4^3 z_5^3 z_6^2 z_7^4 z_9^2,$$

$$\varphi_{b_3} = z_1 z_2^3 z_4^2 z_6^3 z_7^2 z_9^4, \qquad \varphi_{b_4} = z_2^3 z_3 z_4^3 z_6^2 z_7 z_9^5,$$

$$\varphi_{b_5} = z_1^2 z_4 z_5^3 z_6^3 z_7^5 z_8, \qquad \varphi_{b_6} = z_1^3 z_2 z_5^2 z_6^3 z_7^4 z_8^2,$$

$$\varphi_{b_7} = z_1^3 z_2 z_3^3 z_5^2 z_7 z_8^5, \qquad \varphi_{b_8} = z_1^2 z_3^3 z_4 z_5^3 z_7^2 z_8^4,$$

$$\varphi_{b_9} = z_1^3 z_2^2 z_3^3 z_5 z_8^5 z_9, \qquad \varphi_{b_{10}} = z_2^2 z_3^3 z_4^3 z_5 z_8^2 z_9^4,$$

$$\varphi_{b_{11}} = z_2^3 z_3^2 z_4^3 z_6 z_8 z_9^5, \qquad \varphi_{b_{12}} = z_1^3 z_2^3 z_3^2 z_6 z_8^4 z_9^2,$$

$$\varphi_{b_{13}} = z_3^3 z_4^3 z_5^3 z_7^2 z_8^2 z_9^2, \qquad \varphi_{b_{14}} = z_1^3 z_2^3 z_6^3 z_7^2 z_8^2 z_9^2.$$

Projective embedding of smooth toric variety

$$\varphi_{a_1} = z_1 z_2 z_4^2 z_5^2 z_6^3 z_7^4 z_9^2, \qquad \varphi_{a_2} = z_1 z_2^2 z_4^2 z_5 z_6^3 z_7^3 z_9^3,$$
$$\varphi_{a_3} = z_1^2 z_2^3 z_4 z_6^3 z_7^2 z_8 z_9^3, \qquad \varphi_{a_4} = z_1^3 z_2^2 z_5 z_6^3 z_7^3 z_8^2 z_9,$$
$$\varphi_{a_5} = z_2 z_3 z_4^3 z_5^2 z_6^2 z_7^3 z_9^3, \qquad \varphi_{a_6} = z_2^2 z_3 z_4^3 z_5 z_6^2 z_7^2 z_9^4,$$
$$\varphi_{a_7} = z_1^3 z_2^3 z_3^1 z_6^2 z_7 z_8^3 z_9^2, \qquad \varphi_{a_8} = z_1^3 z_2 z_3 z_5^2 z_6^2 z_7^3 z_8^3,$$
$$\varphi_{a_9} = z_1^2 z_3 z_4 z_5^3 z_6^2 z_7^4 z_8^2, \qquad \varphi_{a_{10}} = z_1 z_2^3 z_3^2 z_4^2 z_6 z_8^2 z_9^4,$$
$$\varphi_{a_{11}} = z_1^2 z_2^3 z_3^2 z_4 z_6 z_8^3 z_9^3, \qquad \varphi_{a_{12}} = z_1^3 z_2 z_3^2 z_5^2 z_6 z_7^2 z_8^4,$$
$$\varphi_{a_{13}} = z_1^2 z_3^2 z_4 z_5^3 z_6 z_7^3 z_8^3, \qquad \varphi_{a_{14}} = z_3^2 z_4^3 z_5^3 z_6 z_7^3 z_8 z_9^2,$$
$$\varphi_{a_{15}} = z_1 z_3^3 z_4^2 z_5^3 z_7^2 z_8^3 z_9, \qquad \varphi_{a_{16}} = z_2 z_3^3 z_4^3 z_5^2 z_7 z_8^2 z_9^3,$$
$$\varphi_{a_{17}} = z_1 z_2^2 z_3^3 z_4^2 z_5 z_8^3 z_9^3, \qquad \varphi_{a_{18}} = z_1^2 z_2^2 z_3^3 z_4 z_5 z_8^4 z_9^2.$$

We see that the dimension of the projective embedding is much more than the dimension of the affine embedding.

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