Ideals, ADR's, and mixing reals

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joint work with Y. Khomskii[a] and Z. Vidnyánszky[b]

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The classical setting

Given a family

$$\mathscr{H} = \{H_0, H_1, \ldots, H_\alpha, \ldots\} \subseteq [\omega]^\omega.$$

Do there exist infinite subsets $A_\alpha \subseteq H_\alpha$ such that the family $\mathscr{A} = \{A_0, A_1, \ldots, A_\alpha, \ldots\}$ is ***almost disjoint*** (**AD**), that is, $A_\alpha \cap A_\beta$ is finite for every $\alpha \neq \beta$? In this case, we say that $\mathscr{A}$, or more precisely, the map $H_\alpha \mapsto A_\alpha$ is an ***almost disjoint refinement*** (**ADR**) of $\mathscr{H}$.

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Proposition (noticed by many people)

If $\mathscr{H} \subseteq [\omega]^\omega$ with $|\mathscr{H}| < \mathfrak{c}$, then $\mathscr{H}$ has an ADR.

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Theorem (Balcar, Vojtáš)

Every ultrafilter on ω has an ADR.

The idealized version

Given an ideal $\mathscr{I}$ on ω ($\mathrm{Fin} := [\omega]^{<\omega} \subseteq \mathscr{I}$ and $\omega \notin \mathscr{I}$), and a family

$$\mathscr{H} = \{H_0, H_1, \ldots, H_\alpha, \ldots\} \subseteq \mathscr{I}^+ := \mathscr{P}(\omega) \setminus \mathscr{I}.$$

Do there exist **$\mathscr{I}$-positive** subsets $A_\alpha \subseteq H_\alpha$ such that the family $\mathscr{A} = \{A_0, A_1, \ldots, A_\alpha, \ldots\}$ is ***$\mathscr{I}$-almost disjoint*** (**$\mathscr{I}$-AD**), that is, $A_\alpha \cap A_\beta \in \mathscr{I}$ for every $\alpha \neq \beta$? In this case, we say that $\mathscr{A}$ is an ***$\mathscr{I}$-almost disjoint refinement*** (**$\mathscr{I}$-ADR**) of $\mathscr{H}$.

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Proposition

If $\mathscr{I}$ is ***everywhere meager***, that is,

$$\mathscr{I} \restriction X := \{A \subseteq X : A \in \mathscr{I}\} \text{ is meager in } \mathscr{P}(X)$$

for every $X \in \mathscr{I}^+$ (e.g. $\mathscr{I}$ is analytic or coanalytic), and $\mathscr{H} \subseteq \mathscr{I}^+$ with $|\mathscr{H}| < \mathfrak{c}$, then $\mathscr{H}$ has an $\mathscr{I}$-ADR.

Everywhere meager ideals

Proposition

If $\mathscr{I}$ is ew.meager and $\mathscr{H} \in [\mathscr{I}^+]^{<\mathfrak{c}}$, then $\mathscr{H}$ has an $\mathscr{I}$-ADR.

Proof: First we show that if $\mathscr{I}$ is a meager ideal then there is a perfect $(\mathscr{I}, \mathrm{Fin})$***-AD*** family, that is, a (perfect) $\mathscr{I}$-AD family $\mathscr{B}$ such that $|A \cap B| < \omega$ for every $\{A, B\} \in [\mathscr{B}]^2$.

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Applying Talagrand's characterization, there is a partition $(P_n)_{n\in\omega}$ of ω into finite sets such that

$$\left|\{n \in \omega : P_n \subseteq A\}\right| < \omega \text{ for every } A \in \mathscr{I}.$$

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Let $\mathscr{A}$ be a perfect AD family (e.g. the branches of $2^{<\omega}$ on $\mathscr{P}(2^{<\omega})$). For each $A \in \mathscr{A}$ let $A' = \bigcup\{P_n : n \in A\} \in \mathscr{I}^+$, and let $\mathscr{B} = \{A' : A \in \mathscr{A}\}$. The function $\mathscr{P}(\omega) \to \mathscr{P}(\omega)$, $A \mapsto A'$ is injective and continuous hence $\mathscr{B}$ is also perfect.

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If $\mathscr{I}$ is ew.meager and $\mathscr{H} \in [\mathscr{I}^+]^{<\mathfrak{c}}$, then $\mathscr{H}$ has an $\mathscr{I}$-ADR.

