Theories for Feasible Set Functions

Arnold Beckmann

joint work with Sam Buss, Sy-David Friedman, Moritz Müller and Neil Thapen

(work in progress)

Department of Computer Science
College of Science, Swansea University
Wales, UK

NUS, 16 April 2015

Motivation

Recent proposals for restrictions of primitive recursive set functions to feasible computation:

Safe Recursive Set Functions [B., Buss, Sy Friedman, accepted JSL 2015, revision on webpages]

Predicatively Computable Set Functions [Arai, AML vol. 54 (2015), pp. 471–485]

Cobham Recursive Set Functions [B., Buss, Sy Friedman, Müller, Thapen, work in progress]

Theories for feasible computation

Kripke-Platek set theory KP consists of axioms Extensionality, Pair, Union, Set Foundation, along with schemas of Δ_0-Collection, Δ_0-Separation, and Foundation for definable classes.

M.Rathjen: A Proof-Theoretic Characterization of the Primitive Recursive Set Functions. JSL 57(3), 1992 asked and answered:

> *Is there, by analogy with PA, a neat subsystem of* ZF *which characterises the primitive recursive set functions?*

Theorem (Rathjen'92)

Let KP^- *be* KP *without Foundation for definable classes.*
The Σ_1*-definable functions in* $\mathrm{KP}^- + \Sigma_1$*-*$\in$*-Induction*
are exactly the primitive recursive set functions.

Σ_1-$\in$-Induction: $\forall x(\forall y{\in}x\, \varphi(y) \to \varphi(x)) \to \forall x \varphi(x)$ for $\varphi \in \Sigma_1$.

Inspired by this we will consider the question:

> *Is there, by analogy with Bounded Arithmetic, a neat subsystem of* ZF *which characterises feasible set functions?*

Analogy to bounded arithmetic: In bounded arithmetic a la Buss:

bounded quantifiers: $\forall x \leq t$, $\exists x \leq t$

sharply bounded quantifiers: $\forall x \leq |t|$, $\exists x \leq |t| \quad (|t| \approx \log t)$

In set theory:

bounded quantifiers: $\forall x \in y$, $\exists x \in y$

Hypothesis

Bounded set quantification corresponds to *sharply bounded arithmetic quantification.*

Reason: For feasible set functions like SRSF, rank on sets plays role of binary length in arithmetic.

For ordinals, bounded set quantification thus behaves like sharply bounded quantification.

New Bounded Set Quantification

Add a new relation to set theory, $x \preceq y$, for *rank comparison*:

$$x \preceq y \quad \text{iff} \quad \text{rank}(x) \leq \text{rank}(y)$$

Define *rank bounded quantification*:

$\forall x \preceq t\ \varphi(x)$ abbreviates $\forall x(x \preceq t \rightarrow \varphi(x))$

$\exists x \preceq t\ \varphi(x)$ abbreviates $\exists x(x \preceq t \wedge \varphi(x))$

Definition

Σ_1^b set formulas are of the form $\exists x \preceq t\ \varphi(x)$ for $\varphi \in \Delta_0$.

Bounded Set Theories

Expand language with some function symbols to support bootstrapping (otherwise rank bounded quantification is too weak to define e.g. polynomials).

Definition

Let $\mathcal{L}_{\mathrm{FST}}$ expand $L_\in$ by $\qquad 0, \mathsf{Succ}, +, \times, \mathsf{rank} \ldots$

Let KP^- be KP without Foundation.

Let $\mathrm{KP}^{\mathrm{b}}_1$ be KP^- plus Σ^b_1-$\in$-Induction.

Theorem

The Σ_1 definable set functions of $\mathrm{KP}^{\mathrm{b}}_1$ are exactly those in SRSF.

