Braids and some other groups arising in geometry and topology

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Lecture 4

7. Geometrical definition of braids

Braids naturally arise as objects in 3-space. Let us consider two parallel planes Q_0 and Q_1 in $\mathbb{R}^3$, which contain two ordered sets of points A_1, ..., $A_n \in Q_0$ and B_1, ..., $B_n \in Q_1$. These points are lying on parallel lines L_A and L_B respectively. The space between the planes Q_0 and Q_1 we denote by Π.

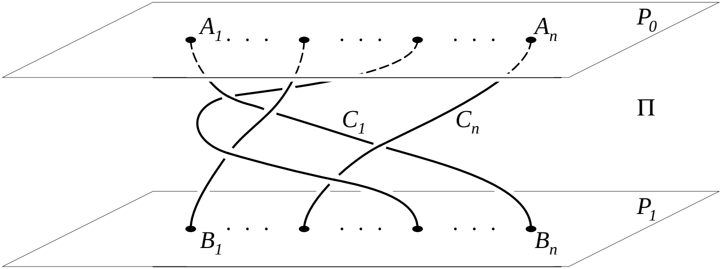

Figure: 1

Suppose that the point B_i is lying under the point A_i, as a result of the orthogonal projection of the plane Q_0 onto the plane Q_1. Let us connect the set of points A_1, ..., A_n with the set of points B_1, ..., B_n by simple non-intersecting curves C_1, ..., C_n lying in the space Π and such that each curve meets only once each parallel plane Q_t lying in the space Π (see Figure 1).
This object is called a *geometric braid* and the curves are called the *strings* or *strands* of a geometric braid.
Two geometric braids β and β' on n strings are *isotopic* if β can be continuously deformed into β' in the class of braids (with the ends fixed).
The relation of isotopy is an equivalence relation on the class of geometric braids on n strings. The corresponding equivalence classes are called *braids on n strings*.

On the set Br_n of braids the structure of a group introduces as follows.

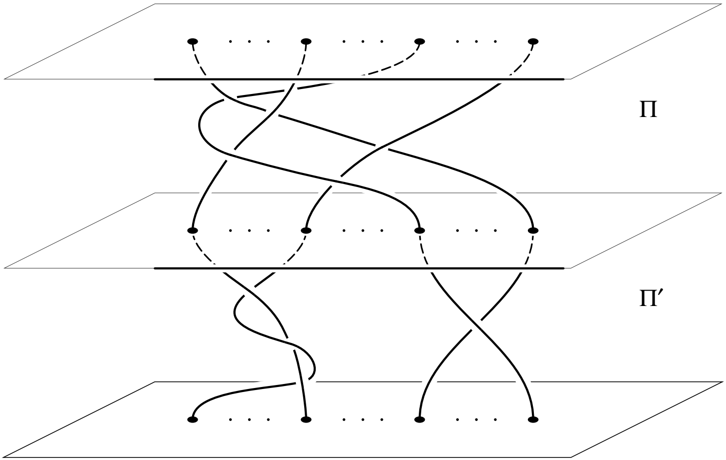

Figure: 2

We put a copy Π' of the domain Π under the Π in such a way that Q_0' coincides with Q_1 and each A_i coincides with B_i and it is possible to glue braids β and β'. After rescaling the height of domain $\Pi \cup \Pi'$ to the height of Π this gluing gives a composition of braids $\beta\beta'$ (Fig. 2).
Unit element is the equivalence class containing a braid of n parallel intervals, the braid β^{-1} inverse to β is defined by reflection of β with respect to the plane $Q_{1/2}$. A string C_i of a braid β connects the point A_i with the pont B_{k_i} defining the permutation S^β. If this permutation is identical then the braid β is called *pure*.
The map $\beta \to S^\beta$ defines an epimorphism τ_n of the braid group Br_n on the permutation group Σ_n with the kernel consisting of all pure braids:

$$1 \to P_n \to Br_n \xrightarrow{\tau_n} \Sigma_n \to 1. \tag{1}$$

8. Configuration spaces

If we look at the Figure 1, then this picture can be interpreted as a graph of a loop in the *configuration space* of n points on a plane, that is the space of unordered sets of n points on a plane, see Figure 3. So, it is possible to interpret the braid group as the fundamental group of the configuration space.

