Strengthened Ramsey's theorem, finitary Ramsey's theorem and their iteration

Keita Yokoyama

JAIST / UC Berkeley

Sets and Computations, April 20, 2015

Outline

(Infinite) Ramsey's theorem

Ramsey's theorem is well-studied in reverse mathematics.

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
  (H is said to be a *homogeneous* set for P.)
- $\mathrm{RT}^n := \forall k\, \mathrm{RT}^n_k$. (In this talk, we may say RT^n_∞.)
- $\mathrm{RT} := \forall n\, \mathrm{RT}^n$. (In this talk, we may say $\mathrm{RT}^\infty_\infty$.)

Over RCA_0, we have the following:

- $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.
- $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Thus, we have

$$\mathrm{RT}^1_2 \le \mathrm{RT}^1 \le \mathrm{RT}^2_2 \le \mathrm{RT}^2 \le \mathrm{RT}^3_2 \le \mathrm{RT}^3 \le \mathrm{RT}^4_2 \le \dots$$

(Infinite) Ramsey's theorem

Ramsey's theorem is well-studied in reverse mathematics.

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
  (H is said to be a *homogeneous* set for P.)
- $\mathrm{RT}^n := \forall k\ \mathrm{RT}^n_k$. (In this talk, we may say RT^n_∞.)
- $\mathrm{RT} := \forall n\ \mathrm{RT}^n$. (In this talk, we may say $\mathrm{RT}^\infty_\infty$.)

Over RCA_0, we have the following:

- $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.
- $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Thus, we have

$$\mathrm{RT}^1_2 \leq \mathrm{RT}^1 \leq \mathrm{RT}^2_2 \leq \mathrm{RT}^2 \leq \mathrm{RT}^3_2 \leq \mathrm{RT}^3 \leq \mathrm{RT}^4_2 \leq \dots$$

(Infinite) Ramsey's theorem

Ramsey's theorem is well-studied in reverse mathematics.

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
  (H is said to be a *homogeneous* set for P.)
- $\mathrm{RT}^n := \forall k\, \mathrm{RT}^n_k$. (In this talk, we may say RT^n_∞.)
- $\mathrm{RT} := \forall n\, \mathrm{RT}^n$. (In this talk, we may say $\mathrm{RT}^\infty_\infty$.)

Over RCA_0, we have the following:

- $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.
- $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Thus, we have

$$\mathrm{RT}^1_2 \le \mathrm{RT}^1 \le \mathrm{RT}^2_2 \le \mathrm{RT}^2 \le \mathrm{RT}^3_2 \le \mathrm{RT}^3 \le \mathrm{RT}^4_2 \le \dots$$

(Infinite) Ramsey's theorem

Ramsey's theorem is well-studied in reverse mathematics.

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
  (H is said to be a *homogeneous* set for P.)
- $\mathrm{RT}^n := \forall k\ \mathrm{RT}^n_k$. (In this talk, we may say RT^n_∞.)
- $\mathrm{RT} := \forall n\ \mathrm{RT}^n$. (In this talk, we may say $\mathrm{RT}^\infty_\infty$.)

Over RCA_0, we have the following:

- $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.
- $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Thus, we have

$$\mathrm{RT}^1_2 \leq \mathrm{RT}^1 \leq \mathrm{RT}^2_2 \leq \mathrm{RT}^2 \leq \mathrm{RT}^3_2 \leq \mathrm{RT}^3 \leq \mathrm{RT}^4_2 \leq \dots$$

Ramsey's theorem and independent statements from PA

It is well-known that several finite variations of Ramsey's theorem provide independent statements from Peano Arithmetic (PA).

- The first such example was found by Paris (in paper 1978). An "iteration version of Finite Ramsey's theorem with relatively largeness".
- A simplification by Harrington (in manuscript 1977). "Paris-Harrington Principle (PH): Finite Ramsey's theorem with relatively largeness".

