Universal Graphs of cardinality $\aleph_1$ without Universal Functions of cardinality $\aleph_1$

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A function of two variables $F(x, y)$ is said to be *universal* if for every other function $G(x, y)$, with the same domain and range, there exists a function $e(x)$ such that $G(x, y) = F(e(x), e(y))$. To be a bit more precise:

Definition

A function $F : A \times A \to B$ is said to be universal if for every other function $G : A \times A \to B$ there exists a function $e : A \to A$ such that

$$G(x, y) = F(e(x), e(y))$$

for all $(x, y) \in A \times A$.

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It is recorded in the Scottish book (problem 132) that Sierpiński had asked if there is a Borel function which is universal in the case that $A = B = \mathbb{R}$. He had shown that, assuming the Continuum Hypothesis, there exists a Borel function $F : \mathbb{R}^2 \to \mathbb{R}$ which is universal.

During the 2012 Fields Semester on Set Theory and Forcing Axioms the paper (LMSW) — *Universal Functions*, authored by P. Larson, A. Miller, J. Steprāns and W. Weiss — was completed. The following results are established in (LMSW).

Theorem (LMSW)

It is consistent that there is no universal function on $\mathbb{R}$, regardless of where or not it is Borel.

Theorem (LMSW)

If $\mathfrak{t} = \mathfrak{c}$ and every $X \in [\mathbb{R}]^{<\mathfrak{c}}$ is a Q-set then there is a universal function on $\mathbb{R}$.

In particular, $MA_{\aleph_1}$ implies that there is a universal function on $\mathbb{R}$. However, the existence of Borel universal functions is connected the theory of abstract rectangles studied by Miller.

Theorem (LMSW)

If $2^{<\mathfrak{c}} = \mathfrak{c}$ then the following are equivalent:

- *There is a universal function on $\mathbb{R}$ that is Borel.*
- *Every subset of $\mathbb{R}^2$ belongs the σ-algebra generated by rectangles.*

Theorem (LMSW)

It is consistent with $MA_{\aleph_1}$ that there is no Borel universal function.

In particular, in this model there is universal function on $\mathbb{R}$, but no Borel such function.

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When one generalizes universal functions to asymmetric domains the behaviour under $MA_{\aleph_1}$ is also of interest.

Definition

A function $F : A \times B \to C$ is said to be universal if for every other function $G : A \times B \to C$ there exists functions $e_A : A \to A$ and $e_B : B \to B$ such that

$$G(x, y) = F(e_A(x), e_B(y))$$

for all $(x, y) \in A \times B$.

Theorem (LMSW)

$MA_{\aleph_1}$ that there is a universal function $F : \omega \times \omega_1 \to \omega_1$.

Theorem (LMSW)

In the standard model of $MA_{\aleph_1}$ obtained by finite support iteration there is no universal function $F : \omega_1 \times \omega_1 \to \omega$.

Definition

A function $\Phi\colon [\omega_1]^2 \to \omega$ has Property R if

- *whenever $k \in \omega$ and $\{\{a_\xi, b_\xi\} : \xi \in \omega_1\}$ is a family of disjoint pairs from ω_1 with each $a_\xi \leq b_\xi$, there are distinct ξ and η such that $\Phi(\{a_\xi, a_\eta\}) \geq \Phi(\{b_\xi, b_\eta\}) \geq k$;*
- *for each $\xi \in \omega_1$ and $k \in \omega$ there are only finitely many $\eta \in \xi$ such that $\Phi(\{\xi, \eta\}) = k$.*

A function with Property R is consistent with $\mathfrak{b} > \aleph_1$.

Theorem (LMSW)

If $\mathfrak{b} > \aleph_1$ and there exists a function $\Phi\colon [\omega_1]^2 \to \omega$ with Property R then there is no universal function from $\omega_1 \times \omega_1$ to ω.

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Theorem (Justin Moore)

Under the Proper Forcing Axiom there are no functions with property R.

While the argument using $\mathfrak{b} > \aleph_1$ and Property R establishes that there are no universal functions from $\omega_1 \times \omega_1 \to \omega$, it does not rule out the existence of a universal functions from $\omega_1 \times \omega_1 \to 2$. A result of Saharon Shelah addresses this question.

Definition

A graph (V, E) is said to be universal (for $\aleph_1$) if given any graph (U, F) such that $|U| = \aleph_1$ there is a function $\Phi : U \to V$ such that $\{x, y\} \in F$ if and only if $\{\Phi(x), \Phi(y)\} \in E$. The function Φ will be called an embedding in this case.

