Maximal orthogonal families in the Sacks extension

Asger Törnquist (Copenhagen)
Joint work with David Schrittesser

asgert@math.ku.dk

Singapore, April 10, 2015

Orthogonal measures

Orthogonal measures

Let X be a standard Borel space:

Orthogonal measures

Let X be a standard Borel space:

- $P(X)$ denotes the set of **Borel probability measures** on X;

Orthogonal measures

Let X be a standard Borel space:

- $P(X)$ denotes the set of **Borel probability measures** on X;
- Two measures $\mu, \nu \in P(X)$ are said to be **orthogonal**, written

$$\mu \perp \nu,$$

just in case:

Orthogonal measures

Let X be a standard Borel space:

- $P(X)$ denotes the set of **Borel probability measures** on X;
- Two measures $\mu, \nu \in P(X)$ are said to be **orthogonal**, written

$$\mu \perp \nu,$$

just in case: there is $A \subseteq X$ Borel such that

$$\mu(A) = 1$$

and

Orthogonal measures

Let X be a standard Borel space:

- $P(X)$ denotes the set of **Borel probability measures** on X;
- Two measures $\mu, \nu \in P(X)$ are said to be **orthogonal**, written

$$\mu \perp \nu,$$

just in case: there is $A \subseteq X$ Borel such that

$$\mu(A) = 1$$

and

$$\nu(A) = 0.$$

The space P(X)

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

In the case when $X = 2^\omega$, there is a convenient alternative way to view $P(X)$, namely by identifying it with the set of functions

$$f : 2^{<\omega} \to [0,1]$$

satisfying

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

In the case when $X = 2^\omega$, there is a convenient alternative way to view $P(X)$, namely by identifying it with the set of functions

$$f : 2^{<\omega} \to [0, 1]$$

satisfying

1. $f(\emptyset) = 1$;

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

In the case when $X = 2^\omega$, there is a convenient alternative way to view $P(X)$, namely by identifying it with the set of functions

$$f : 2^{<\omega} \to [0,1]$$

satisfying

1. $f(\emptyset) = 1$;
2. $f(s) = f(s^\frown 0) + f(s^\frown 1)$ for all $s \in 2^{<\omega}$.

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

In the case when $X = 2^\omega$, there is a convenient alternative way to view $P(X)$, namely by identifying it with the set of functions

$$f : 2^{<\omega} \to [0,1]$$

satisfying

1. $f(\emptyset) = 1$;
2. $f(s) = f(s^\frown 0) + f(s^\frown 1)$ for all $s \in 2^{<\omega}$.

FACTS:

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

In the case when $X = 2^\omega$, there is a convenient alternative way to view $P(X)$, namely by identifying it with the set of functions

$$f : 2^{<\omega} \to [0,1]$$

satisfying

1. $f(\emptyset) = 1$;
2. $f(s) = f(s^\frown 0) + f(s^\frown 1)$ for all $s \in 2^{<\omega}$.

FACTS:

- Kolmogorov's theorem guarantees that for each such f there is a unique $\mu^f \in P(2^\omega)$ such that $\mu^f(N_s) = f(s)$.

The space P(X)

The space $P(X)$ can be given a standard Borel structure.

In the case when $X = 2^\omega$, there is a convenient alternative way to view $P(X)$, namely by identifying it with the set of functions

$$f : 2^{<\omega} \to [0, 1]$$

satisfying

1. $f(\emptyset) = 1$;
2. $f(s) = f(s^\frown 0) + f(s^\frown 1)$ for all $s \in 2^{<\omega}$.

FACTS:

- Kolmogorov's theorem guarantees that for each such f there is a unique $\mu^f \in P(2^\omega)$ such that $\mu^f(N_s) = f(s)$.
- The subset of $[0, 1]^{2^{<\omega}}$ satisfying 1. and 2. above is closed in the product topology, so Polish.

Sets of orthogonal measures

Sets of orthogonal measures

Definition

- $\mathcal{A} \subseteq P(X)$ is called an **orthogonal family** if any two distinct measures in $\mathcal{A}$ are orthogonal.

Sets of orthogonal measures

Definition

- $\mathcal{A} \subseteq P(X)$ is called an **orthogonal family** if any two distinct measures in $\mathcal{A}$ are orthogonal.
- An orthogonal family is **maximal** if it is maximal under inclusion.

Sets of orthogonal measures

Definition

- $\mathcal{A} \subseteq P(X)$ is called an **orthogonal family** if any two distinct measures in $\mathcal{A}$ are orthogonal.
- An orthogonal family is **maximal** if it is maximal under inclusion.
- We abbreviate "maximal orthogonal family" by "mof".

Sets of orthogonal measures

Definition

- $\mathcal{A} \subseteq P(X)$ is called an **orthogonal family** if any two distinct measures in $\mathcal{A}$ are orthogonal.
- An orthogonal family is **maximal** if it is maximal under inclusion.
- We abbreviate "maximal orthogonal family" by "mof".

