Regularity properties and derived forcing properties

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Joint work with Jindřich Zapletal.

- D. Chodounský, J. Zapletal, *Why Y-c.c.*, to appear in Ann. Pure Appl. Logic; arXiv:1409.4596

- T. Yorioka, *Todorcevic orderings as examples of ccc orderings without adding random reals*, Comment. Math. Univ. Carolin. 56,1 (2015) 125–132.

Outline

Definitions

Definition

A poset P satisfies *Y-c.c.* if for every countable elementary submodel $M \prec H_\theta$, $P \in M$, and every condition $q \in P$ there is a filter $\mathcal{F} \in M$ on $\mathrm{RO}(P)$ such that $\{ p \in \mathrm{RO}(P) \cap M : p \geq q \} \subseteq \mathcal{F}$.

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Definition

A poset P is *Y-proper* if for every countable elementary submodel $M \prec H_\theta$, $P \in M$, and every condition $p \in P \cap M$ there is a *Y-master* condition $q \leq p$ which is a master condition for M and such that for every $r \leq q$ there is a filter $\mathcal{F} \in M$ on $\mathrm{RO}(P)$ such that $\{ s \in \mathrm{RO}(P) \cap M : s \geq r \} \subseteq \mathcal{F}$.

Properties

Lemma

σ-centered $\Rightarrow$ *Y-c.c.* $\Rightarrow$ *c.c.c.*

Fact

strongly proper $\Rightarrow$ *Y-proper* $\Rightarrow$ *proper*

Properties

Lemma

σ-centered $\Rightarrow$ Y-c.c. $\Rightarrow$ c.c.c.

Fact

strongly proper $\Rightarrow$ Y-proper $\Rightarrow$ proper

Proposition

An atomless Y-proper forcing adds an unbounded real.

Corollary

A Y-proper forcing does not add random reals.

Properties

Theorem (ω_1-aproximation property)

Let P be a Y-proper poset, κ be a cardinal, and let f be a name for a function in κ^κ. If $P \Vdash \dot{f} \restriction a \in V$ for each $a \in [\kappa]^\omega \cap V$, then $P \Vdash \dot{f} \in V$.

Corollary

A Y-proper forcing does not add branches of uncountable cofinality into trees.

Theorem

Let X be a second countable topological space and $H \subseteq [X]^2$ be an open set – a graph. If P is Y-proper, then every H-anticlique in a P-generic extension is covered by a ground model countable set of H-anticliques.

Corollary

A Y-proper forcing cannot force an instance of OCA *for a clopen graph.*

Properties

Theorem

Y-proper forcings preserve ω_1-covers of compact Polish spaces consisting of G_δ sets.

Corollary

A Y-proper forcing cannot separate a gap in $\mathcal{P}(\omega)$ of uncountable cofinality.

Corollary

A Y-proper forcing does not increase $\mathrm{cov}(\mathcal{N})$.

Examples

Let X be a set, $\pi \subset [X]^2$

1. $P_\pi = \{ p \in [X]^{<\omega} : [p]^2 \subset \pi \}$
   $q \leq p \quad \text{iff} \quad q \supseteq p$
2. $Q_\pi = [X]^{<\omega}$
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If Q_π is c.c.c., then both Q_π and P_π are Y-c.c.

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Corollary

The following forcings are Y-c.c.;

- *specializing an ω_1-Aronszajn tree,*
- *freezing an (ω_1, ω_1)-gap,*
- *poset of finite sets of functions with positive oscillation,*
- *Todorcevic poset $T(Y)$ as defined by Balcar & Pazák & Thümmel (provided that $T(Y)$ is c.c.c.).*

Examples

The following are Y-proper;

- Laver forcing, Miller forcing,
- poset for forcing an instance of PID,
- poset for killing an S-space,
- forcing for killing Tukey types between $\omega \times \omega_1$ and $[\omega_1]^{<\omega}$.

Iterations

Theorem

Y-c.c. is preserved under finite support forcing iteration.

Meta-theorem

Let Φ *be a property of complete Boolean algebras such that*

$$\mathrm{ZFC} \vdash \Phi \Rightarrow \textit{c.c.c., and } \Phi \textit{ is preserved under the finite support iteration and complete sub-algebras.}$$

Let κ *be a regular uncountable cardinal such that* $\diamondsuit_{\kappa^+}(\mathrm{cof}\,\kappa)$ *holds. There is a complete Boolean algebra satisfying* Φ *and forcing* $\mathrm{MA}_\kappa(\Phi)$.