Theorem (Shelah)

Assuming the following two hypotheses:

1. *For every $\mathcal{F} \subseteq [\omega_1^{\omega_1}]^{2^{\aleph_0}}$ there exist two functions f and g in $\mathcal{F}$ such that $\{\xi \in \omega_1 \mid f(\xi) = g(\xi)\}$ is stationary.*
2. *There exist f_ξ for every limit ordinal $\xi \in \omega_1$ such that*
   - *$f_\xi : \omega \to \xi$ is increasing and cofinal in ξ*
   - *for every club $C \subseteq \omega_1$ there is a club X such that for each $\xi \in X$ there is some n such that $f_\xi(k) \in C$ for all $k \geq n$.*

there is no universal graph on ω_1.

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Corollary

It is consistent with MA that there is no universal graph on ω_1.

Begin with model of $\diamondsuit$ and GCH and force with ccc partial order of cardinality $\aleph_4$ to obtain a model of MA and $2^{\aleph_0} = \aleph_4$. The second hypothesis of the Theorem is true because it holds in the ground model satisfying $\diamondsuit$ and clubs in the forcing extension contain clubs in the ground model.

To see that the first hypothesis is true, let $\{\dot{f}_\mu\}_{\mu \in \omega_4}$ be names for functions from ω_1 to ω_1. For each $\mu \in \omega_4$ choose a function $w_\mu : \omega_1 \to \omega_1$ and conditions $p_{\mu,\xi}$ such that

$$p_{\mu,\xi} \Vdash \text{"}\dot{f}_\mu(\xi) = w_\mu(\xi)\text{"}$$

for all $\xi \in \omega_1$.

For each pair $\mu \neq \theta$ let $\dot{C}_{\mu,\theta}$ be a name for a club such that $1 \Vdash$ "$(\forall \xi \in \dot{C}_{\mu,\theta}) \dot{f}_\mu(\xi) \neq \dot{f}_\theta(\xi)$". Using the ccc there is a club $D_{\mu,\theta}$ in the ground model such that $1 \Vdash$ "$D_{\mu,\theta} \subseteq \dot{C}_{\mu,\theta}$".

First let $E \subseteq \omega_4$ be of cardinality $\aleph_4$ such that there is a function w such that $w_\mu = w$ for all $\mu \in E$. Since the ground model satisfies $\aleph_4 \to [\aleph_1]^2_{\aleph_2}$ it follows that there is an uncountable set $B \subseteq E$ and a club D such that $D_{\mu,\theta} = D$ for $\{\mu, \theta\} \in [B]^2$. Let $\delta \in D$. Using the ccc there are distinct μ and θ in B such that there is p such that $p \leq p_{\mu,\delta}$ and $p \leq p_{\theta,\delta}$. This contradicts that $\delta \in D$ and $p \Vdash$ "$w(\xi) = w_\mu(\xi) = \dot{f}_\mu(\xi) \neq \dot{f}_\theta(\xi) = w_\theta(\xi) = w(\xi)$".

The model theoretic universality of graphs can be deceiving when considering the relationship between the existence of abstract universal functions and the existence of universal models. The key difference is that if one were to consider a universal function as the model of some theory, then embedding would require embedding the range as well as the domain of the function. This is different than the notion of universality being considered here since the values in the range remain fixed. One needs a constant for each member of the domain to achieve this model theoretically.

Nevertheless, there is insight to be gained from the model theoretic perspective. It is well known that saturated models are universal in the sense of elementary substructures and that saturated models of cardinality κ exist if $\kappa^{<\kappa} = \kappa$.

The following definitions describe possible variations on universality.

Definition

A function $U : \kappa \times \kappa \to \kappa$ will now be called **Sierpiński universal** *if for every $f : \kappa \times \kappa \to \kappa$ there exists $h : \kappa \to \kappa$ such that $f(\alpha, \beta) = U(h(\alpha), h(\beta))$ for all α and β.*

Definition

A function $U : \kappa \times \kappa \to \kappa$ is **model theoretically universal** *if for every $f : \kappa \times \kappa \to \kappa$ there exists $h : \kappa \to \kappa$ one-to-one such that $h(f(\alpha, \beta)) = U(h(\alpha), h(\beta))$ for all α and β.*

Definition

A function $U : \kappa \times \kappa \to \kappa$ is **weakly universal** *if for every $f : \kappa \times \kappa \to \kappa$ there exist $h : \kappa \to \kappa$ and $k : \kappa \to \kappa$ one-to-one such that $k(f(\alpha, \beta)) = U(h(\alpha), h(\beta))$ for all α and β.*

Question

Is the existence of a model theoretically universal function from $\kappa \times \kappa$ to κ equivalent to the existence of a Sierpiński universal one? Does the existence of either one imply the existence of the other?