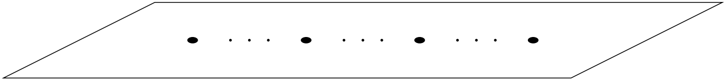

Figure: 3

Formally it is done as follows. The symmetric group Σ_m acts naturally on the Cartesian power $(\mathbb{R}^2)^m$ of the space $\mathbb{R}^2$:

$$w(y_1, \ldots, y_m) = (y_{w^{-1}(1)}, \ldots, y_{w^{-1}(m)}), \quad w \in \Sigma_m. \tag{2}$$

Denote by $F(\mathbb{R}^2, m)$ the space of m-tuples of pairwise different points in $\mathbb{R}^2$:

$$F(\mathbb{R}^2, m) = \{(p_1, \ldots, p_m) \in (\mathbb{R}^2)^m : p_i \neq p_j \text{ for } i \neq j\}.$$

This is the space of regular points of this action. We call the orbit space of this action $B(\mathbb{R}^2, m) = F(\mathbb{R}^2, m)/\Sigma_m$ the *configuration space of n points on a plane*. The braid group Br_m is the fundamental group of configuration space

$$Br_m = \pi_1(B(\mathbb{R}^2, m)).$$

The pure braid group P_m is the is the fundamental group of the space $F(\mathbb{R}^2, m)$. The covering

$$p : F(\mathbb{R}^2, m) \to B(\mathbb{R}^2, m)$$

defines the exact sequence:

$$1 \to \pi_1(F(\mathbb{R}^2, m)) \xrightarrow{p_*} \pi_1(B(\mathbb{R}^2, m)) \to \Sigma_n \to 1.$$

which is equivalent to sequence (1).

9. Artin presentation for braid group

Usually braids are depicted by projections on the plane passing through the lines L_A and L_B. It is supposed to be in general position so that there is only finite number of double points of intersection which are lying on pairwise different levels and intersections are transversal. The simplest braid σ_i corresponds to the transposition $(i, i+1)$.

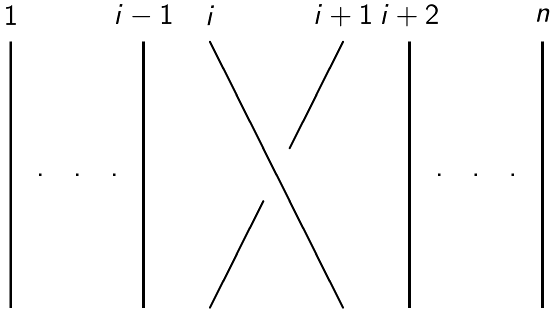

Figure: 4

Artin presentation of the braid group Br_n has generators σ_i, $i = 1, ..., n-1$ and relations:

$$\begin{cases} \sigma_i \sigma_j & = \sigma_j \, \sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

It is the algebraic expression of the fact that any isotopy of braids can be broken down into "elemntary moves" of two types that correspond to two types of relations.
If we add a vertical interval to the system of curves on we can get a canonical inclusion j_n of the group Br_n into the group Br_{n+1}

$$j_n : Br_n \to Br_{n+1}.$$

The canonical presentation the symmetric group Σ_n has the generators s_i, $i = 1, ..., n-1$ and relations:

$$\begin{cases} s_i s_j & = s_j\, s_i, \quad \text{if} \ \ |i-j| > 1, \\ s_i s_{i+1} s_i & = s_{i+1} s_i s_{i+1} \\ s_i^2 = 1, & \end{cases}$$

The homomorphism τ_n is given by the formula

$$\tau_n(\sigma_i) = s_i, \quad i = 1, \ldots, n-1.$$

It is possible to consider braids as classes of equivalence of *braid diagrams* which are generic projections of three dimensional braids on a plane. The classes of equivalence are defined by the *Reidemeister moves* depicted at Figure 5.

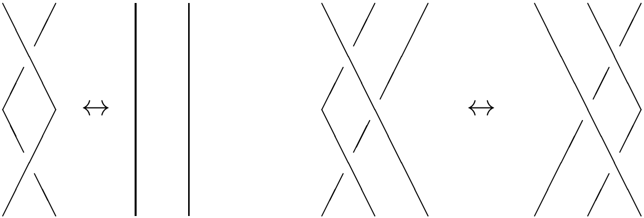

Figure: 5

10. Mapping class groups of a punctured disc

Another important approach to the braid group bases on the following fact.

Theorem

The braid group Br_n is the mapping class group of a punctured disc D_n with its boundary fixed.

11. Braid groups as automorphism groups of free groups

Braid group may be also considered as a subgroup of the automorphism group of a free group.
Let F_n be the free group of rank n with the set of generators $\{x_1, \dots, x_n\}$. Denote by $\operatorname{Aut} F_n$ the automorphism group of F_n.
We define a map from the braid group Br_n into $\operatorname{Aut} F_n$ as follows.
Let $\overline{\sigma}_i \in \operatorname{Aut} F_n, i = 1, 2, \dots, n-1$, be given by the formula which describes its action on generators:

$$\begin{cases} x_i & \mapsto x_{i+1}, \\ x_{i+1} & \mapsto x_{i+1}^{-1} x_i x_{i+1}, \\ x_j & \mapsto x_j, j \neq i, i+1. \end{cases} \tag{3}$$

Let us define a map ν of the generators σ_i, $i = 1, \dots, n-1$ of the braid group Br_n to these automorphisms:

$$\nu(\sigma_i) = \overline{\sigma}_i. \tag{4}$$

Theorem

Formulas (4) define correctly a homomorphism

$$\nu : Br_n \to \mathrm{Aut}\, F_n.$$

which is a monomorphism.