Here, $X \subseteq_{\mathrm{fin}} \mathbb{N}$ is *relatively large* if $|X| > \min X$. Then,

- PH^n_k: for any $a \in \mathbb{N}$, there exists $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that for any $P : [X]^n \to k$, there exists a homogeneous set $H \subseteq X$ which is relatively large and $\min H > a$.
- $\mathrm{PH}^n \equiv \forall k \mathrm{PH}^n_k$, and $\mathrm{PH} \equiv \forall n \mathrm{PH}^n$.

Infinite vs finite Ramsey's theorem

Observation

Infinite Ramsey's theorem implies corresponding finite Ramsey's theorem (with some largeness notion), but the usual proof requires weak König's lemma / compactness argument.

This cause the following.

Fact

Iterated version of infinite Ramsey's theorem cannot prove iterated version of finite Ramsey's theorem.

Note that this happens because of the lack of Σ^1_1-induction, but infinite Ramsey's theorem as itself does not prove such a strong induction.

Infinite vs finite Ramsey's theorem

Observation

Infinite Ramsey's theorem implies corresponding finite Ramsey's theorem (with some largeness notion), but the usual proof requires weak König's lemma / compactness argument.

This cause the following.

Fact

Iterated version of infinite Ramsey's theorem cannot prove iterated version of finite Ramsey's theorem.

Note that this happens because of the lack of Σ^1_1-induction, but infinite Ramsey's theorem as itself does not prove such a strong induction.

What is essential for iterated Ramsey's theorem?

A technical interest:

Question

Is there any good notion extending Ramsey's theorem which implies PH-like statement "naturally" and thus iteration available without any stronger induction?

$\Rightarrow$ Strengthen Ramsey's theorem for coloring families (Y)

$\Rightarrow$ "Finitary version" of Ramsey's theorem (Pelupessy, Murakami)

What is essential for iterated Ramsey's theorem?

A technical interest:

Question

Is there any good notion extending Ramsey's theorem which implies PH-like statement "naturally" and thus iteration available without any stronger induction?

- $\Rightarrow$ Strengthen Ramsey's theorem for coloring families (Y)
- $\Rightarrow$ "Finitary version" of Ramsey's theorem (Pelupessy, Murakami)

Outline

Strengthened Ramsey's theorem

We first consider a strengthened version of Ramsey's theorem.

Definition (coloring family)

- An (n, k)-finite coloring is a function $P : [F]^n \to k$ where $F = \mathrm{dom}(P) \subseteq_{\mathrm{fin}} \mathbb{N}$.
- An (n, k)-coloring family is a set $\mathcal{P}$ of (n, k)-finite colorings which is closed under restriction.
- $X \subseteq \mathbb{N}$ is in the domain of $\mathcal{P}$ (write $X \in [\mathrm{dom}\,\mathcal{P}]$) if for any $Y \subseteq_{\mathrm{fin}} X$, there exists $P \in \mathcal{P}$ such that $Y = \mathrm{dom}(P)$.
- $H \subseteq \mathbb{N}$ is said to be homogeneous for $\mathcal{P}$ if $H \in [\mathrm{dom}\,\mathcal{P}]$ then there exists a constant function $P \in [\mathcal{P}]$ (i.e., $P = \bigcup_i P_i$ for some $\{P_i\} \subseteq \mathcal{P}$) such that $H = \mathrm{dom}(P)$.

Note that an (infinite) coloring $P : [\mathbb{N}]^n \to k$ can be considered as a coloring family $\mathcal{P} = \{P \upharpoonright [X]^n \mid X \subseteq_{\mathrm{fin}} \mathbb{N}\}$.

Strengthened Ramsey's theorem

Definition (Strengthened Ramsey's theorem)

RT^{n+}_k assets the following:

for any (n,k)-coloring family $\mathcal{P}$, there exists an infinite homogeneous set H for $\mathcal{P}$.

- RT^{n+}_k is a generalization of RT^n_k.
- RT^{n+}_k directly implies PH^n_k as follows:
  For given $a \in \mathbb{N}$, we want $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that any $P : [X]^n \to k$ has a (†)relatively large homogeneous set H with $\min H > a$. If not, put $\mathcal{P} := \{P \mid P \text{ has no } (\dagger)\}$, then $\mathbb{N} \in [\mathrm{dom}\,\mathcal{P}]$. By RT^{n+}_k, $\mathcal{P}$ has an infinite homogeneous set, which leads a contradiction.
- RT^{n+}_k splits into RT^n_k plus a version of "Ramsey type König's lemma", and we have $\mathrm{RT}^n_k \le \mathrm{RT}^{n+}_k \le \mathrm{RT}^n_k + \mathrm{WKL}$.

