Braids and some other groups arising in geometry and topology

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Lecture 6

17. Classifying space for Thompson group F

We recall that a space X is a *classifying space* for a group G if $\pi_1(X) \cong G$ and the universal cover $\widetilde{X}$ of X is contractible. This is equivalent to saying X is a $K(G,1)$ space, meaning that $\pi_k(X)$ is trivial when $k \neq 1$ and is G when $k = 1$. This condition that the higher homotopy groups vanish is called being *aspherical*.

We now define the space $\mathcal{C}F$. Let $\mathcal{C}F_n$ denote the space of all n-tuples of real numbers, $(t_1, \dots, t_n)$, such that:

1. the entries are non-decreasing, i.e., $t_1 \leq \cdots \leq t_n$, and
2. for all $i = 1, \dots, n-2$, $t_{i+2} - t_i \geq 1$.

This second condition should be thought of as saying that no three distinct entries are too close together. For example, $(1, 1, 2)$ is a point in $\mathcal{C}F_3$, but $(1, 1, 3/2)$ is not.

Let

$$\mathcal{C}F = \left(\coprod_{n=1}^{\infty} \mathcal{C}F_n \right) / \sim$$

denote the disjoint union of the spaces $\mathcal{C}F_n$, subject to the identifications

$$(t_1, \ldots, t_i, \ldots, t_n) \sim (t_1, \ldots, t_i, t_i, \ldots, t_n)$$

whenever $(t_1, \ldots, t_i, t_i, \ldots, t_n) \in \mathcal{C}F_{n+1}$, that is whenever t_i is at least 1 away from its neighbors.

For each n, $\mathcal{C}F_n$ is contractible, for instance to the point $p_n = (1, \ldots, n)$ in $\mathcal{C}F_n$. The contraction is given by the homotopy that at time $0 \leq t \leq 1$ takes the point $\vec{t} = (t_1, \ldots, t_n)$ to $p_n t + (1 - t)\vec{t}$.

The fundamental group of $\mathcal{C}F$ (which will be F) will come from the identifications arising from $\sim$. Points in $\mathcal{C}F_n$ are identified with points in $\mathcal{C}F_{n+1}$ in at most n ways, once for each allowed bifurcation of an entry in $\mathcal{C}F_n$, so non-trivial elements of $\pi_1(\mathcal{C}F)$ can arise, for example, by splitting one entry and then merging into a different entry.

Then non-trivial relations in $\pi_1(\mathcal{C}F)$ arise from the fact that splits and merges that are far enough apart may happen in any order.

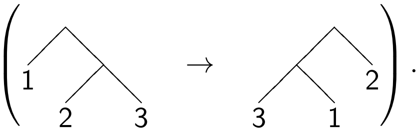
Theorem

The space $\mathcal{C}F$ is a classifying space for F.

Braids and Thompson groups

18. Informal introduction to braided Thompson group BV

As we saw elements of Thompson group V can coded by pairs of labeled binary trees. For example:



Elements of Thompson group V can also be described with the help of the standard Georg Cantor set K.

Let us recall the standard "deleted middle thirds" description of the Cantor set K. The set K is a limit of a sequence of collections of closed intervals in the unit interval $[0, 1]$.

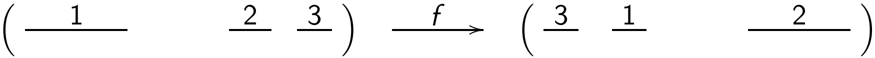
The first few collections are

$$
\begin{gathered}
\{[0,1]\} \\
\{[0,\tfrac{1}{3}]\ ,\ [\tfrac{2}{3},1]\} \\
\{[0,\tfrac{1}{9}]\ ,\ [\tfrac{2}{9},\tfrac{1}{3}]\ ,\ [\tfrac{2}{3},\tfrac{7}{9}]\ ,\ [\tfrac{8}{9},1]\} \\
\{[0,\tfrac{1}{27}]\ ,\ [\tfrac{2}{27},\tfrac{1}{9}]\ ,\ [\tfrac{2}{9},\tfrac{7}{27}]\ ,\ [\tfrac{8}{27},\tfrac{1}{3}]\ ,\ [\tfrac{2}{3},\tfrac{19}{27}]\ ,\ [\tfrac{20}{27},\tfrac{7}{9}]\ ,\ [\tfrac{8}{9},\tfrac{25}{27}]\ ,\ [\tfrac{26}{27},1]\} \\
\vdots
\end{gathered}
\tag{1}
$$

