Braids and some other groups arising in geometry and topology

Vladimir Vershinin

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1. Geometrical definition of Thompson group F

By *dyadic rational numbers* we mean rational numbers of the form $p2^q$, where $p, q \in \mathbb{Z}$.

Let F be the set of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2.

Since derivatives are positive where they exist, elements of F preserve orientation.

Let $f \in F$, and let

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$$

be the points at which f is not differentiable. Then since $f(0) = 0$,

$$f(x) = a_1 x \text{ for } x_0 < x < x_1,$$

where a_1 is a power of 2.

Likewise, since $f(x_1)$ is a dyadic rational number,

$$f(x) = a_2 x + b_2 \text{ for } x_1 < x < x_2,$$

where a_2 is a power of 2 and b_2 is a dyadic rational number. It follows inductively that

$$f(x) = a_i x + b_i \quad \text{for} \quad x_i < x < x_{i+1}$$

and $i = 1, \ldots, n$, where a_i is a power of 2 and b_i is a dyadic rational number.

It follows that $f^{-1} \in F$ and that f maps the set of dyadic rational numbers bijectively to itself. From this it follows that F is closed under composition of functions. Thus F is a subgroup of the group of all homeomorphisms from $[0, 1]$ to $[0, 1]$. This group is Thompson's group F.

As we are dealing with functions words in the group F we are reading from right to left.

Example 1. Let A and B be the following functions:

$$A(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2}, \\ x - \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4}, \\ 2x - 1, & \frac{3}{4} \le x \le 1, \end{cases}$$

$$B(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4}, \\ x - \frac{1}{8}, & \frac{3}{4} \le x \le \frac{7}{8}, \\ 2x - 1, & \frac{7}{8} \le x \le 1. \end{cases}$$

Figure: The generators A and B

Define functions $X_0, X_1, X_2, \ldots$ in F so that

$$X_0 = A$$

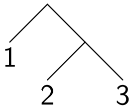
and

$$X_n = A^{-(n-1)}\, B\, A^{n-1} \quad \text{for} \quad n > 1.$$

2. Tree diagrams

Define an *ordered rooted binary tree* to be a tree S such that
i) S has a root v_0,
ii) if S consists of more than v_0, then v_0 has valence 2,
iii) if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}$, $e_{v,R}$ which contain v and are not contained in the shortest path from v_0 to v.

The edge $e_{v,L}$ is called a *left edge* of S, and $e_{v,R}$ is called a *right edge* of S. Vertices with valence 0 (in case of the trivial tree) or 1 in S will be called leaves of S. There is a canonical left-to-right linear ordering on the leaves of S. The right side of S is the maximal arc of right edges in S which begins at the root of S. The left side of S is defined analogously.



Example of an ordered rooted binary tree

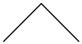
An *ordered rooted binary sub-tree* T of an ordered rooted binary tree S is an ordered rooted binary tree which is a sub-tree of S whose left edges are left edges of S, whose right edges are right edges of S, but whose root need not be the root of S.

Define a *standard dyadic interval* in $[0, 1]$ to be an interval of the form $[\frac{a}{2^n}, \frac{a+1}{2^n}]$ where a, n are non-negative integers with $a < 2^n - 1$.

There is a tree of standard dyadic intervals, $\mathcal{T}$, which is defined as follows. The vertices of $\mathcal{T}$ are the standard dyadic intervals in $[0,1]$. An edge of $\mathcal{T}$ is a pair (I,J) of standard dyadic intervals I and J such that either I is the left half of J, in which case (I,J) is a left edge, or I is the right half of J, in which case (I,J) is a right edge. The tree $\mathcal{T}$ is an ordered rooted binary tree.

Define a $\mathcal{T}$-*tree* to be a finite ordered rooted binary subtree of $\mathcal{T}$ with root $[0, 1]$. Call the $\mathcal{T}$-tree with just one vertex the *trivial* $\mathcal{T}$-tree. For every non-negative integer n, let $\mathcal{T}_n$ be the $\mathcal{T}$-tree with $n + 1$ leaves whose right side has length n.

Define a *caret* to be an ordered rooted binary subtree of $\mathcal{T}$ with exactly two edges:



A partition of $[0,1]$ is called a *standard dyadic partition* if and only if the intervals of the partition are standard dyadic intervals.

There is a canonical bijection between standard dyadic partitions and $\mathcal{T}$-trees.

Lemma

Let $f \in F$. Then there exists a standard dyadic partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that f is linear on every interval of the partition and

$$0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = 1$$

is a standard dyadic partition.

A *tree diagram* is an ordered pair (R, S) of $\mathcal{T}$-trees such that R and S have the same number of leaves :

$$R \to S.$$

The tree R is called the *domain tree* of the diagram, and S is called the *range tree* of the diagram.

Given $f \in F$ there exist standard dyadic partitions P and Q such that f is linear on the intervals of P and maps them to the intervals of Q. To f is associated the tree diagram (R, S), where R is the $\mathcal{T}$-tree corresponding to P and S is the $\mathcal{T}$-tree corresponding to Q.

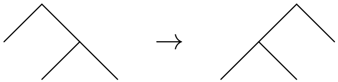
If there do not exist such carets in R, S, then the tree diagram (R, S) is said to be *reduced*.

For $f \in F$ intervals P and Q are not unique. Given one tree diagram (R, S) for f, another can be constructed by adjoining carets to R and S at the leaf with the same number i. Conversely such carets can be deleted.