Let $\mathcal{E}_4$ be the theory in the language of a single 4-ary relation A that is an equivalence relation between the first two and last two coordinates. In other words, it has the following axioms:

- $A(a, b, c, d) \to A(c, d, a, b)$
- $A(a, b, a, b)$
- $A(a, b, c, d)$ & $A(c, d, e, f) \to A(a, b, e, f)$

The transitivity condition on A implies that $\mathcal{E}_4$ does not have the 3-amalgamation property, so Mekler's argument cannot be applied to produce a universal model for this theory of cardinality $\aleph_1$ along with $2^{\aleph_0} > \aleph_1$.

Nevertheless, the following observation highlights the connection between Sierpiński universality and model theoretic universality.

Theorem

There is a universal model for $\mathcal{E}_4$ of cardinality κ if and only if there is a function $U : \kappa \times \kappa \to \kappa$ which is weakly universal.

However, Mekler's argument can be used to show that it is consistent with $2^{\aleph_0} > \aleph_1$ that there is a Sierpiński universal function from $\omega_1 \times \omega_1$ to ω_1. Moreover, the existence of a Sierpiński universal function from $\omega_1 \times \omega_1$ to ω_1 is equivalent to the existence of a Sierpiński universal function from $\omega_1 \times \omega_1$ to ω.

The existence of a Sierpiński universal function from $\omega_1 \times \omega_1$ to 2, however, is equivalent to the existence of a weakly (and, hence also model theoreticly) universal function from $\omega_1 \times \omega_1$ to 2 because of the scarcity of embedding from 2 to 2. Moreover, the existence of a model theoretic universal function from $\omega_1 \times \omega_1$ to 2 is equivalent to the existence of a universal graph on ω_1. The following question was raised in (LMSW)

Question

Does the existence of a (Sierpiński) universal function from $\omega_1 \times \omega_1$ to 2 imply the existence of a Sierpiński universal function from $\omega_1 \times \omega_1$ to ω? What about the existence of a weakly or model theoretically universal function from $\omega_1 \times \omega_1$ to ω?

The first of these questions is answered by the following:

Theorem (Shelah & S.)

It is consistent that there is a universal function from $\omega_1 \times \omega_1$ to 2 yet there is no Sierpiński universal function from $\omega_1 \times \omega_1$ to ω.

The following lemma plays a key role:

Lemma

If there is are sequences of natural numbers $\{n_i\}_{i\in\omega}$ and $\{m_i\}_{i\in\omega}$ such that

1. $m_i < n_i < m_{i+1}$
2. *for each infinite $W \subseteq \omega$ and $\mathcal{F} \subseteq \prod_{i\in W}[n_i]^{m_i}$ such that $|\mathcal{F}| \leq \aleph_1$ there is $g \in \prod_{i\in W} n_i$ such that $g(k) \notin f(k)$ for all $f \in \mathcal{F}$ and for all but finitely many $k \in W$*
3. $\mathfrak{b} = \aleph_1$

then there is no Sierpiński universal $c : [\omega_1]^2 \to \omega$.

Let $\mathcal{U}$ be a family of increasing functions from ω to ω that is $\leq^*$ unbounded and such that $|\mathcal{U}| = \aleph_1$. Let $B_\eta : \eta \to \omega$ be a bijection for each $\eta \in \omega_1$.

Suppose that $c : [\omega_1]^2 \to \omega$ is a universal function. If $u \in \mathcal{U}$, $\eta \in \xi \in \omega_1$ and $j \in \omega$ let

$$f_{u,\eta,\xi}(j) = \left\{ c(\{\xi, B_\eta^{-1}(k)\}) \;\middle|\; k \leq m_{u(j)} \right\}$$

and use the hypothesis of the lemma to find a function $g_{u,\eta} \in \prod_{i\in\omega} n_{u(i)}$ such that $g_{u,\eta}(j) \notin f_{u,\eta,\xi}(j)$ for every $\xi \in \omega_1$ and for all but finitely many $j \in \omega$. Let $\psi : \mathcal{U} \times \omega_1 \to \omega_1$ be a bijection and define

$$b : \{\{i, \alpha\} \mid i \in \omega \text{ and } \alpha \in \omega_1 \setminus \{i\}\} \to \omega$$

by $b(\{j, \psi(u, \eta)\}) = g_{u,\eta}(j)$.

Now suppose that $e : \omega_1 \to \omega_1$ is an embedding of the partial function b into c. Let η be such that $e(j) \in \eta$ for all $j \in \omega$ and let $u \in \mathcal{U}$ be such that there are infinitely many k such that $B_\eta(e(k)) \leq m_{u(k)}$. Choose j so large that $g_{u,\eta}(j) \notin f_{u,\eta,e(\psi(u,\eta))}(j)$ and such that $B_\eta(e(j)) \leq m_{u(j)}$. Then

$$b(\{j, \psi(u,\eta)\}) = g_{u,\eta}(j) \neq c(\{e(\psi(u,\eta)), B_\eta^{-1}(B_\eta(e(j)))\})$$

$$= c(\{e(\psi(u,\eta)), e(j)\})$$

contradicting that e is an embedding.