Proof (ctnd.): Now let $\mathscr{H} = \{H_\alpha : \alpha < \kappa\} \subseteq \mathscr{I}^+$ $(\kappa < \mathfrak{c})$. Fix an $\mathscr{I}$-AD family $\{A_\xi : \xi < \kappa^+\}$ on H_0 and for every $\alpha < \kappa$ let

$$T_\alpha = \{\xi < \kappa^+ : H_\alpha \cap A_\xi \in \mathscr{I}^+\}.$$

By induction on $C = \{\alpha < \kappa : |T_\alpha| = \kappa^+\} (\ni 0)$ we can pick

$$\xi_\alpha \in T_\alpha \setminus \left(\bigcup \{T_\beta : |T_\beta| \leq \kappa\} \cup \{\xi_{\alpha'} : \alpha' \in \alpha \setminus C\}\right)$$

and let $E_\alpha = H_\alpha \cap A_{\xi_\alpha} \in \mathscr{I}^+$.

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$$\xi_\alpha \in T_\alpha \setminus \left(\bigcup \{T_\beta : |T_\beta| \leq \kappa\} \cup \{\xi_{\alpha'} : \alpha' \in \alpha \setminus C\} \right)$$

and let $E_\alpha = H_\alpha \cap A_{\xi_\alpha} \in \mathscr{I}^+$. Then the family $\{E_\alpha : \alpha \in C\}$ is an $\mathscr{I}$-ADR of $\{H_\alpha : \alpha \in C\}$. We can continue this procedure on $\{H_\beta : \beta \in \kappa \setminus C\}$ because $E_\alpha \cap H_\beta \in \mathscr{I}$ for every $\alpha \in C$ and $\beta \in \kappa \setminus C$.

Some related question

Proposition

If $\mathscr{I}$ is ew.meager and $\mathscr{H} \in [\mathscr{I}^+]^{<\mathfrak{c}}$, then $\mathscr{H}$ has an $\mathscr{I}$-ADR.

Question

Assume that there is a perfect $(\mathscr{I}, \mathrm{Fin})$-AD family. Is $\mathscr{I}$ meager?

No.

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Question

Assume that there are perfect $(\mathscr{I}, \mathrm{Fin})$-AD families on every $X \in \mathscr{I}^+$. Is $\mathscr{I}$ (everywhere) meager?

No under $\mathfrak{b} = \mathfrak{c}$. In ZFC?

Some related question

Question

What can we say about $\underset{\sim}{\Sigma}^1_2$ etc ideals?

In L there is a Σ^1_2 (i.e. Δ^1_2) prime ideal $\mathscr{J}$. Clearly, all $\mathscr{J}$-AD families are of size ≤ 1.

We can also easily construct a Δ^1_2 ideal $\mathscr{I}$ from $\mathscr{J}$ such that there are infinite $\mathscr{I}$-AD families but all of them are countable: Copy $\mathscr{J}$ to the elements of an infinite partition of ω, and let $\mathscr{I}$ be the ideal generated by these copies.

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Question

Does there exist a $\underset{\sim}{\Sigma}^1_2$ ideal $\mathscr{I}$ such that every $\mathscr{I}$-AD family is countable BUT $\mathscr{I}$ is nowhere maximal?

Without the complexity condition, Yes in ZFC (O. Selim).

Refinements of $\mathscr{I}^+ \cap V$

Theorem (Brendle | Balcar, Pazák)

Let $V \subseteq W$ be transitive models with $(2^\omega)^V \neq (2^\omega)^W$. Then

$$W \models \text{“}[\omega]^\omega \cap V \text{ has an ADR.”}$$

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Theorem

Let $V \subseteq W$ be transitive models with $\omega_1^W \subseteq V$ but $(2^\omega)^V \neq (2^\omega)^W$, and let $\mathscr{I}$ be an analytic or coanalytic ideal coded in V. Then

$$W \models \text{“}\mathscr{I}^+ \cap V \text{ has an } \mathscr{I}\text{-ADR.”}$$

Examples of Borel ideals

F_σ ideals:

- Summable ideals, e.g. $\mathscr{I}_{1/n} = \{A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty\}$.
- Tsirelson ideals (Farah, Solecki, Veličković).
- The eventually different ideals: $\mathscr{ED} = \left\{A \subseteq \omega \times \omega : \limsup_{n \to \infty} \left|\{k : (n,k) \in A\}\right| < \infty\right\}$ and $\mathscr{ED}_{\mathrm{fin}} = \mathscr{ED} \restriction \Delta$ where $\Delta = \{(n,m) \in \omega \times \omega : m \leq n\}$.
- The van der Waerden ideal: $\mathscr{W} = \{A \subseteq \omega : A$ does not contain arbitrary long AP's$\}$.
- The random graph ideal: $\mathrm{Ran} = \mathrm{id}(\{$homogeneous subsets of the random graph$\})$.
- The ideal of graphs with finite chromatic number: $\mathscr{G}_{\mathrm{fc}} = \{E \subseteq [\omega]^2 : \chi(\omega, E) < \omega\}$.
- Solecki's ideal: $\mathscr{S} = \mathrm{id}\{\{A \in \mathrm{Clopen}(2^\omega) : \lambda(A) = 1/2$ and $x \in A\} : x \in 2^\omega\}$.