QUESTION (Mauldin, circa 1980):

Sets of orthogonal measures

Definition

- $\mathcal{A} \subseteq P(X)$ is called an **orthogonal family** if any two distinct measures in $\mathcal{A}$ are orthogonal.
- An orthogonal family is **maximal** if it is maximal under inclusion.
- We abbreviate "maximal orthogonal family" by "mof".

QUESTION (Mauldin, circa 1980):

Can a mof in $P(2^\omega)$ be analytic?

Answer to Mauldin's question

Answer to Mauldin's question

Mauldin's question was answered fairly soon:

Answer to Mauldin's question

Mauldin's question was answered fairly soon:

Theorem (Preiss-Rataj, 1985)

There are no analytic mofs in $P(2^\omega)$.

Answer to Mauldin's question

Mauldin's question was answered fairly soon:

Theorem (Preiss-Rataj, 1985)

There are no analytic mofs in $P(2^\omega)$.

Remark. A new proof of this theorem, which uses Hjorth's theory of turbulence, was found by Kechris and Sofronidis around 2000.

Answer to Mauldin's question

Mauldin's question was answered fairly soon:

Theorem (Preiss-Rataj, 1985)

There are no analytic mofs in $P(2^\omega)$.

Remark. A new proof of this theorem, which uses Hjorth's theory of turbulence, was found by Kechris and Sofronidis around 2000.

WARNING: For the remainder of the talk, we will study mofs only in $P(2^\omega)$.

What about co-analytic, then?

What about co-analytic, then?

Preiss and Rataj's theorem is best possible, in some sense, because of the following:

What about co-analytic, then?

Preiss and Rataj's theorem is best possible, in some sense, because of the following:

Theorem (Fischer-T., 2009)

If all reals are constructible then there is a Π^1_1 mof in $P(2^\omega)$.

What about co-analytic, then?

Preiss and Rataj's theorem is best possible, in some sense, because of the following:

Theorem (Fischer-T., 2009)

If all reals are constructible then there is a Π^1_1 *mof in* $P(2^\omega)$*.*

Here Π^1_1 means *lightface co-analytic*.

A corollary of the proof

A corollary of the proof

The *proof* of the previous theorem shows something better:

A corollary of the proof

The *proof* of the previous theorem shows something better:

Corollary

If there is a Σ^1_2 *mof then there is a* Π^1_1 *mof.*

A corollary of the proof

The *proof* of the previous theorem shows something better:

Corollary

If there is a Σ^1_2 mof then there is a Π^1_1 mof.

(We will need this, repeatedly, later.)

Cohen and Random destroy mofs

Cohen and Random destroy mofs

It turns out that mofs are very unstable beings:

Cohen and Random destroy mofs

It turns out that mofs are very unstable beings:

Theorem (Fischer-T., 2009)

If there is a Cohen real over L, then there are no Π^1_1 *mofs.*

Cohen and Random destroy mofs

It turns out that mofs are very unstable beings:

Theorem (Fischer-T., 2009)

If there is a Cohen real over L, then there are no Π^1_1 mofs.

Theorem (Fischer-S.D.Friedman-T., 2011)

If there is a Random real over L, then there are no Π^1_1 mofs.

Cohen and Random destroy mofs

It turns out that mofs are very unstable beings:

Theorem (Fischer-T., 2009)

If there is a Cohen real over L, then there are no Π^1_1 mofs.

Theorem (Fischer-S.D.Friedman-T., 2011)

If there is a Random real over L, then there are no Π^1_1 mofs.

QUESTION (Fischer-T., 2009): Does the existence of a Π^1_1 mof imply that all reals are constructible?

The obvious move?

The obvious move?

To answer this question, there is an obvious response:

The obvious move?

To answer this question, there is an obvious response:

1. Study the problem in the Sacks extension of L, which in some sense is the mildest forcing extension of L.

The obvious move?

To answer this question, there is an obvious response:

1. Study the problem in the Sacks extension of L, which in some sense is the mildest forcing extension of L.
2. Specifically, try to construct at Π^1_1 (or just Σ^1_2) mof in L which is *indestructible* for Sacks forcing, i.e., remains a mof in the Sacks extension.

The obvious move?

To answer this question, there is an obvious response:

1. Study the problem in the Sacks extension of L, which in some sense is the mildest forcing extension of L.
2. Specifically, try to construct at Π^1_1 (or just Σ^1_2) mof in L which is *indestructible* for Sacks forcing, i.e., remains a mof in the Sacks extension.

The second part, however, is doomed to fail:

The obvious move?

To answer this question, there is an obvious response:

1. Study the problem in the Sacks extension of L, which in some sense is the mildest forcing extension of L.
2. Specifically, try to construct at Π^1_1 (or just Σ^1_2) mof in L which is *indestructible* for Sacks forcing, i.e., remains a mof in the Sacks extension.

The second part, however, is doomed to fail:

Proposition (T.)