Outline of talk

1. Safe Recursive Set Functions
   - Safe Recursive Set Functions SRSF
   - Definability Characterisations of SRSF
2. Set Theories for SRSF
   - Set Theories for SRSF
3. Characterizing SRSF
   - Defining SRSF in $\mathrm{KP}^{\mathrm{b}}_1$
   - Applications
4. Conclusion

Safe Set Functions

(Inspired by *S.Bellantoni and S.A.Cook: A new recursion-theoretic characterization of the polytime functions. Comput. Complexity, 2(2):97-110, 1992.*)

Safe Set Functions:

$$f(x_1, \dots, x_k \,/\, a_1, \dots, a_\ell)$$

denotes a function on sets, whose arguments are typed into normal positions $x_1, \dots, x_k$, and safe positions $a_1, \dots, a_\ell$.

Idea: The Safe Recursive Set Functions are obtained by imposing the above typing scheme onto Primitive Recursive Set Functions.

Safe Recursive Set Functions

The Safe Recursive Set Functions (SRSF) are the smallest class containing i) – iii), and being closed under iv) – vi).

i) $\pi_j^{n,m}(x_1, \ldots, x_n \,/\, x_{n+1}, \ldots, x_{n+m}) = x_j$, for $1 \leq j \leq n + m$.

ii) $\mathrm{diff}(/\, a, b) = a \setminus b$

iii) $\mathrm{pair}(/\, a, b) = \{a, b\}$

iv) **(Rudimentary Union Scheme)**
$f(\vec{x} \,/\, \vec{a}, b) = \bigcup_{z \in b} g(\vec{x} \,/\, \vec{a}, z)$

v) **(Safe Composition Scheme)**
$f(\vec{x} \,/\, \vec{a}) = h(\vec{r}(\vec{x} \,/) \,/\, \vec{t}(\vec{x} \,/\, \vec{a}))$

vi) **(Safe Set Recursion Scheme)**
$f(x, \vec{y} \,/\, \vec{a}) = h(x, \vec{y} \,/\, \vec{a}, \{f(z, \vec{y} \,/\, \vec{a}) \colon z \in x\})$

Bounding Ranks

Examples

Successor, addition and multiplication on ordinals

$$\mathsf{Succ}(/\,\alpha) = \alpha + 1,\ \mathsf{Add}(\beta\,/\,\alpha) = \alpha + \beta,\ \mathsf{Mult}(\alpha, \beta\,/) = \alpha \cdot \beta$$

are in SRSF.

But ordinal exponentiation is *not* safe recursive:

Theorem

Let f be a safe recursive set function. There is a polynomial q_f such that

$$\mathrm{rank}(f(\vec{x}\,/\,\vec{a})) \quad \leq \quad \max(\mathrm{rank}(\vec{a})) + q_f(\mathrm{rank}(\vec{x}))$$

for all sets $\vec{x}$, $\vec{a}$.

SRSF and the M-Hierarchy

$\vec{x}$ tuples of sets encoded as set sequences, $\star$ sequence concatenation

Let $\mathrm{Succ}(T) = T \cup \{T\}$;
$G_1, \ldots, G_{10}$ functions used by Gödel to define L.

Definition

$$
\begin{aligned}
M_0^T &= T \\
M_{\alpha+1}^T &= \mathrm{Succ}(M_\alpha^T) \cup \bigcup_{1 \le i \le 10} \mathrm{range}(G_i(\mathrm{Succ}(M_\alpha^T)) \times G_i(\mathrm{Succ}(M_\alpha^T))) \\
M_\lambda^T &= \bigcup_{\alpha<\lambda} M_\alpha^T \qquad \text{for limit } \lambda
\end{aligned}
$$

Definition

For sets $\vec{x}, \vec{y}$ and $0 < n < \omega$ define $\mathrm{SR}_n^*(\vec{x} \,/\, \vec{y}) := M_{n+\mathrm{rank}(\vec{x})^n}^{\mathrm{tc}(\vec{x} \star \vec{y})}$

Theorem (Sy Friedman, '11)

SRSF *functions are exactly the* $f(\vec{x} \,/\, \vec{y})$ *which are uniformly definable in* $\mathsf{SR}^*_n(\vec{x} \,/\, \vec{y})$ *for some finite n.*

Corollary

The SR functions on ω*-strings coincide with those computable by an infinite-time Turing machine in time* ω^n *for some finite n, and were considered by Deolaliker, Hamkins, Schindler, Welch and others.*

Bounded Set Theories

rank comparison: $x \preceq y$ iff $\mathrm{rank}(x) \leq \mathrm{rank}(y)$

Rank bounded quantification: $\forall x \preceq t\ \varphi(x)$ and $\exists x \preceq t\ \varphi(x)$

Σ_1^b formulas of form $\exists x \preceq t\ \varphi(x)$ for $\varphi \in \Delta_0$.