Examples of Borel ideals

$F_{\sigma\delta}$ ideals:

- (Generalized) Density ideals, e.g. $\mathcal{Z} = \left\{A \subseteq \omega : \frac{|A \cap n|}{n} \to 0\right\}$.
- The uniform density zero ideal: $\mathcal{Z}_u = \left\{A \subseteq \omega : \frac{\max\{|A \cap [k, k+n)| : k \in \omega\}}{n} \to 0\right\}$.
- The trace of the null ideal: $\mathrm{tr}(\mathcal{N}) = \{A \subseteq 2^{<\omega} : \lambda\{x \in 2^\omega : \exists^\infty n \; x \restriction n \in A\} = 0\}$.
- The ideal of nowhere dense subsets of the rationals: $\mathrm{Nwd} = \{A \subseteq \mathbb{Q} : \mathrm{int}(\overline{A}) = \emptyset\}$.
- Banach space ideals (Louveau, Veličković).

$F_{\sigma\delta\sigma}$ ideals:

- The ideal generated by convergent sequences in $\mathbb{Q} \cap [0,1]$: $\mathrm{Conv} = \{A \subseteq \mathbb{Q} \cap [0,1] : |\{\text{acc. points of } A \text{ (in } \mathbb{R})\}| < \omega\}$.
- The Fubini product of Fin by itself: $\mathrm{Fin} \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \forall^\infty n \in \omega \, |\{k : (n,k) \in A\}| < \omega\}$.

Examples of Borel ideals

In general, it is easy to see that there are no G_δ (i.e. $\underset{\sim}{\Pi}^0_2$) ideals but we know the following:

Theorem (Calbrix)

There are $\underset{\sim}{\Sigma}^0_\alpha$- and $\underset{\sim}{\Pi}^0_\alpha$-complete ideals for every $\alpha \geq 3$.

Examples of projective ideals

Example (Zafrany)

For every $x \in \omega^\omega$ let $I_x = \{s \in \omega^{<\omega} : x \restriction |s| \not\leq s\}$. Then the ideal on $\omega^{<\omega}$ generated by $\{I_x : x \in \omega^\omega\}$ is $\underset{\sim}{\Sigma}^1_1$-complete.

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Example (Hrušák?, Meza-Alcántara?)

The ideal of graphs without infinite complete subgraphs,

$$\mathscr{G}_c = \{E \subseteq [\omega]^2 : \forall\ X \in [\omega]^\omega\ [X]^2 \not\subseteq E\} \text{ is } \underset{\sim}{\Pi}^1_1\text{-complete.}$$

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Theorem

There exist $\undertilde{\Sigma}^1_n$ and $\undertilde{\Pi}^1_n$-complete ideals for every $n \geq 1$.

Proof (idea): Let $\mathscr{A}$ be a perfect AD family. If $\mathscr{B}$ is a $\undertilde{Q}^1_n$-complete subset of $\mathscr{A}$, then $\mathrm{id}(\mathscr{B})$ is also $\undertilde{Q}^1_n$-complete.

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Theorem

The following ideal ($\mathscr{I}_V$) is $\underset{\sim}{\Pi}^1_1$-complete:

$$\{A \subseteq \omega \times \omega : \forall\, X, Y \in [\omega]^\omega\ \exists\, X' \in [X]^\omega\ \exists\, Y' \in [Y]^\omega\ A \cap (X' \times Y') = \emptyset\}.$$

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Proof: For $X, Y \in [\omega]^\omega$ let $T^{\uparrow}(X, Y) = \{(n, m) \in X \times Y : n < m\}$ and let $T^{\downarrow}(X, Y) = \{(n, m) \in X \times Y : n > m\}$. We show that a set $A \subseteq \omega \times \omega$ is $\mathscr{J}_V$-positive iff there are $X, Y \in [\omega]^\omega$ such that

$$X \times Y \subseteq A \text{ or } A \cap (X \times Y) = T^{\uparrow}(X, Y) \text{ or } A \cap (X \times Y) = T^{\downarrow}(X, Y).$$

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Let $A \in \mathcal{J}_V^+$, i.e. $\exists\ X = \{x_0 < x_1 < \ldots\}, Y = \{y_0 < y_1 < \ldots\} \in [\omega]^\omega$ such that $A \cap (X' \times Y') \neq \emptyset$ for every infinite $X' \subseteq X$ and $Y' \subseteq Y$. We can assume that $x_0 < y_0 < x_1 < y_1 < \ldots$. Let $c\colon [\omega]^2 \to 2 \times 2$ be the following coloring: If $n < m$ then $c(n, m) = (\chi_A(x_n, y_m), \chi_A(x_m, y_n))$. $\rightsquigarrow H \in [\omega]^\omega$ c-hom. $\rightsquigarrow H_X, H_Y \in [H]^\omega$ such that $H_X \cap H_Y = \emptyset$ and $X' = \{x_n : n \in H_X\}$ and $Y' = \{y_m : m \in H_Y\}$ are also alternating.