If there is a non-constructible real, then no Σ^1_2 orthogonal family contained in L can be maximal.

The obvious move?

To answer this question, there is an obvious response:

1. Study the problem in the Sacks extension of L, which in some sense is the mildest forcing extension of L.
2. Specifically, try to construct at Π^1_1 (or just Σ^1_2) mof in L which is *indestructible* for Sacks forcing, i.e., remains a mof in the Sacks extension.

The second part, however, is doomed to fail:

Proposition (T.)

If there is a non-constructible real, then no Σ^1_2 orthogonal family contained in L can be maximal.

...let us sketch the proof of this:

Sketch of proof

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).
- Let
$$F = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).
- Let

$$F = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$

- This set is Σ^1_2, so we may uniformize it by some $f : Y \to \mathcal{A}$ which is also Σ^1_2.

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).
- Let
$$F = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$
- This set is Σ^1_2, so we may uniformize it by some $f : Y \to \mathcal{A}$ which is also Σ^1_2.
- For each $\nu \in \mathcal{A}$, there can only be **countably many** μ such that $(\mu, \nu) \in F$, so it follows that f is countable-to-1.

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).
- Let
$$F = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$
- This set is Σ^1_2, so we may uniformize it by some $f : Y \to \mathcal{A}$ which is also Σ^1_2.
- For each $\nu \in \mathcal{A}$, there can only be **countably many** μ such that $(\mu, \nu) \in F$, so it follows that f is countable-to-1.
- Since Y is Δ^1_1-isomorphic to 2^ω, we now have a Σ^1_2 countable-to-1 function from 2^ω into the constructible reals.

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).
- Let
$$F = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$
- This set is Σ^1_2, so we may uniformize it by some $f : Y \to \mathcal{A}$ which is also Σ^1_2.
- For each $\nu \in \mathcal{A}$, there can only be **countably many** μ such that $(\mu, \nu) \in F$, so it follows that f is countable-to-1.
- Since Y is Δ^1_1-isomorphic to 2^ω, we now have a Σ^1_2 countable-to-1 function from 2^ω into the constructible reals.
- This implies that all reals are constructible. (Use the Mansfield-Solovay perfect set theorem!)

Sketch of proof

- Suppose $\mathcal{A} \subseteq L$ is a Σ^1_2 mof.
- Let $Y \subseteq P(2^\omega)$ be a Π^0_1 Cantor set of orthogonal measures (such can always be found).
- Let

$$F = \{(\mu, \nu) \in Y \times \mathcal{A} : \mu \not\perp \nu\}.$$

- This set is Σ^1_2, so we may uniformize it by some $f : Y \to \mathcal{A}$ which is also Σ^1_2.
- For each $\nu \in \mathcal{A}$, there can only be **countably many** μ such that $(\mu, \nu) \in F$, so it follows that f is countable-to-1.
- Since Y is Δ^1_1-isomorphic to 2^ω, we now have a Σ^1_2 countable-to-1 function from 2^ω into the constructible reals.
- This implies that all reals are constructible. (Use the Mansfield-Solovay perfect set theorem!) □

Main theorem

Main theorem

The difficulty of there not being any Sacks indestructible Σ^1_2 mofs can be overcome:

Main theorem

The difficulty of there not being any Sacks indestructible Σ^1_2 mofs can be overcome:

Theorem (Schrittesser-T., 2015)

In $L[s]$, where s is a single Sacks real over L, there is a (lightface!) Π^1_1 mof.

Main theorem

The difficulty of there not being any Sacks indestructible Σ^1_2 mofs can be overcome:

Theorem (Schrittesser-T., 2015)

In $L[s]$, where s is a single Sacks real over L, there is a (lightface!) Π^1_1 mof.

The theorem follows from a more general statement about sets that are *maximal discrete* for a relation.

A general theorem

A general theorem

Definition

If $\mathcal{G}$ is a symmetric relation on a set X, then $\mathcal{A} \subseteq X$ is $\mathcal{G}$-**discrete** if

$$(\forall x, y \in \mathcal{A})\ x \neq y \implies x \not\mathcal{G}\ y.$$

A general theorem

Definition

If $\mathcal{G}$ is a symmetric relation on a set X, then $\mathcal{A} \subseteq X$ is $\mathcal{G}$-**discrete** if

$$(\forall x, y \in \mathcal{A})\ x \neq y \implies x \not{\mathcal{G}}\ y.$$

Theorem (Schrittesser-T., 2015)

Let $\mathcal{G}$ be a symmetric Δ^1_1 relation on ω^ω (or some other recursively presented Polish space). Then in $L[s]$, the Sacks extension of L, there is a maximal $\mathcal{G}$-discrete Σ^1_2 set in ω^ω.

