# Univalent Morphisms

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Let me recall first the basic concepts related to the notion of a simplicial set. For a non-negative integer $n \geq 0$ let $[n]$ be the set

$$[n] = \{0, \ldots, n\}$$

Let $MonFun([m],[n])$ be the set of non-decreasing functions from $[m]$ to $[n]$ i.e. the set of functions $f : [m] \to [n]$ such that for all $x, y \in [m]$ satisfying $x \leq y$ one has $f(x) \leq f(y)$.

Let $\Delta$ be the category whose set of objects $Ob(\Delta)$ is the set of sets of the form $[n]$ for $n \geq 0$ and whose set of morphisms is

$$Mor(\Delta) = \amalg_{[m],[n] \in Ob(\Delta)} MonFun([m],[n])$$

The domain and codomain functions as well as the identity and composition functions are defined in the obvious way.

Note that for $[m], [n] \in Ob(\Delta)$ the set

$$\Delta([m],[n]) = \{f \in Mor(\Delta) \,|\, dom(f) = [m] \ and \ codom(f) = [n]\}$$

of morphisms from $[m]$ to $[n]$ in $\Delta$ is not the set $MonFun([m],[n])$ but the set of pairs of the form $([m],([n],f))$ where $f \in MonFun([m],[n])$.

Sending $([m],([n],f))$ to $f$ one obtains a bijection of the form

$$\Delta([m],[n]) \to MonFun([m],[n])$$

and sending $f$ to $([m],([n],f))$ one obtains the inverse bijection.

It is customary to use these bijections to "identify" the sets $\Delta([m],[n])$ and $MonFun([m],[n])$ and I will do the same in this lecture. I will allow myself many other similar imprecisions because otherwise the exposition would become very hard to follow.

When one starts to formalize mathematics using a computer proof assistant one has to be explicit about such details.

The example that you have just seen is one of the reasons why it is so hard to formalize categorical constructions in set theory.

Indeed, in set theory, it is in general impossible, given a set $Ob$ and a family of sets $Mor(X, Y)$ for $X, Y \in Ob$ to define a global set $Mor$ with two functions $dom, codom : Mor \to Ob$ such that

$$Mor(X, Y) = \{f \in Mor \,|\, dom(f) = X \text{ } and \text{ } codom(f) = Y\}$$

for all $X$ and $Y$.

For example, suppose that there are two objects $X$ and $Y$ with $X \neq Y$ and $Mor(X, X) = Mor(Y, Y) \neq \emptyset$. It is an exercise to prove that in this case it is impossible to find a set $Mor$ and functions

$$dom, codom : Mor \to Ob$$

such that

$$Mor(X, X) = \{f \in Mor \,|\, dom(f) = X \text{ } and \text{ } codom(f) = X\}$$

and

$$Mor(Y, Y) = \{f \in Mor \,|\, dom(f) = Y \text{ } and \text{ } codom(f) = Y\}$$

In the univalent foundations this particular problem of building a category when the sets of morphisms between objects are given is easily resolved since there one can directly operate with families of sets (and more generally families of types). So in the univalent foundations we can have a category $\Delta$ where

$$\Delta(X, Y) = MonFun(X, Y)$$

in the sense of the "strict" or "substitutional" equality.

However, univalent foundations are so complex that they can not be intuitively seen to be consistent.

We can only show that univalent foundations are consistent relative to set theory and this requires proving it mathematically.

Of course, one can not use univalent foundations to prove that univalent foundations are consistent. Nor is it sufficient to use informal mathematics for this because it is too important and complex and must be verified formally before it can be trusted.

Therefore, the most important task in the univalent foundations today is to prove all the theorems that are needed in order to show that these new foundations are consistent relative to set theory.

These theorems have to be formulated and proved with meticulous precision in the framework of set theory. Some of these theorems have already been proved in the "formalization ready" style in a series of my papers that can be found on the arXiv.

Now they have to be formalized and formally verified in set theory or in an equally simple and reliable theory such as HOL.

The results related to the univalent morphisms that I will speak about today is another part of what will have to be proved in the same style. Today I will only outline some of these results on a very informal level.

**Definition**. A simplicial set is a a contravariant functor

$$X : \Delta^{op} \to Sets$$

from the category $\Delta$ to the category of sets. One denotes the category of simplicial sets by $\Delta^{op}Sets$.

A good book about simplicial sets that also contains a lot of other useful material is by Peter Gabriel and Michael Zisman. You can easily find it on the web.