Elements of V are defined using covers of K by pairwise disjoint intervals chosen from the collections in (1).

Given a pair of such covers with the same number of intervals (necessarily finite) and a one-to-one correspondence between them, an element of V is created by taking each interval in the first cover affinely in an orientation preserving way onto the corresponding interval in the second cover and restricting this map to K.

The restriction is a homeomorphism of K. The group V is the set of all such maps. Below, we show one the map f where we indicate the bijection by numbering the intervals.



The map f is the restriction of the following affine surjections

$$[0, \tfrac{1}{3}] \to [\tfrac{2}{9}, \tfrac{1}{3}], \qquad [\tfrac{2}{3}, \tfrac{7}{9}] \to [\tfrac{2}{3}, 1], \qquad [\tfrac{8}{9}, 1] \to [0, \tfrac{1}{9}]$$

to the portions of K contained in the given intervals.

The structure of the left tree indicates that the interval $[0, 1]$ (corresponding to the root at the top of the tree) is to be split, and the resulting right interval $[\frac{2}{3}, 1]$ is to be split again. This describes the intervals in the domain of f. The right tree codes the splittings needed to describe the intervals in the range of f.

To obtain an element of BV, we embed in the plane $\mathbb{R}^2$ as a subset of the x-axis.

Let K be covered by a collection I of pairwise disjoint intervals from (1) and also a similar collection J with the same number of intervals.

An element of BV will take intervals in I to intervals in J exactly as described above, but the move will be accomplished by an isotopy of $\mathbb{R}^2$ with compact support.

That is, the move will be accomplished by braiding if we view the isotopy as a level preserving homeomorphism from $\mathbb{R}^2 \times [0, 1]$ to itself and letting the braid strands be the images of the components of $K \times [0, 1]$.

A restriction that must be observed is that during the isotopy, each interval in I must have its image during the isotopy parallel to the x-axis at all times.

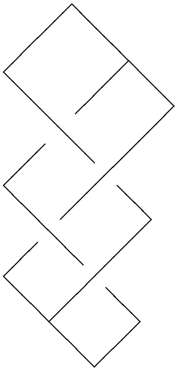
Isotopies u and v are equivalent if there is a level preserving isotopy of $\mathbb{R}^2 \times [0, 1]$ from u to v (adhering to the restriction that the images of intervals from I be kept parallel to the x-axis throughout) that are fixed on the Cantor set at the 0 and 1 levels.

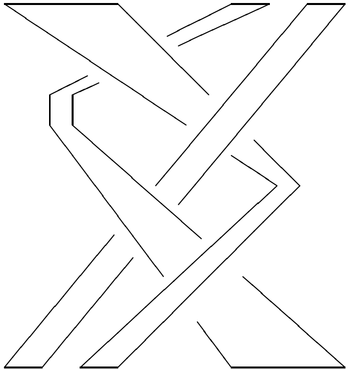
Thus BV is seen to be a subgroup of a braid group on a Cantor set of strands.

The surjection from BV to V is obtained by taking each element of BV to the homeomorphism of K obtained at the end of the isotopy.

An element of BV can also be coded by pairs of binary trees, but now the connection from the leaves of the first to the leaves of the second is given by a braid and not a bijection. This is most easily pictured by drawing the second tree upside down below the first and drawing the braid connecting the leaves between the two trees.

As an example, the following is one element of BV that maps to the element f of V in the example above. We draw both the trees and braid encoding of the element as well as a picture of a braiding of intervals.