Definition

Let $\mathcal{L}_{\mathrm{FST}}$ expand $L_\in$ by $0, \mathrm{Succ}, +, \times, \mathrm{rank} \ldots$

Let KP^- be KP without Foundation, and define

$\mathrm{KP}_1^{\mathrm{b}}$ to be KP^- plus Σ_1^b-$\in$-Induction.

Let KP^{--} be KP without Δ_0-Collection and Foundation, and define

T_1 to be KP^{--} plus Δ_0-b-Collection plus Σ_1^b-$\in$-Induction.

Δ_0-b-Collection is

$$\forall x \in a\ \exists y \preceq t(x)\ \varphi(x,y) \ \rightarrow\ \exists z\ \forall x \in a\ \exists y \in z\ \varphi(x,y)$$

with $\varphi(x,y) \in \Delta_0$.

Adapt Barwise

The first chapters of Barwise *Admissible Sets and Structures* can be adapted to T_1 for Σ_1^b and Π_1^b instead of Σ_1 and Π_1:

Definition

Σ^b smallest class containing Δ_0 and closed under conjunction, disjunction, bounded quantification and existential rank bounded quantification.
Π^b dually defined.

Definition

Given formula φ and variable a not occurring in φ, write $\varphi^{(a)}$ for result of replacing each *unbounded* quantifier (this includes rank bounded quantifiers) by *bounded* quantifier; that is
replace $\exists x$ by $\exists x \in a$, and $\forall x$ by $\forall x \in a$.

We observe that $\varphi^{(a)}$ is a Δ_0-formula. If φ is Δ_0 then $\varphi^{(a)} = \varphi$.
The following statements are already logically valid for Σ^b-formulas φ.

1. $\varphi^{(a)} \wedge a \subseteq b \rightarrow \varphi^{(b)}$
2. $\varphi^{(a)} \rightarrow \varphi$

where $a \subseteq b$ abbreviates the formula $\forall x \in a\, (x \in b)$.

Theorem (The Σ^b Reflection Principle)

For every Σ^b formula φ there exists an $\mathcal{L}_{\mathrm{FST}}$-term t whose variables are amongst the free variables of φ such that:

$$T_1 \vdash \varphi \leftrightarrow \exists x \preceq t\, \varphi^{(x)} .$$

In particular, any Σ^b formula is equivalent to some Σ^b_1 formula in T_1.

Theorem (The Σ^b Bounded Collection Principle)

For any Σ^b formula φ the following is a theorem of T_1:
If $\forall x \in a\, \exists y \preceq b\, \varphi(x,y)$, then there is a set c such that $c \preceq \mathrm{rank}(b) + 1$, $\forall x \in a\, \exists y \in c\, \varphi(x,y)$ and $\forall y \in c\, \exists x \in a\, \varphi(x,y)$.

Theorem (Δ^b Separation)

For any Σ^b formula $\varphi(x)$ and Π^b formula $\psi(x)$, the following is a theorem of T_1:
If for all $x \in a$, $\varphi(x) \leftrightarrow \psi(x)$, then there is a set $b = \{x \in a\colon \varphi(x)\}$.

Theorem (Σ^b Replacement)

For each Σ^b formula $\varphi(x, y)$ the following is a theorem of T_1:
If $\forall x \in a\, \exists! y \preceq b\, \varphi(x, y)$, then there is a function f, with $\mathrm{dom}(f) = a$, such that $\forall x \in a\, \varphi(x, f(x))$.