This theorem gives a solution of the word problem for the braid groups.

The free group F_n is a fundamental group of a disc D_n without n points and the generator x_i corresponds to a loop going around the i-th point. The braid group Br_n as the mapping class group of a disc D_n with its boundary fixed acts on the fundamental group of D_n. This action is described by the formulas (3) where x_i correspond to the canonical loops on D_n which form the generators of the fundamental group. Geometrically this action is depicted in the Figure 6.

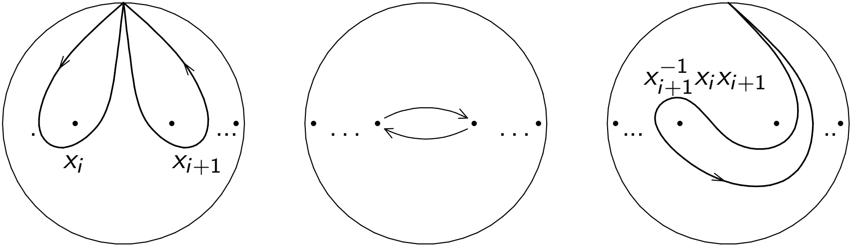

Figure: 6

12. Configuration spaces of manifolds

The notion of configuration space of Section 3 can be naturally generalized for a manifold as follows.
Let Y be a connected topological manifold and let W be a finite group acting on Y. A point $y \in Y$ is called *regular* if its stabilizer $\{w \in W : wy = y\}$ is trivial, i.e., consists only of the unit of the group W. The set $\tilde{Y}$ of all regular points is open. Suppose that it is connected and nonempty. The subspace $B(Y, W)$ of the space of all orbits $Orb(Y, W)$ consisting of the orbits of all regular points is called the *space of regular orbits*. There is a free action of W on $\tilde{Y}$ and the projection $p : \tilde{Y} \to \tilde{Y}/W = B(Y, W)$ defines a covering. Let us consider the initial segment of the long exact sequence of this covering:

$$1 \to \pi_1(\tilde{Y}, y_0) \xrightarrow{p_*} \pi_1(B(Y, W), p(y_0)) \to W \to 1. \qquad (5)$$

The fundamental group $\pi_1(B(Y,W),p(y_0))$ of the space of regular orbits is called the *braid group of the action of W on Y* and is denoted by $Br(Y,W)$. The fundamental group $\pi_1(\tilde{Y},y_0)$ is called the *pure braid group of the action of W on Y* and is denoted by $P(Y,W)$. The spaces $\tilde{Y}$ and $B(Y,W)$ are path connected, so the pair of these groups is defined uniquely up to isomorphism and we may omit mentioning the base point y_0 in the notations.
For any space Y the symmetric group Σ_m acts on the Cartesian power Y^m of the space Y by the formulas (2). We denote by $F(Y,m)$ the space of m-tuples of pairwise different points in Y:

$$F(Y,m)=\{(p_1,...,p_m)\in Y^m : p_i\neq p_j \text{ for } i\neq j\}.$$

This is the space of regular points of this action. In the case when Y is a connected topological manifold M without boundary and $\dim M \geq 2$, the space of regular orbits $B(M^m, \Sigma_m)$ is open, connected and nonempty. We call $B(M^m, \Sigma_m)$ the *configuration space of the manifold* M and denote by $B(M, m)$. The braid group $Br(M^m, \Sigma_m)$ is called the *braid group on m strings of the manifold* M and is denoted by $Br(m, M)$. Analogously, we call the group $P(M^m, \Sigma_m)$ the *pure braid group on m strings of the manifold* M and denote it by $P(m, M)$. These definitions of braid groups were given by R. Fox and L. Neuwirth.

Let $(q_i)_{i\in\mathbb{N}}$ be a fixed sequence of distinct points in the manifold M and put $Q_m = \{q_1, ..., q_m\}$. We use

$$Q_{m,l} = (q_{l+1}, ..., q_{l+m}) \in F(M \setminus Q_l, m)$$

as the standard base point of the space $F(M \setminus Q_l, m)$. If $k < m$ we define the projection

$$\mathrm{proj} : F(M \setminus Q_l, m) \to F(M \setminus Q_l, k)$$

by the formula: $\mathrm{proj}(p_1, ..., p_m) = (p_1, ..., p_k)$. The following statements were proved by E. Fadell and L. Neuwirth.