Remark

Note that the proof does not show that model theoretically or weakly universal functions do not exist.

The partial order to be used will need a quickly growing sequences of natural numbers $\{n_i\}_{i\in\omega}$ and $\{m_i\}_{i\in\omega}$ as in the lemma. Associated to each pair (n_i, m_i) will be a norm $\| \ \|_i$ on the subsets of n_i.

Let $G_0 \subseteq [\omega_1]^2$ and $G_1 \subseteq [\omega_1]^2$ be graphs on ω_1. Define $\mathbb{P}(G_0, G_1)$ to consist of triples (T, F, η) such that:

1. $T \subseteq \bigcup_{k\in\omega} \prod_{j\in k} n_j$ is a subtree
2. there is $\mathbf{Root}(T) \in T$ such that $s \supseteq \mathbf{Root}(T)$ for all $s \in T$ such that $|s| \geq \mathbf{Root}(T)$ and $\mathbf{Root}(T)$ is maximal with this property
3. F is a one-to-one function with domain $\{s \in T \mid \mathbf{Root}(T) \subseteq s\}$ and $F(t)$ is a finite partial function from ω_1 to ω_1
4. if $t \subsetneq s$ then $\mathbf{domain}(F(t)) \cap \mathbf{domain}(F(s)) = \emptyset$
5. $\bigcup_{j\leq|t|} F(t \restriction j)$ is a finite partial embedding from ω_1 to ω_1
6. the set $\{\mathbf{domain}(F(t)) \mid t \in T\}$ is pairwise disjoint
7. $\|F_{\mathbf{Root}(T)}\|_{\mathbf{Root}(T)} \geq 1$
8. $\lim_{t\in T} \|F_t\|_{|t|} = \infty$
9. $\eta \in \omega_1$.

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Define $(T^*, F^*, \eta^*) \leq (T, F, \eta)$ if and only if

1. $T^* \subseteq T$
2. $\eta^* \geq \eta$
3. $F^*(t) \supseteq F(t)$ and $F^*(t)(\delta) > \eta$ for each $t \in T$ and each $\delta \in \mathbf{domain}(F^*(t) \setminus F(t))$
4. $F^*(\mathbf{Root}(T^*)) \supseteq \bigcup\{F(\mathbf{Root}(T^*) \upharpoonright j) \mid |\mathbf{Root}(T)| \leq j \leq |\mathbf{Root}(T^*)|\}$.

The definition of $\| \ \|_{|t|}$ guarantees that $|F(t)| < m_{|t|}$ for $t \supsetneq \textbf{Root}(T)$ but there is no bound on the size of $F(\textbf{Root}(T))$.

Definition

If $\Gamma \subseteq \mathbb{P}(G_0, G_1)$ is generic define $E_\Gamma = \bigcup_{(T,F,\eta)\in\Gamma} F(\textbf{Root}(T))$.

It is routine that to see that E_Γ is a partial embedding, but it is harder to see that its domain is all of ω_1. The following lemma is the key.

Lemma

Let $\mathbb{P}(G_0, G_1)$ be such that if $w_\zeta : \eta \to 2$ is defined by $w_\zeta(\alpha) = G_0(\{\alpha, \zeta\})$ then $\{w_\zeta \mid \zeta > \eta\} \cap B$ is not null for any Borel set $B \subseteq 2^\eta$ such that $\Lambda(B) > 0$ where Λ is Lebesgue measure on 2^η. Then for any $\xi \in \omega_1$ and $(T, F, \eta) \in \mathbb{P}(G_0, G_1)$ there is $(T^*, F^*, \eta^*) \leq (T, F, \eta)$ such that $\xi \in \textbf{domain}(F(\textbf{Root}(T)))$.

It is not difficult, assuming $2^{\aleph_0} = \aleph_1$, to construct a universal graph G_1 having the stronger hypothesis of the previous lemma. Moreover, it can be shown that $\mathbb{P}(G_0, G_1)$ will be

- proper
- ω^ω bounding
- satisfy the Laver property

and so outer measure will be preserved in the countable support iteration. It is then routine to show that G_1 will be universal in the ω_2 iteration extension that enumerates all possible G_0.

Moreover, the hypothesis of the first lemma can be obtained by forcing with a partial order satisfying the above three properties to add, for each infinite $W \subseteq \omega$ and $\mathcal{F} \subseteq \prod_{i \in W} [n_i]^{m_i}$, a function $g \in \prod_{i \in W} n_i$ such that $g(k) \notin f(k)$ for all $f \in \mathcal{F}$ and for all but finitely many $k \in W$. So there will be no Sierpiński universal function from $\omega_1 \times \omega_1$ to ω.