Theorem (Strong Σ^b Replacement)

For each Σ^b formula $\varphi(x, y)$ the following is a theorem of T_1:
If $\forall x \in a\, \exists! y \preceq b\, \varphi(x, y)$, then there is a function f, with $\mathrm{dom}(f) = a$, such that

1. $\forall x \in a\, f(x) \neq \emptyset$;
2. $\forall x \in a\, \forall y \in f(x)\, \varphi(x, y)$.

Defined relations

Definition

Let $\varphi(x_1, \ldots, x_n)$ be a Σ^b formula of $\mathcal{L}_{\mathrm{FST}}$ and $\psi(x_1, \ldots, x_n)$ be a Π^b formula of $\mathcal{L}_{\mathrm{FST}}$ such that $T_1 \vdash \varphi \leftrightarrow \psi$. Let R be a new n-ary relation symbol and define R by

$$\forall x_1 \ldots \forall x_n \big[\mathrm{R}(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)\big] \tag{R}$$

R is then called a Δ^b *relation symbol* of T_1.

Lemma

Let T_1 *be formulated in* $\mathcal{L}_{\mathrm{FST}}$ *and let* R *be a* Δ^b *relation symbol of* T_1*. Let* T_1' *be* T_1 *as formulated in* $\mathcal{L}_{\mathrm{FST}}(\mathrm{R})$*, plus the defining axiom* (R) *above.*

1. *For every formula* $\theta(\vec{x}, \mathrm{R})$ *of* $\mathcal{L}_{\mathrm{FST}}(\mathrm{R})$*, there is a formula* $\theta_0(\vec{x})$ *of* $\mathcal{L}_{\mathrm{FST}}$ *such that*

   $$T_1 + (\mathrm{R}) \vdash \theta(\vec{x}, \mathrm{R}) \leftrightarrow \theta_0(\vec{x})$$

   Moreover, if θ *is a* Σ^b *formula of* $\mathcal{L}_{\mathrm{FST}}(\mathrm{R})$ *then* θ_0 *is a* Σ^b*-formula of* $\mathcal{L}_{\mathrm{FST}}$*.*
2. *For every* Δ_0 *formula* $\theta(\vec{x}, \mathrm{R})$ *of* $\mathcal{L}_{\mathrm{FST}}(\mathrm{R})$*, there are* Σ^b *and* Π^b *formulas* $\theta_0(\vec{x})$*,* $\theta_1(\vec{x})$ *of* $\mathcal{L}_{\mathrm{FST}}$ *such that*

   $$T_1 + (\mathrm{R}) \vdash \theta(\vec{x}, \mathrm{R}) \leftrightarrow \theta_0(\vec{x}) \quad \text{and} \quad T_1 + (\mathrm{R}) \vdash \theta(\vec{x}, \mathrm{R}) \leftrightarrow \theta_1(\vec{x})$$
3. T_1' *is a conservative extension of* T_1*.*

Definition

Let $\varphi(x_1, \ldots, x_n, y)$ be a Σ^b formula of $\mathcal{L}_{\mathrm{FST}}$ and $t(x_1, \ldots, x_n)$ a term of $\mathcal{L}_{\mathrm{FST}}$ such that

$$T_1 \vdash \forall x_1, \ldots, x_n \exists! y \preceq t(x_1, \ldots, x_n) \varphi(x_1, \ldots, x_n, y) \ .$$

Let F be a new n-ary function symbol and define F by

$$\forall x_1 \ldots x_n, y \big[\mathrm{F}(x_1, \ldots, x_n) = y \ \leftrightarrow \ \varphi(x_1, \ldots, x_n, y) \big] \qquad \text{(F)}$$

F is then called a Σ^b *function symbol* of T_1.