A general theorem

Definition

If $\mathcal{G}$ is a symmetric relation on a set X, then $\mathcal{A} \subseteq X$ is $\mathcal{G}$-**discrete** if

$$(\forall x, y \in \mathcal{A})\ x \neq y \implies x \not{\mathcal{G}}\ y.$$

Theorem (Schrittesser-T., 2015)

Let $\mathcal{G}$ be a symmetric Δ^1_1 relation on ω^ω (or some other recursively presented Polish space). Then in $L[s]$, the Sacks extension of L, there is a maximal $\mathcal{G}$-discrete Σ^1_2 set in ω^ω.

NOMENCLATURE: We abbreviate maximal $\mathcal{G}$-discrete set by $\mathcal{G}$-mds or simpy mds.

How the general theorem implies the theorem about mofs in $L[s]$

How the general theorem implies the theorem about mofs in $L[s]$

The theorem about mofs follows from this general theorem about $\mathcal{G}$-discrete sets since:

How the general theorem implies the theorem about mofs in $L[s]$

The theorem about mofs follows from this general theorem about $\mathcal{G}$-discrete sets since:

- We can apply the general theorem to the relation “is not orthogonal” in $P(2^\omega)$ to get a Σ^1_2 mof;

How the general theorem implies the theorem about mofs in $L[s]$

The theorem about mofs follows from this general theorem about $\mathcal{G}$-discrete sets since:

- We can apply the general theorem to the relation "is not orthogonal" in $P(2^\omega)$ to get a Σ^1_2 mof;
- Use the fact that the existence of a Σ^1_2 mof implies the existence of a Π^1_1 mof.

Strategy

Strategy

Recall our general theorem again:

Strategy

Recall our general theorem again:

Theorem (Schrittesser-T., 2015)

Let $\mathcal{G}$ be a symmetric Δ^1_1 relation on ω^ω (or some other recursively presented Polish space). Then in $L[s]$, the Sacks extension of L, there is a maximal $\mathcal{G}$-discrete Σ^1_2 set in ω^ω.

Strategy

Recall our general theorem again:

Theorem (Schrittesser-T., 2015)

Let $\mathcal{G}$ be a symmetric Δ^1_1 relation on ω^ω (or some other recursively presented Polish space). Then in $L[s]$, the Sacks extension of L, there is a maximal $\mathcal{G}$-discrete Σ^1_2 set in ω^ω.

- To prove the above, we will build (in L) a $\mathcal{G}$-mds inductively by *sometimes* adding a *single* new element which is not $\mathcal{G}$-related any of the things that have already been added, and *sometimes* adding an *entire perfect $\mathcal{G}$-discrete set*, all element of which are not $\mathcal{G}$-related to everything previously added.

Strategy

Recall our general theorem again:

Theorem (Schrittesser-T., 2015)

Let $\mathcal{G}$ be a symmetric Δ^1_1 relation on ω^ω (or some other recursively presented Polish space). Then in $L[s]$, the Sacks extension of L, there is a maximal $\mathcal{G}$-discrete Σ^1_2 set in ω^ω.

- To prove the above, we will build (in L) a $\mathcal{G}$-mds inductively by *sometimes* adding a *single* new element which is not $\mathcal{G}$-related any of the things that have already been added, and *sometimes* adding an *entire perfect $\mathcal{G}$-discrete set*, all element of which are not $\mathcal{G}$-related to everything previously added.
- A (the?) key ingredient is **Galvin's Ramsey theorem** for Polish spaces.

Setting up for the proof

Setting up for the proof

- Sacks forcing, $\mathbb{S}$, is of course forcing with perfect subtrees of $2^{<\omega}$.

Setting up for the proof

- Sacks forcing, $\mathbb{S}$, is of course forcing with perfect subtrees of $2^{<\omega}$.
- Sacks forcing has *continuous reading of names* for reals, in the following sense: If $p \in \mathbb{S}$, $\dot{x}$ an $\mathbb{S}$-name, and $p \Vdash \dot{x} \in \omega^\omega$, then there is a continuous function $\eta : 2^\omega \to \omega^\omega$ and $q \leq p$ such that

  $$q \Vdash \dot{x} = \eta(x_G),$$

  where x_G is the canonical name for the generic.

Galvin's theorem

Galvin's theorem

Theorem (Galvin, 1968)

Let X be a nonempty perfect Polish space, and suppose

$$[X]^2 = P_0 \cup P_1,$$

where P_0, P_1 have the Baire property. Then there is a Cantor set $C \subseteq X$ such that $[C]^2 \subseteq P_0$ or $[C]^2 \subseteq P_1$.

A corollary of Galvin's theorem

A corollary of Galvin's theorem

Corollary

Let $\mathcal{G}$ be a symmetric Borel (binary) relation on ω^ω, and let $\eta : 2^\omega \to \omega^\omega$ be continuous (or just Borel).

A corollary of Galvin's theorem

Corollary

Let $\mathcal{G}$ be a symmetric Borel (binary) relation on ω^ω, and let $\eta : 2^\omega \to \omega^\omega$ be continuous (or just Borel).