One denotes the functor represented by $[n]$ by $\Delta^n$ and calls it the n-dimensional simplex. The 0-dimensional simplex is called the point and often denoted by $pt$ and the 1-dimensional simplex $\Delta^1$ is called the interval and often denoted by $I^1$.

Every simplicial set is "glued" from simplexes - the precise statement is that there is a construction that, for every simplicial set, provides a diagram of simplexes and an isomorphism from the colimit of this diagram to the original simplicial set.

By mapping simplexes $\Delta^n$ to the topological spaces

$$\Delta^n_{top} = \{(x_0, \ldots, x_n) \in \mathbf{R}^{n+1}_{\geq 0} \mid \sum_i x_i = 1\}$$

and then extending this mapping to all simplicial sets such as to preserve the way in which general simplicial sets are glued from simplexes one obtains a functor

$$|\ \ | : \Delta^{op}Sets \to Top$$

that is called the functor of geometric realization.

This functor takes $pt$ to a one point space, $I^1$ to a space homeomorphic to the unit interval $[0,1]$ and preserves not only the way things are glued together (colimits) but also products.

One can use this functor to translate the intuition related to spaces, such as the intuition of the classical homotopy theory, into concepts related to simplicial sets.

From now on I will assume that the listeners are familiar with this translation and that I can use the main concepts of homotopy theory in the category of simplicial sets. I will sometimes say “space” instead of “simplicial set” as is customary in this field.

One construction that will be important below and that is not widely known is the construction of the relative Hom-object. For two morphisms $f : Y \to X$, $f' : Y' \to X$ one defines $\underline{Hom}_X((Y, f), (Y', f'))$ as the object whose sets of n-simplexes are given by

$$\underline{Hom}_X((Y, f), (Y', f'))_n = \amalg_{a:\Delta^n \to X} Hom_X((\Delta^n, a) \times_X (Y, f), (Y', f'))$$

and whose action on morphisms of $\Delta$ is defined in the obvious way.

The morphism that maps $(a, f)$ to $a$ is denoted as

$$f \diamond f' : \underline{Hom}_X((Y, f), (Y', f')) \to X$$

and together with this morphism this space is the internal Hom-object in the slice category $\Delta^{op} Sets/X$.

One has:

1. $f \diamond f'$ is a Kan fibration if $f'$ is a Kan fibration,
2. if $X = pt$ and $\pi_Y : Y \to pt$, $\pi_{Y'} : Y' \to pt$ are the unique morphisms then
$$\underline{Hom}_X((Y, \pi_Y), (Y', \pi_{Y'})) \cong S(Y, Y')$$
where $S(Y, Y')$ is the usual simplicial function space,
3. in general, a point (zero simplex) of $\underline{Hom}_X((Y, f), (Y', f'))$ is given by a point $x$ in $X$ and a morphism $f^{-1}(x) \to (f')^{-1}(x)$.

Let us remind the following standard definition.

**Definition 1** *Let $f : Y \to X$ and $f' : Y' \to X$ be two morphisms and $g : Y \to Y'$ a morphism over $X$. Then $g$ is called a fiber-wise weak equivalence if for any $x \in X$ the corresponding morphism between the homotopy fibers of $f$ and $f'$ is a weak equivalence.*

For two morphisms $f : Y \to X$ and $f' : Y' \to X$ let $Eq_X(f, f')$ be the space of fiber-wise weak equivalences from $Y$ to $Y'$ over $X$.

It is fibered over $X$ such that the fiber of $Eq_X(f, f') \to X$ over $x \in X$ is (homotopy equivalent to) the space of homotopy equivalences between the homotopy fibers $f^{-1}(x)$ and $(f')^{-1}(x)$.

In the construction of $Eq_X(f, f')$ one should first replace $f$ and $f'$ by equivalent Kan fibrations

$$f_{Kan} : Y_{Kan} \to X$$

$$f'_{Kan} : Y'_{Kan} \to X$$

Then consider the intermediate object

$$Int = \underline{Hom}_X((Y_{Kan}, f_{Kan}), (Y'_{Kan}, f'_{Kan}))$$

A point of this object is a pair $(x, g)$ where $x$ is a point of $X$ and $g : f_{Kan}^{-1}(x) \to (f'_{Kan})^{-1}(x)$ is a morphism. One verifies easily that if $(x, g)$ is such that $g$ is a homotopy equivalence and $(x', g')$ is in the same connected component of $Int$ as $(x, g)$ then $g$ is a homotopy equivalence. One defines $Eq_X(f, f')$ as the union of such connected components of $Int$ that contain points $(x, g)$ where $g$ is a homotopy equivalence.