Theorem

The triple $\mathrm{proj} : F(M \setminus Q_l, m) \to F(M \setminus Q_l, k)$ *is a locally trivial fiber bundle with fiber* $\mathrm{proj}^{-1} Q_{k,l}$ *homeomorphic to* $F(M \setminus Q_{k+l}, m-k)$. *If* $l \geq 1$, *then* proj *admits a cross-section.*

Proof. Let us denote $F(M \setminus Q_l, m)$ by $F_{l,m}$. We give a proof for the case $k = 1$. We first prove that proj is locally trivial. Let U be a neighborhood of q_{l+1} (homeomorphic to Euclidean ball), which avoids Q_l. Furthermore let

$$\theta : U \times \overline{U} \to \overline{U}$$

denote a map with the following properties. Setting $\theta_x(y) = \theta(x, y)$ we require

1. $\theta_x : \overline{U} \to \overline{U}$ is a homeomorphism having $\partial \overline{U}$ fixed.
2. $\theta_x(x) = q_{l+1}$.

Map θ has the obvious extension

$$\theta : U \times M \to M$$

by setting $\theta(x, y) = y$ for $y \notin U$.

The required local product representation

$$\begin{tikzcd}
U \times F_{l+1,m-1} \arrow[rr, "\phi"] \arrow[dr] & & \mathrm{proj}^{-1}(U) \arrow[dl, "\mathrm{proj}"] \\
& U &
\end{tikzcd}$$

is obtained by setting

$$\phi(x, p_2, \dots p_m) = (x, \theta_x^{-1}(p_2), \dots, \theta_x^{-1}(p_m)).$$

Now we prove that proj admits a cross-section if $l \geq 1$. We remark that $\mathrm{proj} : F(M \setminus Q_l, m) \to M \setminus Q_l$ admits a cross-section if and only if there exist $m-1$ fixed point free maps

$$f_2, \ldots, f_m : M \setminus Q_l \to M \setminus Q_l$$

which are non-coincident, i.e.,

$$f_i(x) \neq f_j(x), \quad i \neq j, \quad x \in M \setminus Q_l.$$

We construct such a family of maps. Let V be a neighborhood of q_1 (homeomorphic to Euclidean unit ball), whose closure avoids q_i, $i \geq 2$. Let W denote a ball inside V of radius $\frac{1}{2}$ and $y_2, \ldots, y_m$ mutually disjoint points ∂W. On $\overline{V} \setminus q_1$ define

$$f_i(x) = \|x\| \, y_i, \quad 2 \leq i \leq m, \quad x \in \overline{V} \setminus q_1,$$

and extend to $F(M \setminus Q_l$ by setting

$$f_i(x) = y_i, \quad x \notin V. \quad \square$$

Remark.
There exist spaces without fixed point free maps, for example, real projective plane $\mathbb{R}P^2$. This follows from Lefschetz fixed-point theorem.
Let $f : X \to X$ be a continuous map from a compact triangulable space X to itself. Define the Lefschetz number Λ_f of f by

$$\Lambda_f := \sum_{k \geq 0} (-1)^k \mathrm{Tr}(f_* | H_k(X, \mathbb{Q})),$$

the alternating (finite) sum of the matrix traces of the linear maps induced by f on the $H_k(X, \mathbb{Q})$, the singular homology of X with rational coefficients.

Theorem

If $\Lambda_f \neq 0$ then f has at least one fixed point, i.e. there exists at least one x in X such that $f(x) = x$.

Consideration of the sequence of fibrations

$$F(M \setminus Q_{m-1}, 1) \to F(M \setminus Q_{m-2}, 2) \to M \setminus Q_{m-2},$$

$$F(M \setminus Q_{m-2}, 2) \to F(M \setminus Q_{m-3}, 3) \to M \setminus Q_{m-3},$$

$$\cdots ,$$

$$F(M \setminus Q_1, m-1) \to F(M, m) \to M$$

leads to the following statement.

Theorem

For any manifold M

$$\pi_i(F(M \setminus Q_1, m-1)) = \oplus_{k=1}^{m-1} \pi_i(M \setminus Q_k)$$

for $i \geq 2$. If proj : $F(M, m) \to M$ *admits a section then*

$$\mathrm{proj}_i \, \pi_i(F(M, m)) = \oplus_{k=0}^{m-1} \pi_i(M \setminus Q_k), \;\; i \geq 2.$$

Corollary

If M is the Euclidean r-space, then

$$\pi_i(F(M,m)) = \oplus_{k=0}^{m-1}\pi_i(\underbrace{S^{r-1}\vee ... \vee S^{r-1}}_{k}), \ \ i \geq 2.$$

Corollary

If M is the Euclidean 2-space, then $F(\mathbb{R}^2,m)$ is the $K(P_m,1)$-space and $B(\mathbb{R}^2,m)$ is the $K(Br_m,1)$-space.