Then: For any $p \in \mathbb{S}$ there is $q \leq p$ such that **either**

A corollary of Galvin's theorem

Corollary

Let $\mathcal{G}$ be a symmetric Borel (binary) relation on ω^ω, and let $\eta : 2^\omega \to \omega^\omega$ be continuous (or just Borel).

Then: For any $p \in \mathbb{S}$ there is $q \leq p$ such that **either**

(1) $\eta(x) \mathcal{G} \eta(y)$ *for all* $x, y \in [q]$;

A corollary of Galvin's theorem

Corollary

Let $\mathcal{G}$ be a symmetric Borel (binary) relation on ω^ω, and let $\eta : 2^\omega \to \omega^\omega$ be continuous (or just Borel).

Then: For any $p \in \mathbb{S}$ there is $q \leq p$ such that **either**

$$\text{(1)} \quad \eta(x) \ \mathcal{G} \ \eta(y) \text{ for all } x, y \in [q];$$

or

$$\text{(2)} \quad \eta(x) \ \not\mathcal{G} \ \eta(y) \text{ for all } x, y \in [q];$$

A corollary of Galvin's theorem

Corollary

Let $\mathcal{G}$ be a symmetric Borel (binary) relation on ω^ω, and let $\eta : 2^\omega \to \omega^\omega$ be continuous (or just Borel).

Then: For any $p \in \mathbb{S}$ there is $q \leq p$ such that **either**

$$(1) \quad \eta(x) \; \mathcal{G} \; \eta(y) \text{ for all } x, y \in [q];$$

or

$$(2) \quad \eta(x) \; \not\mathcal{G} \; \eta(y) \text{ for all } x, y \in [q];$$

(We will call q a **Galvin witness** *to p and η.)*

A corollary of Galvin's theorem

Corollary

Let $\mathcal{G}$ be a symmetric Borel (binary) relation on ω^ω, and let $\eta : 2^\omega \to \omega^\omega$ be continuous (or just Borel).

Then: For any $p \in \mathbb{S}$ there is $q \leq p$ such that **either**

$$\text{(1)} \quad \eta(x) \ \mathcal{G} \ \eta(y) \text{ for all } x, y \in [q];$$

or

$$\text{(2)} \quad \eta(x) \ \not{\mathcal{G}} \ \eta(y) \text{ for all } x, y \in [q];$$

(We will call q a **Galvin witness** *to p and η.)*

Proof.

This is exactly Galvin's theorem applies to

$$\{(x, y) \in 2^\omega : \eta(x) \ \mathcal{G} \ \eta(y)\}$$

□

The construction, 1

The construction, 1

We now turn to the proof of our general theorem.

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

- q' is a subtree of q (not necessarily perfect);

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

- q' is a subtree of q (not necessarily perfect);
- q is a Galvin witness for η, i.e. one of the two alternatives of the corollary hold for q.

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

- q' is a subtree of q (not necessarily perfect);
- q is a Galvin witness for η, i.e. one of the two alternatives of the corollary hold for q.
- If alternative (1) holds then q' is the left-most branch of q;

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

- q' is a subtree of q (not necessarily perfect);
- q is a Galvin witness for η, i.e. one of the two alternatives of the corollary hold for q.
- If alternative (1) holds then q' is the left-most branch of q;
- If alternative (2) holds, then $q = q'$.

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in \subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

- q' is a subtree of q (not necessarily perfect);
- q is a Galvin witness for η, i.e. one of the two alternatives of the corollary hold for q.
- If alternative (1) holds then q' is the left-most branch of q;
- If alternative (2) holds, then $q = q'$.
- $\mathcal{A}^0$ will be constructed in stages, $\mathcal{A}^0_\xi$, $\xi < \omega_1$.

The construction, 1

We now turn to the proof of our general theorem.

Work (for now) in L. We will construct, inductively, a Σ^1_2 set $\mathcal{A}^0$, consisting of triples

$$(q, q', \eta) \in\subseteq \mathbb{S} \times \mathcal{P}(2^{<\omega}) \times C(2^\omega, \omega^\omega)$$

where (among other properties) we will have:

- q' is a subtree of q (not necessarily perfect);
- q is a Galvin witness for η, i.e. one of the two alternatives of the corollary hold for q.
- If alternative (1) holds then q' is the left-most branch of q;
- If alternative (2) holds, then $q = q'$.
- $\mathcal{A}^0$ will be constructed in stages, $\mathcal{A}^0_\xi$, $\xi < \omega_1$.
- At limit stages we will have $\mathcal{A}^0_\lambda = \bigcup_{\xi<\lambda} A^0_\xi$.