For a morphism $f : Y \to X$ consider the morphisms

$$f \times Id_X : Y \times X \to X \times X$$

$$Id_X \times f : X \times Y \to X \times X$$

Let

$$E(f) = Eq_{X \times X}(f \times Id, Id \times f)$$

and let $pre(f) : E(f) \to X \times X$ be the canonical projection. The fiber of $pre(f)$ over $(x, x')$ is the space of homotopy equivalences between the homotopy fibers of $f$ over $x$ and $x'$. In particular, $pre(f)$ has a canonical section $w_f$ over the diagonal corresponding to the identity:

$$\begin{array}{ccc} X & \stackrel{w_f}{\longrightarrow} & E(f) \\ \| & & \downarrow{\scriptstyle pre(f)} \\ X & \stackrel{\Delta_X}{\longrightarrow} & X \times X \end{array}$$

**Definition 2** *A morphism $f : Y \to X$ is called univalent if*

$$w_f : X \to E(f)$$

*is a fiber-wise homotopy equivalence over $X \times X$.*

The homotopy fiber of the diagonal $\Delta_X : X \to X \times X$ over $(x, x')$ is the space $P(X; x, x')$ of paths from $x$ to $x'$ in $X$. Therefore, $f$ is univalent if and only if for any $x$, $x'$ the canonical morphism from $P(X; x, x')$ to the space of homotopy equivalences between the homotopy fibers $u^{-1}(x)$ and $u^{-1}(x')$ is an equivalence.

Here are some examples of univalent morphisms :

1. There are only four univalent morphisms between sets. They are $\emptyset \to \emptyset$, $\emptyset \to pt$, $pt \to pt$ and $pt \to pt \coprod pt$. Of these four the last one is the universal one since the other three are obtained from it by pull-back. These four morphisms are also the only univalent morphisms of *h-level* 1.

2. For $n > 0$ the morphism $BS_{n-1} \to BS_n$ where $S_{n-1} \to S_n$ is the standard embedding of symmetric groups, is univalent. The homotopy fiber of this morphism is the set with $n$ elements.

3. For $n \geq 0$ the inclusion of the distinguished point $pt \to K(\mathbf{Z}/2, n)$ is univalent. For $n = 0$ one gets the morphism $pt \to pt \coprod pt$ from the first example and for $n = 1$ one gets the morphism $BS_1 \to BS_2$ of the second example. I do not know at the moment any other examples of univalent morphisms whose domain is the point.

4. A morphism $X \to pt$ is univalent iff $X$ has no symmetries i.e. iff the space of homotopy auto-equivalences of $X$ is contractible. For a group $G$ it means that the morphism $BG \to pt$ is univalent if the center and the group of outer automorphisms of $G$ are trivial. In particular, for $n > 2$ the morphism $BS_n \to pt$ is univalent.

To describe some properties of univalent morphisms it is convenient to use the following concept of h-level that is a generalization of the known concept of homotopy n-type.

**Definition 3** *Define the property of a space to be of h-level $n$ inductively as follows:*

1. *$X$ is of h-level $0$ if and only if $X$ is contractible,*
2. *$X$ is of h-level $n > 0$ if and only if for any $x, x' \in X$ the paths space $P(X; x, x')$ is of h-level $n - 1$.*

**Definition 4** *A morphism $f : Y \to X$ is called a morphism of h-level $n$ if for any $x \in X$ the homotopy fiber $f^{-1}(x)$ is of h-level $n$.*

Let us state some elementary properties of univalent maps.

**Lemma 5** *Consider a (homotopy) cartesian square*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

*such that $f$ is univalent. Then $f'$ is univalent if and only if $g$ is a morphism of h-level* $1$*.*

**Proposition 6** *If for a given univalent $f : Y \to X$ and a given $f' : Y' \to X'$ there exists a (homotopy) cartesian square of the form*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

*then such a square is unique up to an equivalence.*

**Theorem 7** *For any map $f : Y \to X$ there exists a unique homotopy cartesian square*

$$\begin{array}{ccc} Y & \longrightarrow & \widetilde{Un}(f) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle un(f)} \\ X & \xrightarrow{g} & Un(f) \end{array}$$

*such that $un(f)$ is univalent and $g$ is surjective on $\pi_0$.*

There is relatively simple way to prove the previous theorem but it requires the use of the axiom of choice. So far nobody knowns how to give a constructive proof of this theorem and this is one of the important open problems in the univalent foundations.