Lemma

Let T_1 be formulated in $\mathcal{L}_{\mathrm{FST}}$ and let F be a Σ^b function symbol of T_1. Let T_1' be T_1 as formulated in $\mathcal{L}_{\mathrm{FST}}(\mathrm{F})$, plus the defining axiom (F) *above.*

1. *For every formula $\theta(\vec{x},\mathrm{F})$ of $\mathcal{L}_{\mathrm{FST}}(\mathrm{F})$, there is a formula $\theta_0(\vec{x})$ of $\mathcal{L}_{\mathrm{FST}}$ such that*

$$T_1 + (\mathsf{F}) \vdash \theta(\vec{x},\mathrm{F}) \leftrightarrow \theta_0(\vec{x})$$

   Moreover, if θ is a Σ^b formula of $\mathcal{L}_{\mathrm{FST}}(\mathrm{F})$ then θ_0 is a Σ^b-formula of $\mathcal{L}_{\mathrm{FST}}$.

2. *For every Δ_0 formula $\theta(\vec{x},\mathrm{F})$ of $\mathcal{L}_{\mathrm{FST}}(\mathrm{F})$, there are Σ^b and Π^b formulas $\theta_0(\vec{x})$, $\theta_1(\vec{x})$ of $\mathcal{L}_{\mathrm{FST}}$ such that*

$$T_1 + (\mathsf{F}) \vdash \theta(\vec{x},\mathrm{F}) \leftrightarrow \theta_0(\vec{x}) \quad \text{and} \quad T_1 + (\mathsf{F}) \vdash \theta(\vec{x},\mathrm{F}) \leftrightarrow \theta_1(\vec{x})$$

3. *T_1' is a conservative extension of T_1.*

T_1 characterises SRSF

Definition

T_1 Σ_1-defines a set function f if there is a Σ_1 formula φ such that $V \models \forall x \varphi(x, f(x))$ and $T_1 \vdash \forall x \exists ! y \varphi(x, y)$.

T_1 Σ^b_1-defines a set function f if $\varphi \in \Sigma^b_1$.

Theorem (B.14)

The Σ_1-definable functions of T_1 are exactly the safe recursive set functions.

Proof utilises characterisation of SRSF as those functions uniformly definable in SR^* using the M-hierarchy.

Corollary

The Δ_1 definable predicates of T_1 are exctly the predicates in SRSF.

Defining SRSF in T_1

We show that all SRSF functions are Σ_1^b-definable in T_1

Let ψ_n be Σ_1^b-definition of $x \mapsto \mathsf{SR}_n^*(x\,/)$ in T_1.

$f(x\,/) \in \mathrm{SRSF} \quad \Rightarrow \quad$ (uniformly definable in SR^*)

$$f(x\,/) = z \quad \text{iff} \quad \mathsf{SR}_n^*(x\,/) \vDash \varphi(x,z) \qquad \text{by some } \varphi$$

Hence $\quad f(x\,/) = \bigcup \left\{ z \in \mathsf{SR}_n^*(x\,/) \colon \varphi(x,z)^{\mathsf{SR}_n^*(x\,/)} \right\}$

Let $\chi(x,u,v)$ be the Δ_0-formula

$$\begin{aligned} \psi_n(x,u) \quad & \wedge \quad \forall y \in v \; (y \in u \wedge \varphi^u(x,y)) \\ & \wedge \quad \forall y \in u \; (\varphi^u(x,y) \; \rightarrow \; y \in v) \end{aligned}$$

and let $\chi(x,z)$ be $\exists u,v\; (\chi(x,u,v) \wedge z = \bigcup v)$. Then

$V \vDash \forall x\; \chi(x, f(x\,/))$ and $V \vDash \forall x\; \exists! z\; \chi(x,z)$

$T_1 \vdash \forall x,u,u',v,v'\; (\chi(x,u,v) \wedge \chi(x,u',v') \; \rightarrow \; u = u' \wedge v = v')$

using Σ_1-definability of SR_n^* and Extensionality, and

$T_1 \vdash \forall x\; \exists u,v\; \chi(x,u,v)$

using Σ_1-definability of SR_n^* and Δ_0-Separation.

Proof theoretic analysis

Adapt M.Rathjen's argument:

$f(x)$ Σ_1-definable in T_1

$\Rightarrow$ exists $\varphi(x,y,z) \in \Delta_0$ such that $V \models \forall x \exists z \varphi(x, f(x), z)$ and $T_1 \vdash \forall x \exists! y \exists z \varphi(x,y,z)$

$\Rightarrow$ $T_1 \vdash \forall x \exists u \psi(x,u)$ for $\psi(x,u)$ denoting $\varphi(x,(u)_0,(u)_1)$.