The construction, 2

The construction, 2

Let $D \subseteq \omega_1$ be the (unbounded) set of $\delta < \omega_1$ such that

The construction, 2

Let $D \subseteq \omega_1$ be the (unbounded) set of $\delta < \omega_1$ such that

- $L_\delta \models$ZF$^-$;

The construction, 2

Let $D \subseteq \omega_1$ be the (unbounded) set of $\delta < \omega_1$ such that

- $L_\delta \models$ZF$^-$;
- $L_\delta \models$"For every continuous $\eta : 2^\omega \to \omega^\omega$, the set of Galvin witnesses for η is dense in $\mathbb{S}$";

The construction, 2

Let $D \subseteq \omega_1$ be the (unbounded) set of $\delta < \omega_1$ such that

- $L_\delta \models$ZF$^-$;
- $L_\delta \models$"For every continuous $\eta : 2^\omega \to \omega^\omega$, the set of Galvin witnesses for η is dense in $\mathbb{S}$";
- $L_\delta \models$"all sets are countable".

The construction, 2

Let $D \subseteq \omega_1$ be the (unbounded) set of $\delta < \omega_1$ such that

- $L_\delta \models$ZF$^-$;
- $L_\delta \models$"For every continuous $\eta : 2^\omega \to \omega^\omega$, the set of Galvin witnesses for η is dense in $\mathbb{S}$";
- $L_\delta \models$"all sets are countable".

Let δ_ξ, $\xi < \omega_1$, enumerate D increasingly.

The construction, 3

The construction, 3

Suppose $\mathcal{A}_\xi^0$ has been defined, and $A_\xi^0 \subseteq L_{\delta_\xi}$.

The construction, 3

Suppose $\mathcal{A}^0_\xi$ has been defined, and $A^0_\xi \subseteq L_{\delta_\xi}$.

Call $(p, \eta) \in L_{\delta_\xi} \cap \mathbb{S} \times C(2^\omega, \omega^\omega)$ a **candidate** at stage $\xi + 1$ if there is $q \in L_{\delta_{\xi+1}} \cap \mathbb{S}$ such that

The construction, 3

Suppose $\mathcal{A}^0_\xi$ has been defined, and $A^0_\xi \subseteq L_{\delta_\xi}$.

Call $(p, \eta) \in L_{\delta_\xi} \cap \mathbb{S} \times C(2^\omega, \omega^\omega)$ a **candidate** at stage $\xi + 1$ if there is $q \in L_{\delta_{\xi+1}} \cap \mathbb{S}$ such that

1. Every $x \in [q]$ is a Sacks real over L_{δ_ξ};

The construction, 3

Suppose $\mathcal{A}^0_\xi$ has been defined, and $A^0_\xi \subseteq L_{\delta_\xi}$.

Call $(p, \eta) \in L_{\delta_\xi} \cap \mathbb{S} \times C(2^\omega, \omega^\omega)$ a **candidate** at stage $\xi + 1$ if there is $q \in L_{\delta_{\xi+1}} \cap \mathbb{S}$ such that

1. Every $x \in [q]$ is a Sacks real over L_{δ_ξ};
2. q is a Galvin witness for p and η;

The construction, 3

Suppose $\mathcal{A}^0_\xi$ has been defined, and $A^0_\xi \subseteq L_{\delta_\xi}$.

Call $(p, \eta) \in L_{\delta_\xi} \cap \mathbb{S} \times C(2^\omega, \omega^\omega)$ a **candidate** at stage $\xi + 1$ if there is $q \in L_{\delta_{\xi+1}} \cap \mathbb{S}$ such that

1. Every $x \in [q]$ is a Sacks real over L_{δ_ξ};
2. q is a Galvin witness for p and η;
3. For all $(r, r', \eta') \in \mathcal{A}^0_\xi$ we have

$$(\forall x \in [q])(\forall y \in [r'])\ \eta(x) \not\mathcal{G}\ \eta'(y)$$

The construction, 3

Suppose $\mathcal{A}^0_\xi$ has been defined, and $A^0_\xi \subseteq L_{\delta_\xi}$.

Call $(p, \eta) \in L_{\delta_\xi} \cap \mathbb{S} \times C(2^\omega, \omega^\omega)$ a **candidate** at stage $\xi + 1$ if there is $q \in L_{\delta_{\xi+1}} \cap \mathbb{S}$ such that

1. Every $x \in [q]$ is a Sacks real over L_{δ_ξ};
2. q is a Galvin witness for p and η;
3. For all $(r, r', \eta') \in \mathcal{A}^0_\xi$ we have

$$(\forall x \in [q])(\forall y \in [r']) \ \eta(x) \not\mathcal{G} \ \eta'(y)$$

We will call such a q a *stage* $\xi + 1$ *Galvin witness for* (p, η).

The construction, 3

Suppose $\mathcal{A}^0_\xi$ has been defined, and $A^0_\xi \subseteq L_{\delta_\xi}$.