Strengthened Ramsey's theorem

Definition (Strengthened Ramsey's theorem)

RT^{n+}_k assets the following:

for any (n,k)-coloring family $\mathcal{P}$, there exists an infinite homogeneous set H for $\mathcal{P}$.

- RT^{n+}_k is a generalization of RT^n_k.
- RT^{n+}_k directly implies PH^n_k as follows:
  For given $a \in \mathbb{N}$, we want $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that any $P : [X]^n \to k$ has a (†)relatively large homogeneous set H with $\min H > a$. If not, put $\mathcal{P} := \{P \mid P \text{ has no } (\dagger)\}$, then $\mathbb{N} \in [\mathrm{dom}\,\mathcal{P}]$. By RT^{n+}_k, $\mathcal{P}$ has an infinite homogeneous set, which leads a contradiction.
- RT^{n+}_k splits into RT^n_k plus a version of "Ramsey type König's lemma", and we have $\mathrm{RT}^n_k \le \mathrm{RT}^{n+}_k \le \mathrm{RT}^n_k + \mathrm{WKL}$.

Strengthened Ramsey's theorem

Definition (Strengthened Ramsey's theorem)

RT^{n+}_k assets the following:

for any (n,k)-coloring family $\mathcal{P}$, there exists an infinite homogeneous set H for $\mathcal{P}$.

- RT^{n+}_k is a generalization of RT^n_k.
- RT^{n+}_k directly implies PH^n_k as follows:
  For given $a \in \mathbb{N}$, we want $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that any $P : [X]^n \to k$ has a (†)relatively large homogeneous set H with $\min H > a$. If not, put $\mathcal{P} := \{P \mid P$ has no $(\dagger)\}$, then $\mathbb{N} \in [\mathrm{dom}\,\mathcal{P}]$. By RT^{n+}_k, $\mathcal{P}$ has an infinite homogeneous set, which leads a contradiction.
- RT^{n+}_k splits into RT^n_k plus a version of "Ramsey type König's lemma", and we have $\mathrm{RT}^n_k \leq \mathrm{RT}^{n+}_k \leq \mathrm{RT}^n_k + \mathrm{WKL}$.

Strengthened Ramsey's theorem

Definition (Strengthened Ramsey's theorem)

RT^{n+}_k assets the following:

for any (n, k)-coloring family $\mathcal{P}$, there exists an infinite homogeneous set H for $\mathcal{P}$.

- RT^{n+}_k is a generalization of RT^n_k.
- RT^{n+}_k directly implies PH^n_k as follows:
  For given $a \in \mathbb{N}$, we want $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that any $P : [X]^n \to k$ has a $(\dagger)$relatively large homogeneous set H with $\min H > a$. If not, put $\mathcal{P} := \{P \mid P \text{ has no } (\dagger)\}$, then $\mathbb{N} \in [\mathrm{dom}\,\mathcal{P}]$. By RT^{n+}_k, $\mathcal{P}$ has an infinite homogeneous set, which leads a contradiction.
- RT^{n+}_k splits into RT^n_k plus a version of "Ramsey type König's lemma", and we have $\mathrm{RT}^n_k \leq \mathrm{RT}^{n+}_k \leq \mathrm{RT}^n_k + \mathrm{WKL}$.

Finitary version?

What is "finitary" version?

- Inspired by Terrence Tao's essay in his blog in 2007, Gaspar and Kohlenbach (2010) studied several finitary versions of the pigeon hole principle.
- Pelupessy generalized the G/K study and introduced finitary Ramsey's theorem FinRT^n_k, and showed $\mathrm{RT}^n_k \leq \mathrm{FinRT}^n_k \leq \mathrm{RT}^n_k + \mathrm{WKL}$.
- Murakami studies finitary versions of another Ramsey type principle.

