Braids and some other groups arising in geometry and topology

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Young Topologist Seminar, IMS Singapore
(11 - 21 August 2015)

4. Further properties of F

A discrete group G is *amenable* if there is a left-invariant measure μ on G which is **finitely** additive and has total measure 1. That is, G is amenable if there is a function

$$\mu : \{subsets\ of\ \ G\} \to [0,1]$$

such that

1) $\mu(gA) = \mu(A)$ for all $g \in G$ and all subsets A of G,

2) $\mu(G) = 1$,

3) $\mu(AUB) = \mu(A) + \mu(B)$ if A and B are disjoint subsets of G.

It is clear from the definition that a finite group is amenable.

The class of all amenable groups is closed under quotients, subgroups, extensions, and directed unions with respect to inclusion.

Abelian groups are amenable.

We call a group an *elementary amenable* group if it is in the smallest class of groups that contains all Abelian and finite groups and is closed under quotients, subgroups, extensions, and directed unions with respect to inclusion.

Theorem

Free group of two variables $< a, b >$ is not amenable.

Proof.

Suppose otherwise that μ is a finitely additive, left invariant total measure on $< a, b >$. Then $\mu(1) = 0$ as $< a, b >$ is infinite. Let

$$g* = \{h \in K : h \text{ has a freely reduced representative beginning with } g\}.$$

Then

$$< a, b >= \{1\} \cup a * \cup b * \cup a^{-1} * \cup b^{-1} * .$$

On the other hand we have $a^{-1}a* = \{1\} \cup a * \cup b * \cup \ b^{-1}*$, and so

$$\mu(a*) = \mu(a*) + \mu(b*) + \mu(b^{-1}*),$$

and hence $\mu(b*) = \mu(b^{-1}*) = 0$. The same way $\mu(a*) = \mu(b^{-1}*) = 0$ and $\mu(< a, b >) = 0$. $\square$

Theorem

The commutator subgroup $[F, F]$ of F consists of all elements in F which are trivial in neighborhoods of 0 and 1. Furthermore,

$$H_1(F, \mathbb{Z}) = F/[F, F] \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Proof.

Define a group homomorphism

$$\psi : F \to \mathbb{Z} \oplus \mathbb{Z}$$

such that if $f \in F$, then

$$\psi(f) = (a, b),$$

where the right derivative of f at 0 is 2^a and the left derivative of f at 1 is 2^b. Since $\psi(A) = (-1, 1)$ and $\psi(B) = (0, 1)$, ψ is surjective. The group F is generated by A and B, so ψ is a homomorphism of abelianization and its kernel is $[F, F]$. □

Lemma

If $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ *and* $0 = y_0 < y_1 < y_2 < \cdots < y_n = 1$ *are partitions of* $[0, 1]$ *consisting of dyadic rational numbers, then there exists* $f \in F$ *such that* $f(x_i) = y_i$ *for* $i = 0, ..., n$*. Furthermore, if* $x_{i-1} = y_{i-1}$ *and* $x_i = y_i$ *for some* i *with* $0 < i \leq n$*, then* f *can be taken to be trivial on the interval* $[x_i, x_{i+1}]$*.*

Theorem

Every proper quotient group of F is Abelian.

Lemma

Let a, b be dyadic rational numbers with $0 < a < b < 1$ such that $b - a$ is a power of 2. Then the subgroup of F consisting of all functions with support in $[a, b]$ is isomorphic with F by means of the linear conjugation.

Theorem

The commutator subgroup $[F, F]$ of F is a simple group.

Theorem

The sub-monoid of F generated by A, B, B^{-1} is the free product of the sub-monoid generated by A and the subgroup generated by B.

Corollary

Thompson's group F has exponential growth.

Theorem

Every non-Abelian subgroup of F contains a free Abelian subgroup of infinite rank.

Corollary

Thompson s group F does not contain a non-Abelian free group.

Theorem

Thompson's group F is not an elementary amenable group.

Theorem

Thompson's group F is a totally ordered group.