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The following ideal ($\mathscr{I}_V$) is $\underset{\sim}{\Pi}^1_1$-complete:

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Proof (ctnd.): $A \in \mathscr{I}_V^+$ is witnessed by X, Y. There are disjoint $H_X, H_Y \in [\omega]^\omega$ such that $X' = \{x_n : n \in H_X\}$ and $Y' = \{y_m : m \in H_Y\}$ are alternating and $c(n,m) = (\chi_A(x_n, y_m), \chi_A(x_m, y_n))$ $(n < m)$ is constant (k, ℓ) on $[H_X \cup H_Y]^2$. We want to show that $X' \times Y' \subseteq A$ or $A \cap (X' \times Y') = T^{\uparrow}(X', Y')$ or $A \cap (X' \times Y') = T^{\downarrow}(X', Y')$.

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$(k,\ell) = (0,0)$ is impossible because then $A \cap (X' \times Y') = \emptyset$. If $(k,\ell) = (1,1)$ then $X' \times Y' \subseteq A$. If $(k,\ell) = (1,0)$ or $(k,\ell) = (0,1)$ then $A \cap (X' \times Y') = T^{\uparrow}(X', Y')$ or $A \cap (X' \times Y') = T^{\downarrow}(X', Y')$.

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Proof (ctnd.): $A \in \mathcal{J}_V^+$ iff there are $X, Y \in [\omega]^\omega$ such that $X' \times Y' \subseteq A$ or $A \cap (X' \times Y') = T^{\uparrow}(X', Y')$ or $A \cap (X' \times Y') = T^{\downarrow}(X', Y')$. We will construct a Wadge-reduction $\mathcal{K}(\mathbb{Q}) \leq_W \mathcal{J}_V$ (where $\mathbb{Q} = \{x \in 2^\omega : \forall^\infty\ n\ x(n) = 0\}$). Fix an enumeration $2^{<\omega} = \{s_n : n \in \omega\}$ and define $\mathcal{K}(2^\omega) \to \mathcal{P}(\omega \times \omega)$ as follows:

$$C \mapsto A_C = \{(n, m) : [s_n] \cap C \neq \emptyset \text{ and } s_m(n) = 1\}.$$

It is straightforward to check that $C \in \mathcal{K}(\mathbb{Q})$ iff $A_C \in \mathcal{J}_V$.

Analytic and coanalytic ideals in forcing extensions

An obvious but important observation:

If $X \subseteq \mathscr{P}(\omega)$ is an analytic or coanalytic set with definition $\varphi(x, r)$ (where $r \in \omega^\omega$ is a parameter), then the statement

"*X* is an ideal"

is the conjunction of the following formulas:

(i) $\neg\varphi(\omega, r)$ and $\forall\ x \in \mathrm{Fin}\ \varphi(x, r)$,

(ii) $\forall\ x, y\ (x \not\subseteq y$ or $\neg\varphi(y, r)$ or $\varphi(x, r))$,

(iii) $\forall\ x, y\ (\neg\varphi(x, r)$ or $\neg\varphi(y, r)$ or $\varphi(x \cup y, r))$.

In particular, "*X* is an ideal" is a $\Pi^1_2(r)$ statement hence absolute for transitive models $V \subseteq W$ with $\omega_1^W \subseteq V$ and $r \in V$.

Strong ADR's

Question

Assume that $\mathscr{I}$ is everywhere meager and let $\mathscr{H} \in [\mathscr{I}^+]^{<\mathfrak{c}}$. Does $\mathscr{H}$ have an $(\mathscr{I}, \mathrm{Fin})$***-ADR***, that is, an $\mathscr{I}$-ADR $\mathscr{A}$ of $\mathscr{H}$ which is an AD family as well?

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Theorem

Assume MA_κ (or $\kappa < \mathrm{cov}(\mathscr{M})$?) and let $\mathscr{I}$ be an everywhere meager ideal, then every $\mathscr{H} \in [\mathscr{I}^+]^{\leq\kappa}$ has an $(\mathscr{I}, \mathrm{Fin})$-ADR.