Finitary version?

What is "finitary" version?

- Inspired by Terrence Tao's essay in his blog in 2007, Gaspar and Kohlenbach (2010) studied several finitary versions of the pigeon hole principle.
- Pelupessy generalized the G/K study and introduced finitary Ramsey's theorem FinRT^n_k, and showed $\mathrm{RT}^n_k \leq \mathrm{FinRT}^n_k \leq \mathrm{RT}^n_k + \mathrm{WKL}$.
- Murakami studies finitary versions of another Ramsey type principle.

Finitary version?

What is "finitary" version?

- Inspired by Terrence Tao's essay in his blog in 2007, Gaspar and Kohlenbach (2010) studied several finitary versions of the pigeon hole principle.
- Pelupessy generalized the G/K study and introduced finitary Ramsey's theorem FinRT^n_k, and showed $\mathrm{RT}^n_k \leq \mathrm{FinRT}^n_k \leq \mathrm{RT}^n_k + \mathrm{WKL}$.
- Murakami studies finitary versions of another Ramsey type principle.

Finitary version?

What is "finitary" version?

- Inspired by Terrence Tao's essay in his blog in 2007, Gaspar and Kohlenbach (2010) studied several finitary versions of the pigeon hole principle.
- Pelupessy generalized the G/K study and introduced finitary Ramsey's theorem FinRT^n_k, and showed $\mathrm{RT}^n_k \leq \mathrm{FinRT}^n_k \leq \mathrm{RT}^n_k + \mathrm{WKL}$.
- Murakami studies finitary versions of another Ramsey type principle.

Finitary Ramsey's theorem

We generalize PH by using largeness notion given by a finitary operation.

- For a given $F : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$, a set $X \subseteq_{\text{fin}} \mathbb{N}$ is said to be F-large if $|X| > F(X)$.
- "relatively large"↔"r-large" for $r(X) = \min X$.
- $\mathrm{PH}^{n,F}_k$: there exists $X \subseteq_{\text{fin}} \mathbb{N}$ such that for any $P : [X]^n \to k$, there exists a homogeneous set $H \subseteq X$ which is F-large.
- In case F is a constant function, $\mathrm{PH}^{n,F}_k$ corresponds to "finite Ramsey's theorem".

Of course, $\mathrm{PH}^{n,F}_k$ would be false if F is not appropriate. Roughly speaking, F needs to make infinite set "large".

Finitary Ramsey's theorem

We generalize PH by using largeness notion given by a finitary operation.

- For a given $F : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$, a set $X \subseteq_{\mathrm{fin}} \mathbb{N}$ is said to be F-large if $|X| > F(X)$.
- "relatively large"↔"r-large" for $r(X) = \min X$.
- $\mathrm{PH}^{n,F}_k$: there exists $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that for any $P : [X]^n \to k$, there exists a homogeneous set $H \subseteq X$ which is F-large.
- In case F is a constant function, $\mathrm{PH}^{n,F}_k$ corresponds to "finite Ramsey's theorem".

Of course, $\mathrm{PH}^{n,F}_k$ would be false if F is not appropriate. Roughly speaking, F needs to make infinite set "large".

Finitary Ramsey's theorem

We generalize PH by using largeness notion given by a finitary operation.

- For a given $F : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$, a set $X \subseteq_{\mathrm{fin}} \mathbb{N}$ is said to be F-large if $|X| > F(X)$.
- "relatively large"↔"r-large" for $r(X) = \min X$.
- $\mathrm{PH}_k^{n,F}$: there exists $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that for any $P : [X]^n \to k$, there exists a homogeneous set $H \subseteq X$ which is F-large.
- In case F is a constant function, $\mathrm{PH}_k^{n,F}$ corresponds to "finite Ramsey's theorem".

Of course, $\mathrm{PH}_k^{n,F}$ would be false if F is not appropriate. Roughly speaking, F needs to make infinite set "large".

Finitary Ramsey's theorem

We generalize PH by using largeness notion given by a finitary operation.