Corollary

If κ *is a regular uncountable cardinal such that* $\diamondsuit_{\kappa^+}(\mathrm{cof}\,\kappa)$ *holds, then there is a Y-c.c. poset forcing* $\mathrm{MA}_\kappa(\textit{Y-c.c.})$.

Iterations

Proposition

Y-properness is preserved under finite step forcing iteration.

Theorem

Assume there is a supercompact cardinal. Then there is a Y-proper poset forcing FA(*Y-proper*).

Applications

Applications

Question (Bagaria)

Does MA(σ-centered) $+$ every ω_1-Aronszajn tree is special imply MA(σ-linked)?

Questions

Question

Suppose that P, Q are Y-c.c. posets such that $P \times Q$ is c.c.c.
Is $P \times Q$ Y-c.c.?

Question

Is every Y-proper c.c.c. poset Y-c.c.?

Question

Does FA(Y-proper) imply $\mathfrak{c} = \omega_2$?

Question

Let $P = \langle P_n, \dot{Q}_n : n \in \omega \rangle$ be a CS iteration of atomless Y-proper posets. Is this forcing Y-proper?

Question

Is there a Y-c.c. poset not adding a Cohen real?

Generalizations

Definition

A poset P satisfies *Y-c.c.* if for every countable elementary submodel $M \prec H_\theta$, $P \in M$, and every condition $q \in P$ there is a filter $\mathcal{F} \in M$ on $\mathrm{RO}(P)$ such that $\{ p \in \mathrm{RO}(P) \cap M : p \geq q \} \subseteq \mathcal{F}$.

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Variations:

- filter

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- principal filter

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- filter
- principal filter
- σ-complete filter

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- F is a n-linked set
- for every $\{ p_n \colon n \in \omega \} \subset F$ the Boolean value $\liminf p_n \neq 0$
- ...

Generalizations

Variations:

- F is a filter
- F is a principal filter
- F is a σ-complete filter
- F is a n-linked set
- for every $\{ p_n \colon n \in \omega \} \subset F$ the Boolean value $\liminf p_n \neq 0$
- ...
- there is $\varepsilon \in \mathbb{Q}^+$ and a finitely additive probability measure μ such that $\varepsilon < \mu(p)$ for all $p \in F$
- there is $g \in (\omega^\omega)^V$ such that for every $n \in \omega$ and every collection of $g(n)$ many elements of F, there are n many elements in the collection with a common lower bound
- ...

Generalizations

Meta-definition

A property $\Phi(F, \mathbb{B})$ of subsets F of complete Boolean algebras $\mathbb{B}$ is a *regularity property* if the following is provable in ZFC:

1. (nontriviality) $\Phi(\{ 1 \}, \mathbb{B})$ for every complete Boolean algebra $\mathbb{B}$;
2. (closure up) $\Phi(F, \mathbb{B}) \to \Phi(G, \mathbb{B})$ whenever $G = \{ p \in \mathbb{B} \colon \exists q \in F\ q \leq p \}$;
3. (restriction) $\Phi(F, \mathbb{B})$ implies $\Phi(F \cap (\mathbb{B} \restriction p), \mathbb{B} \restriction p)$, and $\Phi(F, \mathbb{B} \restriction p)$ implies $\Phi(F, \mathbb{B})$ for each $p \in \mathbb{B}^+$;
4. (complete subalgebras) if $\mathbb{B}_0$ is a complete subalgebra of $\mathbb{B}_1$: for every $F \subset \mathbb{B}_1$ $\Phi(F, \mathbb{B}_1) \to \Phi(F \cap \mathbb{B}_0, \mathbb{B}_0)$ holds, and for every $F \subset \mathbb{B}_0$ $\Phi(F, \mathbb{B}_0) \to \Phi(F, \mathbb{B}_1)$ holds;

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4. (complete subalgebras) if $\mathbb{B}_0$ is a complete subalgebra of $\mathbb{B}_1$: for every $F \subset \mathbb{B}_1$ $\Phi(F, \mathbb{B}_1) \to \Phi(F \cap \mathbb{B}_0, \mathbb{B}_0)$ holds, and for every $F \subset \mathbb{B}_0$ $\Phi(F, \mathbb{B}_0) \to \Phi(F, \mathbb{B}_1)$ holds;
5. (iteration) if $\dot{\mathbb{B}}_1$ is a $\mathbb{B}_0$-name for a complete Boolean algebra, $F_0 \subset \mathbb{B}_0$, $\dot{F}_1$ a name for a subset of $\mathbb{B}_1$, $\Phi(F_0, \mathbb{B}_0)$ and $1 \Vdash \Phi(\dot{F}_1, \dot{\mathbb{B}}_1)$, then $\Phi(F_0 * \dot{F}_1, \mathbb{B}_0 * \dot{\mathbb{B}}_1)$ where

$$F_0 * \dot{F}_1 = \{\langle p_0, \dot{p}_1 \rangle \in \mathbb{B}_0 * \dot{\mathbb{B}}_1 \colon p_0 \wedge \|\dot{p}_1 \in \dot{F}_1\| \in F_0\}.$$