19. Formal definition to braided Thompson group BV

The group BV has the following presentation. The equalities $v_i = v_0^{1-i} v_1 v_0^{i-1}$ and $\bar{\pi}_i = v_0^{1-i} \bar{\pi}_1 v_0^{i-1}$ for all $i \geq 2$, and $\pi_i = \bar{\pi}_i v_i \bar{\pi}_{i+1}^{-1}$ for all $i \in \mathbb{N}$ are taken as definitions.

$$
\begin{aligned}
BV = v_0, v_1, \bar{\pi}_0, \bar{\pi}_1 \,\Big|\, v_2 v_1 &= v_1 v_3, & v_3 v_1 &= v_1 v_4,\\
\bar{\pi}_2 v_1 &= v_1 \bar{\pi}_3, & \bar{\pi}_3 v_1 &= v_1 \bar{\pi}_4,\\
\pi_0 v_0 &= v_1 \pi_0 \pi_1, & \pi_1 v_1 &= v_2 \pi_1 \pi_2,\\
\pi_0^{-1} v_0 &= v_1 \pi_0^{-1} \pi_1^{-1}, & \pi_1^{-1} v_1 &= v_2 \pi_1^{-1} \pi_2^{-1},\\
\pi_0 v_2 &= v_2 \pi_0, & \pi_0 v_3 &= v_3 \pi_0,\\
\pi_1 v_3 &= v_3 \pi_1, & \pi_1 v_4 &= v_4 \pi_1,\\
\pi_0 &= \bar{\pi}_1^{-1} v_0^{-1} \bar{\pi}_0, & \pi_1 &= \bar{\pi}_2^{-1} v_1^{-1} \bar{\pi}_1,
\end{aligned}
$$

$$\pi_0\pi_2 = \pi_2\pi_0, \qquad \pi_0\pi_3 = \pi_3\pi_0,$$
$$\pi_1\pi_3 = \pi_3\pi_1, \qquad \pi_1\pi_4 = \pi_4\pi_1,$$
$$\pi_0\pi_1\pi_0 = \pi_1\pi_0\pi_1, \pi_1\pi_2\pi_1 = \pi_2\pi_1\pi_2,$$
$$\bar\pi_2\pi_0 = \pi_0\bar\pi_2, \qquad \bar\pi_3\pi_0 = \pi_0\bar\pi_3,$$
$$\bar\pi_3\pi_1 = \pi_1\bar\pi_3, \qquad \bar\pi_4\pi_1 = \pi_1\bar\pi_4,$$
$$\pi_0\bar\pi_1\pi_0 = \bar\pi_1\pi_0\bar\pi_1, \pi_1\bar\pi_2\pi_1 = \bar\pi_2\pi_1\bar\pi_2$$

A presentation for the group V is obtained from that for BV by adding the relations $\bar{\pi}_0^2 = \bar{\pi}_1^2 = \pi_0^2 = \pi_1^2 = 1$.

19. Group $\widehat{BV}$. Another model for braided Thompson group.

Definition

Let $\widehat{BV}$ be a group given by generators a_i, σ_j; $i, j = 1, 2, \ldots$, and the following relations: For $i \geq 1$ and $j \geq i+2$,

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \sigma_i a_j = a_j\sigma_i, \quad a_i a_{j-1} = a_j a_i, \quad a_i\sigma_{j-1} = \sigma_j a_i, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ \sigma_{i+1}\sigma_i a_{i+1} = a_i\sigma_i, \ \sigma_i\sigma_{i+1}a_i = a_{i+1}\sigma_i. \end{cases} \tag{2}$$

Theorem

The subgroup of $\widehat{BV}$ *generated by* σ_* *is (a copy of) the braid group* B_∞*, and the subgroup generated by* a_* *is (a copy of) Thompson's group* F*. These subgroups generate* $\widehat{BV}$*, and their intersection is* $\{1\}$*.*