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Proof: Let $\mathscr{H} = \{H_\alpha : \alpha < \kappa\}$, and define $p \in \mathbb{P} = \mathbb{P}(\mathscr{H})$ iff p is a function, $\mathrm{dom}(p) \in [\kappa]^{<\omega}$, and $p(\alpha) \in [H_\alpha]^{<\omega}$ for every $\alpha \in \mathrm{dom}(p)$; $p \leq q$ iff (i) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$, (ii) $\forall\ \alpha \in \mathrm{dom}(q)\ p(\alpha) \supseteq q(\alpha)$, and

$$\text{(iii) } \forall\ \{\alpha,\beta\} \in [\mathrm{dom}(q)]^2\ p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta).$$

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Assume MA_κ (or $\kappa < \mathrm{cov}(\mathscr{M})$?) and let $\mathscr{I}$ be an everywhere meager ideal, then every $\mathscr{H} \in [\mathscr{I}^+]^{\leq\kappa}$ has an $(\mathscr{I}, \mathrm{Fin})$-ADR.

Proof: Let $\mathscr{H} = \{H_\alpha : \alpha < \kappa\}$, and define $p \in \mathbb{P} = \mathbb{P}(\mathscr{H})$ iff p is a function, $\mathrm{dom}(p) \in [\kappa]^{<\omega}$, and $p(\alpha) \in [H_\alpha]^{<\omega}$ for every $\alpha \in \mathrm{dom}(p)$; $p \leq q$ iff (i) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$, (ii) $\forall\ \alpha \in \mathrm{dom}(q)\ p(\alpha) \supseteq q(\alpha)$, and

$$\text{(iii) } \forall\ \{\alpha, \beta\} \in [\mathrm{dom}(q)]^2\ p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta).$$

$\mathbb{P}$ has the ccc. If G is a $\{\{p \in \mathbb{P} : \alpha \in \mathrm{dom}(p)\} : \alpha < \kappa\}$-generic filter, then let $F_G : \kappa \to \mathscr{P}(\omega)$, $F_G(\alpha) = \bigcup\{p(\alpha) : p \in G\} \subseteq H_\alpha$.

Strong ADR's

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Proof (ctnd.): $p \in \mathbb{P}$ iff $\mathrm{dom}(p) \in [\kappa]^{<\omega}$ and $\forall\ \alpha < \kappa\ p(\alpha) \in [H_\alpha]^{<\omega}$; $p \leq q$ iff (i) $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$, (ii) $\forall\ \alpha \in \mathrm{dom}(q)\ p(\alpha) \supseteq q(\alpha)$, and (iii) $\forall\ \{\alpha, \beta\} \in [\mathrm{dom}(q)]^2\ p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$. If G is a reasonably generic filter, then let $F_G(\alpha) = \bigcup\{p(\alpha) : p \in G\} \subseteq H_\alpha$. Clearly $|F_G(\alpha) \cap F_G(\beta)| < \omega$ for every $\alpha < \beta < \kappa$.

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$F_G(\alpha) \in \mathscr{I}^+$? We show that if G is $(V, \mathbb{P})$-generic then $F_G(\alpha)$ is a Cohen-real in $\mathscr{P}(H_\alpha)$ over V. It is enough because then $F_G(\alpha) \notin \mathscr{I} \restriction H_\alpha$ and it holds under MA_κ as well.

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$F_G(\alpha) \in \mathscr{I}^+$? We show that if G is $(V, \mathbb{P})$-generic then $F_G(\alpha)$ is a Cohen-real in $\mathscr{P}(H_\alpha)$ over V. It is enough because then $F_G(\alpha) \notin \mathscr{I} \restriction H_\alpha$ and it holds under MA_κ as well. Fix an $\alpha < \kappa$, and define the map $e = e_\alpha : \mathbb{P} \to \mathbb{C}(H_\alpha) := \mathrm{Fn}(H_\alpha, 2)$ as follows: $\mathrm{dom}(e(p)) = \bigcup\{p(\beta) \cap H_\alpha : \beta \in \mathrm{dom}(p)\}$, $e(p)(n) = \chi_{p(\alpha)}(n)$.

Strong ADR's

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Strong ADR's

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$$\forall\ p \in \mathbb{P}\ \forall\ s \in \mathbb{C}(H_\alpha)\ \big(s \leq e(p) \to \exists\ p' \leq p\ e(p') = s\big).$$

We know that if G is $(V, \mathbb{P})$-generic then $e[G]$ generates a $(V, \mathbb{C}(H_\alpha))$-generic filter G', and clearly the Cohen real defined from G' is $F_G(\alpha)$, we are done.