Generalizations

Meta-definition

Let $\mathfrak{G}$ be a set with a binary operation $*$. A property $\Phi(g, F, \mathbb{B})$ of subsets F of complete Boolean algebras $\mathbb{B}$ and elements $g \in \mathfrak{G}$ is a $\mathfrak{G}$-*regularity property* if the following is provable in ZFC for each $g \in \mathfrak{G}$:

1. (nontriviality);
2. (closure up);
3. (restriction);
4. (complete subalgebras);

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1. (nontriviality);
2. (closure up);
3. (restriction);
4. (complete subalgebras);

and

5. if $\dot{\mathbb{B}}_1$ is a $\mathbb{B}_0$-name for a complete Boolean algebra, $F_0 \subset \mathbb{B}_0$, $\dot{F}_1$ a name for a subset of $\mathbb{B}_1$, $\Phi(g_0, F_0, \mathbb{B}_0)$ and $1 \Vdash \Phi(\check{g}_1, \dot{F}_1, \dot{\mathbb{B}}_1)$, then $\Phi(g_0 * g_1, F_0 * \dot{F}_1, \mathbb{B}_0 * \dot{\mathbb{B}}_1)$.

Generalizations

Definition

Suppose that $\langle \mathfrak{G}, * \rangle$ is a set with a binary operation. Suppose that Φ is a $\mathfrak{G}$-regularity property of subsets of complete Boolean algebras.

1. A poset P is Φ-*c.c.* if for every countable elementary submodel $M \prec H_\theta$ containing $P, \mathfrak{G}$, and every condition $q \in P$ there is an element $g \in \mathfrak{G} \cap M$ and a set $F \subset \mathrm{RO}(P)$, $F \in M$ such that $\Phi(g, F)$ and $\{ p \in \mathrm{RO}(P) \cap M : p \geq q \} \subseteq F$.
2. P is Φ-*proper* if for every countable elementary submodel $M \prec H_\theta$ containing $P, \mathfrak{G}$ and every condition $p \in P \cap M$ there is a Φ-*master condition* $q \leq p$ which is a master for M, and for every $r \leq q$, there is an element $g \in \mathfrak{G} \cap M$ and a set $F \subset \mathrm{RO}(P)$, $F \in M$ such that $\Phi(g, F)$ and $\{ p \in \mathrm{RO}(P) \cap M : p \geq q \} \subseteq F$.

Generalizations

Theorem

If $\mathsf{ZFC} \Vdash \Phi(F) \to F$ *is* ω*-c.c.* (“Φ *is* ω*-c.c.*”)
then P *is* Φ*-c.c.* $\Rightarrow$ P *is c.c.c.*

Generalizations

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If ZFC $\Vdash \Phi(F) \to F$ *is* ω*-c.c.* *("*Φ *is* ω*-c.c.")*
then P *is* Φ*-c.c.* $\Rightarrow$ P *is c.c.c.*

Theorem (ω_1**-aproximation property)**

Let Φ *be an* ω*-c.c.,* P *be a* Φ*-proper poset,* κ *be a cardinal, and let* $\dot{f}$ *be a name for a function in* κ^κ. *If* $P \Vdash \dot{f} \restriction a \in V$ *for each* $a \in [\kappa]^\omega \cap V$*, then* $P \Vdash \dot{f} \in V$.

Theorem

Let X *be a second countable topological space and* $H \subseteq [X]^2$ *be an open set – a graph. If* Φ *is* ω*-c.c. and* P *is* Φ*-proper, then every* H*-anticlique in a* P*-generic extension is covered by a ground model countable set of* H*-anticliques.*

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Theorem

Φ-c.c. + c.c.c. is preserved under finite support forcing iteration.

Corollary

If κ is a regular uncountable cardinal such that $\diamondsuit_{\kappa^+}(\mathrm{cof}\,\kappa)$ holds, and Φ is a regularity property such that Φ-c.c. $\Rightarrow$ c.c.c., then there is a Φ-c.c. poset forcing $\mathrm{MA}_\kappa(\Phi\text{-c.c.})$.

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Proposition

Φ-properness is preserved under finite step forcing iteration.

Theorem

Assume there is a supercompact cardinal. Then there is a Φ-proper poset forcing $\mathsf{FA}(\Phi\text{-proper})$.