Interpretation Theorem (next slide) shows that there exists finite n such that

$V \models \exists u \in M^{\mathrm{tc}(\{x\})}_{n+\mathrm{rank}(x)^n}\ \psi(x,u)$ for any $x \in V$

$\Rightarrow$ $\mathrm{SR}^*_m(x\,/) \models \exists y\, \exists z\ \varphi(x,y,z)$ for some $m \geq n$ independent of x

$\Rightarrow$ $f(x) = y$ iff $\mathrm{SR}^*_m(x\,/) \models \exists z\ \varphi(x,y,z)$

$\Rightarrow$ $f \in \mathrm{SRSF}$

Interpretation Theorem

For formula ψ (which may contain unbounded quantifiers) write $\psi_v^{\alpha,\beta}$ for replacing each *unbounded* quantifier (this includes range bounded quantifiers) $\forall x$ and $\exists x$ in ψ by $\forall x \in M_\alpha^{\mathrm{tc}(v)}$ and $\exists x \in M_\beta^{\mathrm{tc}(v)}$, respectively.

Theorem (Interpretation Theorem, B.14)

Let $\Gamma(\vec{a})$ be set of $\Delta_0(\Sigma_1)$ formulas with free variables amongst $\vec{a}$.
If $T_1 \vdash \Gamma(\vec{a})$, then there exists polynomial p
such that

$$V \models \bigvee \Gamma(\vec{u})_v^{\alpha,\alpha+p(\mathrm{rank}(\vec{u}))} \text{ for all } \alpha \text{ and } \vec{u}, v \text{ such that } \vec{u} \in M_\alpha^v.$$

Conclusion can be strengthend to be provable in T_0

Collection Rule

Δ_0 Collection Rule: $$\frac{\Gamma,\ b\notin t,\ \exists y\varphi(b,y)}{\Gamma,\ \exists z\,\forall x{\in}t\,\exists y{\in}z\,\varphi(x,y)}$$

with b Eigenvariable, $\varphi \in \Delta_0$ and $\Gamma \subset \Sigma_1 \cup \Pi_1$

Theorem (B.'15)

T_1 is closed under above Δ_0 Collection Rule.

Corollary

$\mathrm{KP}^{\mathrm{b}}_1$ is $\forall\Sigma_1$ conservative over T_1.

Corollary

The Σ_1 definable functions of $\mathrm{KP}^{\mathrm{b}}_1$ are exactly SRSF.

Corollary

The Δ_1 definable predicates of $\mathrm{KP}^{\mathrm{b}}_1$ are exctly the predicates in SRSF.

Open Problems / Work in Progress

- Can KP_1^b be interpreted in some fragement of Bounded Arithmetic (in analogy to fact that $\mathrm{KP}^- + \Sigma_1$-$\in$-Induction can be interpreted in $I\Sigma_1$)
- What is the "right" definition for theory for CRSF? Perform bootstrapping.
- Once the right definitions have been settled, study bounded set theory hierarchy KP_i^b and characterise their Σ_1-definable functions. Links to "usual" complexity theory?

Summary

- Defined a restriction of Kripke-Platek set theory $\mathrm{KP}^{\mathrm{b}}_1$ by defining new "rank" bounded quantifier $\exists x \preceq y$ to define Σ^b_1, and restricting class foundation to Σ^b_1-$\in$-induction.
- Showed that the Σ_1 definable functions in $\mathrm{KP}^{\mathrm{b}}_1$ are exactly the SRSF functions.
- Main proof theoretic tool: the interpretation theorem, that showed that in proofs of $\Sigma_1 \cup \Pi_1$ statements in (a fragment of) $\mathrm{KP}^{\mathrm{b}}_1$, witnesses for existential quantifiers can be found polynomially above given witnesses to universal quantifiers in the M-hierarchy.

Take Away Message:

Combining set theory and complexity theory is fruitful and fun!