- For a given $F : [\mathbb{N}]^{<\mathbb{N}} \to \mathbb{N}$, a set $X \subseteq_{\mathrm{fin}} \mathbb{N}$ is said to be F-large if $|X| > F(X)$.
- "relatively large"↔"r-large" for $r(X) = \min X$.
- $\mathrm{PH}^{n,F}_k$: there exists $X \subseteq_{\mathrm{fin}} \mathbb{N}$ such that for any $P : [X]^n \to k$, there exists a homogeneous set $H \subseteq X$ which is F-large.
- In case F is a constant function, $\mathrm{PH}^{n,F}_k$ corresponds to "finite Ramsey's theorem".

Of course, $\mathrm{PH}^{n,F}_k$ would be false if F is not appropriate. Roughly speaking, F needs to make infinite set "large".

Finitary Ramsey's theorem

- F is said to be asymptotically stable ($F \in AS$) if for any infinite set X, $\forall A_n \nearrow X\ \exists M \in \mathbb{N}\ \forall^\infty n\ F(A_n) \leq M$.
- F is said to be weakly stable ($F \in WS$) if for any infinite set X, $\exists A_n \nearrow X\ \exists M \in \mathbb{N}\ \forall^\infty n\ F(A_n) \leq M$.
- F is said to be strongly stable ($F \in SS$) if for any infinite set X, $\exists M \in \mathbb{N}\ \forall A_n \nearrow X\ \forall^\infty n\ F(A_n) \leq M$.

Definition (Finitary Ramsey's theorem)

The following are equivalent over RCA_0.

- FinRT^n_k: for any $F \in AS$, $\mathrm{PH}^{n,F}_k$ holds. (Pelupessy)
- FinRT^{n+}_k: for any $F \in WS$, $\mathrm{PH}^{n,F}_k$ holds.
- FinRT^{n-}_k: for any $F \in SS$, $\mathrm{PH}^{n,F}_k$ holds.

We may loosely write FinRT^n_k for one of the above.

Finitary Ramsey's theorem

- F is said to be asymptotically stable ($F \in AS$) if for any infinite set X, $\forall A_n \nearrow X \ \exists M \in \mathbb{N} \ \forall^\infty n \ F(A_n) \le M$.
- F is said to be weakly stable ($F \in WS$) if for any infinite set X, $\exists A_n \nearrow X \ \exists M \in \mathbb{N} \ \forall^\infty n \ F(A_n) \le M$.
- F is said to be strongly stable ($F \in SS$) if for any infinite set X, $\exists M \in \mathbb{N} \ \forall A_n \nearrow X \ \forall^\infty n \ F(A_n) \le M$.

Definition (Finitary Ramsey's theorem)

The following are equivalent over RCA_0.

- FinRT^n_k: for any $F \in AS$, $\mathrm{PH}^{n,F}_k$ holds. (Pelupessy)
- FinRT^{n+}_k: for any $F \in WS$, $\mathrm{PH}^{n,F}_k$ holds.
- FinRT^{n-}_k: for any $F \in SS$, $\mathrm{PH}^{n,F}_k$ holds.

We may loosely write FinRT^n_k for one of the above.

Finitary Ramsey's theorem

- F is said to be asymptotically stable ($F \in AS$) if for any infinite set X, $\forall A_n \nearrow X\ \exists M \in \mathbb{N}\ \forall^\infty n\ F(A_n) \le M$.
- F is said to be weakly stable ($F \in WS$) if for any infinite set X, $\exists A_n \nearrow X\ \exists M \in \mathbb{N}\ \forall^\infty n\ F(A_n) \le M$.
- F is said to be strongly stable ($F \in SS$) if for any infinite set X, $\exists M \in \mathbb{N}\ \forall A_n \nearrow X\ \forall^\infty n\ F(A_n) \le M$.

Definition (Finitary Ramsey's theorem)

The following are equivalent over RCA_0.

- FinRT^n_k: for any $F \in AS$, $\mathrm{PH}^{n,F}_k$ holds. (Pelupessy)
- FinRT^{n+}_k: for any $F \in WS$, $\mathrm{PH}^{n,F}_k$ holds.
- FinRT^{n-}_k: for any $F \in SS$, $\mathrm{PH}^{n,F}_k$ holds.