Call $(p, \eta) \in L_{\delta_\xi} \cap \mathbb{S} \times C(2^\omega, \omega^\omega)$ a **candidate** at stage $\xi + 1$ if there is $q \in L_{\delta_{\xi+1}} \cap \mathbb{S}$ such that

1. Every $x \in [q]$ is a Sacks real over L_{δ_ξ};
2. q is a Galvin witness for p and η;
3. For all $(r, r', \eta') \in \mathcal{A}^0_\xi$ we have

$$(\forall x \in [q])(\forall y \in [r'])\ \eta(x) \not\mathcal{G}\ \eta'(y)$$

We will call such a q a *stage* $\xi + 1$ *Galvin witness for* (p, η).

Note: What (3) is essentially saying is that what we are considering to add to our $\mathcal{G}$-mds at this point is not $\mathcal{G}$-related to anything we put in previously.

The construction, 4

- If no candidate exists at stage $\xi + 1$, we do nothing.

The construction, 4

- If no candidate exists at stage $\xi + 1$, we do nothing.
- If a candidate exists at stage $\xi + 1$, let (p, η) be the least such (in L), and let q be the least Galvin witness for p and η, and

The construction, 4

- If no candidate exists at stage $\xi + 1$, we do nothing.
- If a candidate exists at stage $\xi + 1$, let (p, η) be the least such (in L), and let q be the least Galvin witness for p and η, and
  - if alternative (1) holds for q and η, then we let q' be the left-most branch in q;

The construction, 4

- If no candidate exists at stage $\xi + 1$, we do nothing.
- If a candidate exists at stage $\xi + 1$, let (p, η) be the least such (in L), and let q be the least Galvin witness for p and η, and
  - if alternative (1) holds for q and η, then we let q' be the left-most branch in q;
  - if alternative (2) holds, then we let $q' = q$.

The construction, 4

- If no candidate exists at stage $\xi + 1$, we do nothing.
- If a candidate exists at stage $\xi + 1$, let (p, η) be the least such (in L), and let q be the least Galvin witness for p and η, and
  - if alternative (1) holds for q and η, then we let q' be the left-most branch in q;
  - if alternative (2) holds, then we let $q' = q$.
- Let $\mathcal{A}^0_{\xi+1} = \mathcal{A}^0_\xi \cup \{(q, q', \eta)\}$

The construction, 4

- If no candidate exists at stage $\xi + 1$, we do nothing.
- If a candidate exists at stage $\xi + 1$, let (p, η) be the least such (in L), and let q be the least Galvin witness for p and η, and
  - if alternative (1) holds for q and η, then we let q' be the left-most branch in q;
  - if alternative (2) holds, then we let $q' = q$.
- Let $\mathcal{A}^0_{\xi+1} = \mathcal{A}^0_\xi \cup \{(q, q', \eta)\}$

After all this, let

$$\mathcal{A}^0 = \bigcup_{\xi < \omega_1} \mathcal{A}^0_\xi.$$

The construction, 5

The construction, 5

FACT: $\mathcal{A}^0$ is Σ^1_2.

The construction, 5

FACT: $\mathcal{A}^0$ is Σ^1_2.

We let

$$\mathcal{A} = \{x \in \omega^\omega : (\exists (q, q', \eta) \in \mathcal{A}^0)(\exists y \in [q']) \; x = \eta(y)\},$$

which is then also Σ^1_2.

FACT: $\mathcal{A}^0$ is Σ^1_2.

We let

$$\mathcal{A} = \{x \in \omega^\omega : (\exists (q, q', \eta) \in \mathcal{A}^0)(\exists y \in [q'])\ x = \eta(y)\},$$

which is then also Σ^1_2.

We claim that $\mathcal{A}$ is a maximal $\mathcal{G}$-discrete set.

The construction, 5

FACT: $\mathcal{A}^0$ is Σ^1_2.

We let

$$\mathcal{A} = \{x \in \omega^\omega : (\exists (q, q', \eta) \in \mathcal{A}^0)(\exists y \in [q'])\ x = \eta(y)\},$$

which is then also Σ^1_2.

We claim that $\mathcal{A}$ is a maximal $\mathcal{G}$-discrete set.

That $\mathcal{A}$ is $\mathcal{G}$-discrete is clear by construction (and this will hold in any model, not just $L[s]$).

The construction, 5

The construction, 5

Let G be $\mathbb{S}$ - generic over L. Suppose, seeking a contradiction, that $\mathcal{A}$ is not maximal. Let $\dot{x}$ be an $\mathbb{S}$-name and suppose

$$p_0 \Vdash \dot{x} \in \omega^\omega \wedge (\forall y \in \mathcal{A})\ \dot{x} \not\mathcal{G}\ y.$$

The construction, 5

Let G be $\mathbb{S}$ - generic over L. Suppose, seeking a contradiction, that $\mathcal{A}$ is not maximal. Let $\dot{x}$ be an $\mathbb{S}$-name and suppose

$$p_0 \Vdash \dot{x} \in \omega^\omega \wedge (\forall y \in \mathcal{A})\ \dot{x} \not\perp y.$$