Refinements of $\mathscr{I}^+ \cap V$

Theorem (an abuse of Brendle's proof with DST☺)

Let $V \subseteq W$ be transitive models with $\omega_1^W \subseteq V$ but $(2^\omega)^V \neq (2^\omega)^W$, and let $\mathscr{I}$ be an analytic or coanalytic ideal coded in V. Then

$$W \models \text{“}\mathscr{I}^+ \cap V \text{ has an } \mathscr{I}\text{-ADR.”}$$

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Proof: Fix perfect $\mathscr{I}$-AD families $\mathscr{A}_X$ on every $X \in \mathscr{I}^+$ in V. The statement “$\mathscr{A}_X$ is an $\mathscr{I}$-AD family” is $\underset{\sim}{\Pi}^1_2$ (hence absolute):

$$(\forall\ A \in \mathscr{A}_X\ A \in \mathscr{I}^+) \text{ and } (\forall\ A, B \in \mathscr{A}_X\ (A \neq B \rightarrow A \cap B \in \mathscr{I})).$$

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For every $X, Y \in \mathscr{I}^+ \cap V$ let $B(X, Y) = \{A \in \mathscr{A}_X : A \cap Y \in \mathscr{I}^+\}$. Then $B(X, Y)$ is also (co)analytic (it is a cont. preimage of $\mathscr{I}^+$).

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Working in W, fix an enumeration $\{X_\alpha : \alpha < \kappa\}$ of the set $\mathscr{I}^+ \cap V$ where $\kappa = |\mathfrak{c}^V|$. We will construct the desired $\mathscr{I}$-ADR $\{A_\alpha : \alpha < \kappa\}$ and the sequence $(B_\alpha)_{\alpha<\kappa}$ in $\mathscr{I}^+$ by recursion on κ.

Refinements of $\mathscr{I}^+ \cap V$

Theorem

Let $V \subseteq W$, $\omega_1^W \subseteq V$, $(2^\omega)^V \neq (2^\omega)^W$, and let $\mathscr{I} \in V$ be an analytic or coanalytic. Then $W \models$"$\mathscr{I}^+ \cap V$ has an $\mathscr{I}$-ADR."

Proof (ctnd.): $\{X_\alpha : \alpha < \kappa\} = \mathscr{I}^+ \cap V$. $\mathscr{A}_X$= a perfect $\mathscr{I}$-AD on $X \in \mathscr{I}^+ \cap V$. $B(X, Y) = \{A \in \mathscr{A}_X : A \cap Y \in \mathscr{I}^+\} \in \underset{\sim}{\Sigma}_1^1 \cup \underset{\sim}{\Pi}_1^1$. $\rightsquigarrow$ $\mathscr{I}$-ADR $\{A_\alpha : \alpha < \kappa\}$ of $\mathscr{I}^+ \cap V$ and $(B_\alpha)_{\alpha<\kappa}$ in $\mathscr{I}^+$ ($\kappa = |\mathfrak{c}^V|$).

Assume that $\{A_\xi : \xi < \alpha\}$ and $(B_\xi)_{\xi<\alpha}$ are done, and let

$$\gamma_\alpha = \min\{\gamma : B(X_\gamma, X_\alpha) \text{ contains a perfect set}\} \leq \alpha.$$

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Let $V \subseteq W$, $\omega_1^W \subseteq V$, $(2^\omega)^V \neq (2^\omega)^W$, and let $\mathscr{I} \in V$ be an analytic or coanalytic. Then $W \models$"$\mathscr{I}^+ \cap V$ has an $\mathscr{I}$-ADR."

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Assume that $\{A_\xi : \xi < \alpha\}$ and $(B_\xi)_{\xi<\alpha}$ are done, and let

$$\gamma_\alpha = \min\{\gamma : B(X_\gamma, X_\alpha) \text{ contains a perfect set}\} \leq \alpha.$$

Notice that $\gamma_\alpha^V = \gamma_\alpha^W$ because if S is $\utilde{\Sigma}^1_1$ (or $\utilde{\Pi}^1_1$ resp.), then "S contains a perfect subset" is $\utilde{\Pi}^1_2$ ($\utilde{\Sigma}^1_2$ resp.).

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Let $V \subseteq W$, $\omega_1^W \subseteq V$, $(2^\omega)^V \neq (2^\omega)^W$, and let $\mathscr{I} \in V$ be an analytic or coanalytic. Then $W \models$"$\mathscr{I}^+ \cap V$ has an $\mathscr{I}$-ADR."

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We also know that perfect sets coded in V have at least κ many new elements (i.e. from $2^\omega \setminus V$) in W: use the group structure on 2^ω. Let $B_\alpha \in B(X_{\gamma_\alpha}, X_\alpha) \setminus (V \cup \{B_\xi : \xi < \alpha\})$ and $A_\alpha = X_\alpha \cap B_\alpha \in \mathscr{I}^+$.