We may loosely write FinRT^n_k for one of the above.

RT^{n+}_k vs FinRT^n_k

- Trivially, FinRT^n_k is a generalization of PH^n_k.
- On the other hand, FinRT^n_k implies RT^n_k although it looks like a "finite variation".

In fact, we have the following.

Theorem

The following are equivalent over RCA_0.

1. RT^{n+}_k.
2. FinRT^n_k.

Each of them is a technical variant of RT^n_k or PH^n_k, but it seems that they are reasonably stable notion.

- Note that we can define a version of RT^+ or FinRT corresponding to $\mathrm{RT}^n_{<\infty}$ or the full RT, but we need to do that carefully.

RT^{n+}_k vs FinRT^n_k

- Trivially, FinRT^n_k is a generalization of PH^n_k.
- On the other hand, FinRT^n_k implies RT^n_k although it looks like a "finite variation".

In fact, we have the following.

Theorem

The following are equivalent over RCA_0*.*

1. RT^{n+}_k.
2. FinRT^n_k.

Each of them is a technical variant of RT^n_k or PH^n_k, but it seems that they are reasonably stable notion.

- Note that we can define a version of RT^+ or FinRT corresponding to $\mathrm{RT}^n_{<\infty}$ or the full RT, but we need to do that carefully.

RT^{n+}_k vs FinRT^n_k

- Trivially, FinRT^n_k is a generalization of PH^n_k.
- On the other hand, FinRT^n_k implies RT^n_k although it looks like a "finite variation".

In fact, we have the following.

Theorem

The following are equivalent over RCA_0*.*

1. RT^{n+}_k.
2. FinRT^n_k.

Each of them is a technical variant of RT^n_k or PH^n_k, but it seems that they are reasonably stable notion.

- Note that we can define a version of RT^+ or FinRT corresponding to $\mathrm{RT}^n_{<\infty}$ or the full RT, but we need to do that carefully.

RT^{n+}_k vs FinRT^n_k

- Trivially, FinRT^n_k is a generalization of PH^n_k.
- On the other hand, FinRT^n_k implies RT^n_k although it looks like a "finite variation".

In fact, we have the following.

Theorem

The following are equivalent over RCA_0*.*

1. RT^{n+}_k.
2. FinRT^n_k.

Each of them is a technical variant of RT^n_k or PH^n_k, but it seems that they are reasonably stable notion.

- Note that we can define a version of RT^+ or FinRT corresponding to $\mathrm{RT}^n_{<\infty}$ or the full RT, but we need to do that carefully.

Outline

Density notion

We first consider the iterated version of PH introduced by Paris.

Definition (Paris)

- A finite set X is said to be 0-*dense*(n, k) if X is relatively large, *i.e.*, $|X| > \min X$.
- A finite set X is said to be $m+1$-*dense*(n, k) if for any $P : [X]^n \to k$, there exists $Y \subseteq X$ which is m-dense(n, k) and P-homogeneous.

Note that "X is m-dense(n, k)" can be expressed by a Σ^0_0-formula.

Iterated PH

Using the density notion, we can naturally define the iterated version of PH.

Definition

- $m\mathrm{PH}_k^n$: for any $a \in \mathbb{N}$, there exists an m-dense(n, k) set X such that $\min X > a$.
- $\mathrm{ItPH}_k^n \equiv \forall m \, m\mathrm{PH}_k^n$.

- Original Paris's independent statement from PA is ItPH_2^3.
- It is equivalent to the full PH and to the Σ_1-soundness of PA.

Iterated PH

Using the density notion, we can naturally define the iterated version of PH.

Definition

- $m\mathrm{PH}^n_k$: for any $a \in \mathbb{N}$, there exists an m-dense(n, k) set X such that $\min X > a$.
- $\mathrm{ItPH}^n_k \equiv \forall m\, m\mathrm{PH}^n_k$.

- Original Paris's independent statement from PA is ItPH^3_2.
- It is equivalent to the full PH and to the Σ_1-soundness of PA.

