Braids and some other groups arising in geometry and topology

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5. Thompson's group T

Consider S^1 as the interval $[0, 1]$ with the endpoints identified.

Then T is the group of piecewise linear homeomorphisms from S^1 to itself that map images of dyadic rational numbers to images of dyadic rational numbers and that are differentiable except at finitely many images of dyadic rational numbers and on intervals of differentiability the derivatives are powers of 2. Just as we proved that F is a group, it is easy to see that T is indeed a group.

The elements A and B of F induce elements of T, which will still be denoted by A and B.

Example. Let C be the following function:

$$C(x) = \begin{cases} \frac{x}{2} + \frac{3}{4}, & 0 \leq x \leq \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ x - \frac{1}{4}, & \frac{3}{4} \leq x \leq 1. \end{cases}$$

We can associate tree diagrams and unique reduced tree diagrams to elements of T almost exactly as we did to elements of F. The only difference is the following. Elements of F map leftmost leaves of domain trees to leftmost leaves of range trees. When an element of T does not do this, we denote the image in its range tree of the leftmost leaf of its domain tree with a small circle.

Lemma

The elements A, B, and C generate T and satisfy the following relations :

$$
\begin{aligned}
&1)\ [AB^{-1}, A^{-1}BA] = 1,\\
&2)\ [AB^{-1}, A^{-2}BA^2] = 1,\\
&3)\ C = B(A^{-l}CB),\\
&4)\ (A^{-l}CB)(A^{-l}BA) = B(A^{-2}CB^2),\\
&5)\ CA = (A^{-1}CB)^2,\\
&6)\ C^3 = 1.
\end{aligned}
$$

Let T_1 be a group generated by A, B and C as formal symbols and relations of the previous lemma.

Lemma

The homomorphism

$$T_1 \to T$$

that maps the formal symbols A, B, and C to the functions A, B, and C in T is a surjection.

Lemma

The subgroup of T_1 generated by A and B is isomorphic to F.

Define the elements C_n , $n \geq 1$, of T_1 by

$$Cn = A^{-(n-1)} C B^{n-1}.$$

For convenience we define $C_0 = 1$.

Theorem

The group T_1 is simple.

Corollary

Group T_1 is isomorphic to T.

6. Thompson's group V

To define group V we admit bijections of the circle which can discontinuous in a finite number of dyadic points. So, V is the group of right-continuous bijections of S^1 that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers, and such that, on each maximal interval on which the function is differentiable, the function is linear with derivative a power of 2.

We can associate tree diagrams with elements of V as we did for F and T, except that now we need to label the leaves of the domain and range trees to indicate the correspondence between the leaves.

Define

$$\pi_0 : S^1 \to S^1$$

by

$$\pi_0(x) = \begin{cases} \frac{x}{2} + \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\ 2x - 1, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ x, & \frac{3}{4} \leq x \leq 1. \end{cases}$$

Define $\pi_1 = C_2^{-1}\pi_0 C_2$ and $\pi_n = A^{-n+1}\pi_1 A^{n-1}$ for $n > 1$.

Lemma

The elements A, B, C and π_0 generate V and satisfy the following relations:

$$
\begin{aligned}
&1)\ [AB^{-1}, A^{-1}BA] = 1, && 2)\ [AB^{-1}, A^{-2}BA^2] = 1,\\
&3)\ C = B(A^{-l}CB), && 4)\ (A^{-l}CB)(A^{-l}BA) = B(A^{-2}CB^2),\\
&5)\ CA = (A^{-1}CB)^2, && 6)\ C^3 = 1,\\
&7)\ \pi_1^2 = 1, &&\\
&8)\ \pi_1\pi_3 = \pi_3\pi_1, &&\\
&9)\ (\pi_2\pi_1)^3 = 1, &&\\
&10)\ X_3\pi_1 = \pi_1X_3, &&\\
&11)\ \pi_1X_2 = B\pi_2\pi_1, &&\\
&12)\ \pi_2B = B\pi_3, &&\\
&13)\ \pi_1C_3 = C_3\pi_2, &&\\
&14)\ (\pi_1C_3)^3 = 1. &&
\end{aligned}
$$

Let V_1 be a group with generated by A, B, C and π_0 as formal symbols and relations of the previous lemma.

Lemma

In V_1 the elements π_i, $i = 0, 1, 2, ..., n, \ldots$, satisfy the following relations:

$$\begin{cases} \pi_i \pi_j & = \pi_j \, \pi_i, \quad \text{if } \ |i-j| > 1, \\ \pi_i \pi_{i+1} \pi_i & = \pi_{i+1} \pi_i \pi_{i+1}, \\ \pi_i^2 = 1. \end{cases}$$

For each positive integer n, let $\Pi(n)$ be the subgroup of V_1 generated by π_i, $i = 0, 1, 2, ..., n$, and let $\Pi = \cup_{n \in \mathbb{N}} \Pi(n)$.

Let Σ be the infinite group given by its canonical presentation with generators s_i, $i = 0, 1, 2, ..., n, \dots$ and relations:

$$\begin{cases} s_i s_j & = s_j \, s_i, \quad \text{if} \quad |i-j| > 1, \\ s_i s_{i+1} s_i & = s_{i+1} s_i s_{i+1}, \\ s_i^2 = 1. \end{cases}$$

Proposition

The subgroup Π *is isomorphic to* Σ.

Theorem

The group V_1 is simple and isomorphic to V.

Corollary

The group V contains all finite groups.