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$B_\alpha \in B(X_{\gamma_\alpha}, X_\alpha) \setminus (V \cup \{B_\xi : \xi < \alpha\})$ and $A_\alpha = X_\alpha \cap B_\alpha \in \mathscr{I}^+$.
We claim that $\{A_\alpha : \alpha < \kappa\}$ is an $\mathscr{I}$-AD family. Let $\alpha \neq \beta$.
If $\gamma_\alpha = \gamma_\beta = \gamma$ then $B_\alpha, B_\beta \in \mathscr{A}_{X_\gamma}$ and hence $A_\alpha \cap A_\beta \subseteq B_\alpha \cap B_\beta \in \mathscr{I}$.

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If $\gamma_\alpha = \gamma_\beta = \gamma$ then $B_\alpha, B_\beta \in \mathscr{A}_{X_\gamma}$ and hence $A_\alpha \cap A_\beta \subseteq B_\alpha \cap B_\beta \in \mathscr{I}$.
If $\gamma_\alpha < \gamma_\beta$. Then $B(X_{\gamma_\alpha}, X_\beta)$ does not contain perfect subsets. It is enough to see that $B(X_{\gamma_\alpha}, X_\beta)$ is the same set is W. Why? Because then $B_\alpha \notin B(X_{\gamma_\alpha}, X_\beta)$ but $B_\alpha \in \mathscr{A}_{X_{\gamma_\alpha}}$, hence it yields that $A_\alpha \cap A_\beta \subseteq B_\alpha \cap X_\beta \in \mathscr{I}$.

Refinements of $\mathscr{I}^+ \cap V$

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$B_\alpha \in B(X_{\gamma_\alpha}, X_\alpha) \setminus (V \cup \{B_\xi : \xi < \alpha\})$ and $A_\alpha = X_\alpha \cap B_\alpha \in \mathscr{I}^+$. We show that if $\gamma_\alpha < \gamma_\beta$ then $B(X_{\gamma_\alpha}, X_\beta) \cap V = B(X_{\gamma_\alpha}, X_\beta) \cap W$. The set $K := B(X_{\gamma_\alpha}, X_\beta) \in \mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$ does not contain perfect subsets. Applying the Mansfield-Solovay theorem, $K \subseteq L[r]$ (where $K \in \Sigma^1_1(r) \cup \Pi^1_1(r)$, $r \in V$). Assume on the contrary that it contains a new real $E \in K^W \setminus V$, then $E \in L_\alpha[r]^W$ for some $\alpha < \omega_1^W \subseteq V$. But we know that $L_\alpha[r]^W = L_\alpha[r]^V$, a contradiction.

Mixing reals

Definition

Let $\mathbb{P}$ be a forcing notion. We say that an $f \in \omega^\omega \cap V^{\mathbb{P}}$ is a ***mixing real*** over V if $|f[X] \cap Y| = \omega$ for every $X, Y \in [\omega]^\omega \cap V$.

Clearly, it is enough to require that $\forall\, X, Y \in [\omega]^\omega \cap V \; f[X] \cap Y \neq \emptyset$.

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Clearly, it is enough to require that $\forall\, X, Y \in [\omega]^\omega \cap V\ f[X] \cap Y \neq \emptyset$.

Proposition

Let $\mathbb{P}$ be a forcing notion. Then the following are equivalent:

(i) There is a mixing real $f \in \omega^\omega \cap V^{\mathbb{P}}$ over V.

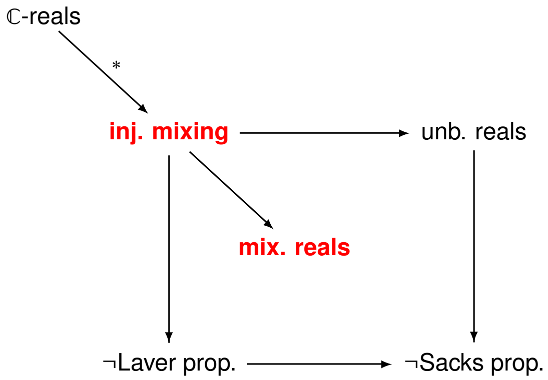
(ii) There is an $f \in \omega^\omega \cap V^{\mathbb{P}}$ s.t. $f[X] = \omega$ for all $X \in [\omega]^\omega \cap V$.

(iii) There is a partition, an ***ω-splitting real***, $(Y_n)_{n\in\omega}$ of ω into infinite sets in $V^{\mathbb{P}}$ such that $\forall\, X \in [\omega]^\omega \cap V\ \forall\, n\ |X \cap Y_n| = \omega$ (i.e. $\forall\, X \in [\omega]^\omega \cap V\ \forall\, n\ X \cap Y_n \neq \emptyset$).