There is a canonical bijection between F and the set of reduced tree diagrams.

Recall that the element A sends intervals of the subdivision:

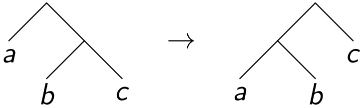
linearly onto intervals of the subdivision:



The tree pair diagram for the generator A of F

This tree pair diagram can be interpreted as a law of associativity of a product $*$:

$$a * (b * c) \to (a * b) * c.$$



The tree pair diagram for the law of associativity

Tree diagrams can be also depicted this way:

Then reduction of a tree diagram consists of removing an opposing pair of carets, like this:

Let $l_0, ..., l_n$ be the leaves of $\mathcal{T}$-tree S in order. For every integer k with $0 \leq k \leq n$ let a_k be the length of the maximal arc of left edges in S which begins at l_k and which does not reach the right side of S. Then a_k is the k-th *expotent* of S.

Theorem

Let R, S be $\mathcal{T}$-trees with $n+1$ leaves for some non-negative integer n. Let $a_0, ..., a_n$ be the exponents of R, and let $b_0, , b_n$ be the exponents of S. Then the function in F with tree diagram (R, S) is

$$X_0^{b_0} X_1^{b_1} \dots X_n^{b_n} X_n^{-a_n} \dots X_1^{-a_1} X_0^{-a_0}.$$

The tree diagram (R,S) is reduced if and only if
i) if the last two leaves of R lie in a caret, then the last two leaves of S do not lie in a caret and
ii) for every integer k with $0 \leq k < n$, if $a_k > 0$ and $b_k > 0$ then either $a_{k+1} > 0$ or $b_{k+1} > 0$.

Corollary

Thompson's group F is generated by A and B.

Corollary

Thompson's group F is generated by A and B. Every non-trivial element of F can be expressed in unique form

$$X_0^{b_0} X_1^{b_1} \dots X_n^{b_n} X_n^{-a_n} \dots X_1^{-a_1} X_0^{-a_0}.$$

where $n, a_0, ..., a_n, b_0, , b_n$ are non-negative integers such that
i) exactly one of a_n and b_n is non-zero,
ii) if $a_k > 0$ and $b_k > 0$ for some integer k with $0 \leq k < n$, then $a_{k+1} > 0$ or $b_{k+1} > 0$. Furthermore, every such function in F is non-trivial.

The form of the above lemma of an element of F is called the *normal form* of this element.

The functions in F of the form

$$X_0^{b_0} X_1^{b_1} \dots X_n^{b_n}$$

with $b_k \geq 0$ for $k = 0, ..., n$ will be called *positive*. The positive elements of F are exactly those with tree diagrams having domain tree $\mathcal{T}_n$ for some non-negative integer n. Inverses of positive elements will be called *negative*.

Lemma

The set of positive elements of F is closed under multiplication.

3. Presentations for F

Let $[y, z] = yzy^{-1}z^{-1}$.

Let F_1 be a group generated by letters α et β, and given by the presentation:

$$< \alpha, \beta \,|\, [\alpha\beta^{-1}, \alpha^{-1}\beta\alpha], \; [\alpha\beta^{-1}, \alpha^{-2}\beta\alpha^2] > . \qquad (1)$$

We rewrite the first relation of F_1 in details:

$$\alpha\beta^{-1}\alpha^{-1}\beta\alpha\beta\alpha^{-1}\alpha^{-1}\beta^{-1}\alpha = e.$$

This is the same as

$$\beta^{-1}\alpha^{-1}\beta\alpha\beta = \alpha^{-2}\beta\alpha^2. \tag{2}$$

We rewrite the second relation of F_1 in details::

$$\alpha\beta^{-1}\alpha^{-2}\beta\alpha^2\beta\alpha^{-1}\alpha^{-2}\beta^{-1}\alpha^2 = e.$$

This is the same as

$$\beta^{-1}\alpha^{-2}\beta\alpha^2\beta = \alpha^{-3}\beta\alpha^3. \tag{3}$$

We define the following notations

$$x_0 = \alpha,$$

$$x_1 = \beta,$$

$$x_i = x_0^{-i+1} x_1 x_0^{i-1}, \; i \geq 2.$$

With these notations equations (2) et (3) ave giving us that presentation (1) is equivalent to the following one

$$\langle x_0, x_1 \mid x_1^{-1} x_2 x_1 = x_3, \; x_1^{-1} x_3 x_1 = x_4 \rangle.$$

Theorem

Group F_1 admits a presentation :

$$\langle x_0, x_1, ..., x_n, ... \mid x_k^{-1} x_n x_k = x_{n+1} \ \ for \ \ k < n \rangle. \tag{4}$$

Let us define a map ϕ of the $\{\alpha, \beta\}$ to the Thompson group F by the formulae

$$\phi(\alpha) = A, \quad \phi(\beta) = B.$$

This map defines a homomorphism of the free group generated by α and β to F. We verify directly that the relators of F_1 are sent to e by ϕ, so we have a homomorphism of groups

$$\phi : F_1 \to F.$$

By definitions we also have

$$\phi(x_i) = X_i.$$

Theorem

Homomorphism

$$\phi : F_1 \to F$$

is an isomorpism. So formulae (1) and (4) give two presentations of the Thompson group.