We may further assume that there is a continuous $\eta : 2^\omega \to \omega^\omega$ such that

$$p_0 \Vdash \eta(x_G) = \dot{x}.$$

The construction, 5

Let G be $\mathbb{S}$ - generic over L. Suppose, seeking a contradiction, that $\mathcal{A}$ is not maximal. Let $\dot{x}$ be an $\mathbb{S}$-name and suppose

$$p_0 \Vdash \dot{x} \in \omega^\omega \wedge (\forall y \in \mathcal{A})\ \dot{x} \not\mathcal{G}\ y.$$

We may further assume that there is a continuous $\eta : 2^\omega \to \omega^\omega$ such that

$$p_0 \Vdash \eta(x_G) = \dot{x}.$$

CLAIM: The set D_η of $q \in \mathbb{S}$ such that for some q' we have $(q, q', \eta) \in \mathcal{A}^0$ is dense below p_0.

The construction, 5

Let G be $\mathbb{S}$ - generic over L. Suppose, seeking a contradiction, that $\mathcal{A}$ is not maximal. Let $\dot{x}$ be an $\mathbb{S}$-name and suppose

$$p_0 \Vdash \dot{x} \in \omega^\omega \wedge (\forall y \in \mathcal{A})\ \dot{x} \not\perp y.$$

We may further assume that there is a continuous $\eta : 2^\omega \to \omega^\omega$ such that

$$p_0 \Vdash \eta(x_G) = \dot{x}.$$

CLAIM: The set D_η of $q \in \mathbb{S}$ such that for some q' we have $(q, q', \eta) \in \mathcal{A}^0$ is dense below p_0.

Proof: Essentially clear by the construction and Galvin's theorem, since every (p, η), where $p \leq p_0$, becomes a candidate at some stage. □

The construction, 6

The construction, 6

CLAIM: The set D_η of $q \in \mathbb{S}$ such that for some q' we have $(q, q', \eta) \in \mathcal{A}^0$ is dense below p_0.

The construction, 6

CLAIM: The set D_η of $q \in \mathbb{S}$ such that for some q' we have $(q, q', \eta) \in \mathcal{A}^0$ is dense below p_0.

Using the claim, we are now done, since then there is $q \in G$ (the Sacks generic over L) for which one of the alternatives hold, and

The construction, 6

CLAIM: The set D_η of $q \in \mathbb{S}$ such that for some q' we have $(q, q', \eta) \in \mathcal{A}^0$ is dense below p_0.

Using the claim, we are now done, since then there is $q \in G$ (the Sacks generic over L) for which one of the alternatives hold, and

- If alternative (1) holds (i.e., the $\mathcal{G}$ is a complete graph on $\eta([q])$), and y is the unique branch of q', then since $x_G \in [q]$ we get

$$\dot{x} = \eta(x_G)\ \mathcal{G}\ \eta(y),$$

  contradicting that $p_0 \Vdash \dot{x}\ \not{\mathcal{G}}\ \mathcal{A}$.

The construction, 6

CLAIM: The set D_η of $q \in \mathbb{S}$ such that for some q' we have $(q, q', \eta) \in \mathcal{A}^0$ is dense below p_0.

Using the claim, we are now done, since then there is $q \in G$ (the Sacks generic over L) for which one of the alternatives hold, and

- If alternative (1) holds (i.e., the $\mathcal{G}$ is a complete graph on $\eta([q])$), and y is the unique branch of q', then since $x_G \in [q]$ we get
$$\dot{x} = \eta(x_G) \; \mathcal{G} \; \eta(y),$$
contradicting that $p_0 \Vdash \dot{x} \not\!\mathcal{G} \; \mathcal{A}$.
- If alternative (2) holds (i.e., $\eta([q])$ is $\mathcal{G}$-discrete), then $x_G \in [q]$, and so $\eta(x_G) \in \mathcal{A}$, again contradicting $p_0 \Vdash \dot{x} \not\!\mathcal{G} \; \mathcal{A}$.

□

End remarks

End remarks

- The proof clearly uses a very special property of Sacks forcing, namely Galvin's theorem, so we must ask:

End remarks

- The proof clearly uses a very special property of Sacks forcing, namely Galvin's theorem, so we must ask:
- **QUESTION**: What happens to Π^1_1 mofs if we add *two* Sacks reals?

End remarks

- The proof clearly uses a very special property of Sacks forcing, namely Galvin's theorem, so we must ask:
- **QUESTION**: What happens to Π^1_1 mofs if we add *two* Sacks reals?
- We also know that if we add a Mathias real to L, then there are no Π^1_1 (or Σ^1_2) mofs.

End remarks

- The proof clearly uses a very special property of Sacks forcing, namely Galvin's theorem, so we must ask:
- **QUESTION**: What happens to Π^1_1 mofs if we add *two* Sacks reals?
- We also know that if we add a Mathias real to L, then there are no Π^1_1 (or Σ^1_2) mofs.
- The analogue of Galvin's theorem (or, if you prefer, the corollary) is false for Laver, Mathias, Silver, Cohen, Hechler.

Thank you.