Mixing reals

Why is this property relevant to almost-disjoint refinements? Fix an AD family $\{A_\alpha : \alpha < \mathfrak{c}\} \in V$, and let $\{X_\alpha : \alpha < \mathfrak{c}\} = [\omega]^\omega \cap V$ be an enumeration. If $f \in \omega^\omega \cap V^{\mathbb{P}}$ is an injective(!) mixing real over V, then $\{f[A_\alpha] \cap X_\alpha : \alpha < \mathfrak{c}\} \in V^{\mathbb{P}}$ is an ADR of $[\omega]^\omega \cap V$.

In the context of classical properties



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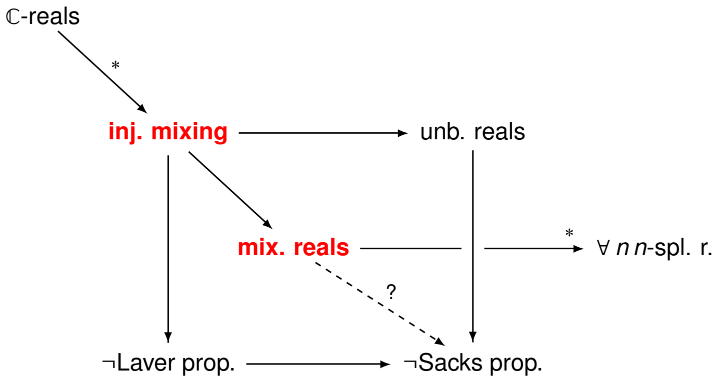

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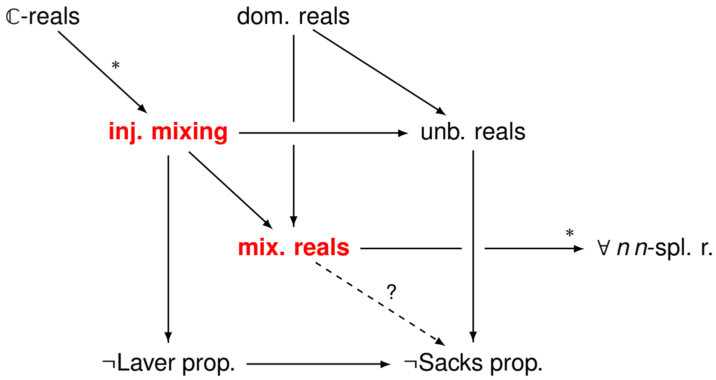
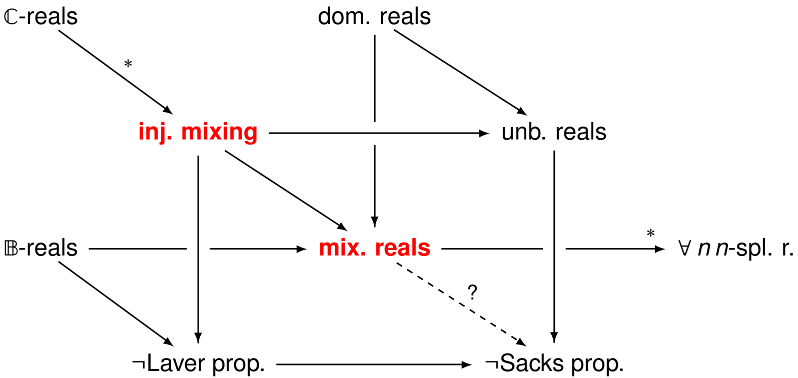


In the context of classical properties



A few more questions

Question

Let $V \subseteq W$ be transitive models with $\omega_1^W \subseteq V$ but $(2^\omega)^V \neq (2^\omega)^W$, and let $\mathscr{I}$ be an analytic or coanalytic ideal coded in V. Does there exist an $(\mathscr{I}, \mathrm{Fin})$-ADR of $\mathscr{I}^+ \cap V$ in W? Or at least an $\mathscr{I}$-ADR $\{A_X : X \in \mathscr{I}^+ \cap V\}$ of $\mathscr{I}^+ \cap V$ such that $A_X \cap A_Y \subseteq B_{X,Y} \in \mathscr{I} \cap V$ for every distinct $X, Y \in \mathscr{I}^+ \cap V$?

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Question

Is it possible that $V \subseteq W$, $\mathrm{Card}^V = \mathrm{Card}^W$ but $\mathscr{P}^V(\omega) \neq \mathscr{P}^W(\omega)$, and $W \models$"$[\omega]^\omega \cap V$ has a projective ADR"?

Thank you for your attention!

(Feel free to answer our questions BUT please

be so kind and do not find mistakes in the proofs ☺)