Iterated PH and the strength of Ramsey's theorem

Iterated version of PH is useful to understand the strength of infinite Ramsey's theorem.

Theorem (Paris, Bovykin/Weiermann)

The following two theories have the same Π^0_2-consequences.

1. $\mathrm{WKL}_0 + \mathrm{RT}^n_k$.
2. $\mathrm{I}\Sigma^0_1 + \{m\mathrm{PH}^n_k \mid m \in \omega\}$.

Note that we can even characterize Π^0_3 or Π^0_4-part of RT^n_k by using some modification of $m\mathrm{PH}$.

Iterated PH and the strength of Ramsey's theorem

Iterated version of PH is useful to understand the strength of infinite Ramsey's theorem.

Theorem (Paris, Bovykin/Weiermann)

The following two theories have the same Π^0_2-consequences.

1. $\mathsf{WKL}_0 + \mathrm{RT}^n_k$.
2. $\mathrm{I}\Sigma^0_1 + \{m\mathrm{PH}^n_k \mid m \in \omega\}$.

Note that we can even characterize Π^0_3 or Π^0_4-part of RT^n_k by using some modification of $m\mathrm{PH}$.

Iterated PH and the strength of Ramsey's theorem

Iterated version of PH also characterize the consistency strength of RT^n_k. The following is a generalization of the original Paris/Harrington's result.

Theorem

The following are equivalent over $\mathrm{I}\Sigma^0_1$.

1. ItPH^n_k.
2. Σ_1*-soundness of* $\mathrm{WKL}_0 + \mathrm{RT}^n_k$.

Thus, we cannot prove ItPH^n_k from RT^n_k.

Question

What is needed to prove ItPH^n_k?

Density notion

We can naturally consider the iterated version of finitary Ramsey's theorem.

Definition

- A finite set X is said to be 0-*dense*(n, k, F) if X is F-large, *i.e.*, $|X| > F(X)$.
- A finite set X is said to be $m+1$-*dense*(n, k, F) if for any $P : [X]^n \to k$, there exists $Y \subseteq X$ which is m-dense(n, k, F) and P-homogeneous.

Note that "X is m-dense(n, k, F)" can be expressed by a Σ^0_0-formula.

Iterated FinRT

Definition

- $m\mathrm{PH}^{n,F}_k$: there exists an m-dense(n,k,F) set.
- $\mathrm{ItPH}^{n,F}_k \equiv \forall m\, m\mathrm{PH}^{n,F}_k$.
- $\mathrm{ItFinRT}^n_k \equiv \forall F \in AS\ \mathrm{ItPH}^{n,F}_k$

Since $\mathrm{ItFinRT}^n_k$ proves the consistency of RT^n_k, it is strictly stronger than RT^n_k.

Question

What is the strength of $\mathrm{ItFinRT}^n_k$?

Iterated FinRT

Definition

- $m\mathrm{PH}^{n,F}_k$: there exists an m-dense(n,k,F) set.
- $\mathrm{ItPH}^{n,F}_k \equiv \forall m\, m\mathrm{PH}^{n,F}_k$.
- $\mathrm{ItFinRT}^n_k \equiv \forall F \in AS\ \mathrm{ItPH}^{n,F}_k$

Since $\mathrm{ItFinRT}^n_k$ proves the consistency of RT^n_k, it is strictly stronger than RT^n_k.

Question

What is the strength of $\mathrm{ItFinRT}^n_k$?

Iterated infinite Ramsey

A naive approach: does multiple applications of RT^n_k imply the iterated versions of PH^n_k or FinRT^n_k?

Definition

- $m\mathrm{RT}^n_k$: for any finite sequence $\langle P_i : [\mathbb{N}]^n \to k \mid i < m \rangle$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is homogeneous for any P_i.
- $\mathrm{ItRT}^n_k \equiv \forall m\, m\mathrm{RT}^n_k$.

However, ItRT^n_k is just equivalent to $\mathrm{RT}^n_{<\infty}$, thus, it cannot prove ItPH^n_k in general.

Iterated infinite Ramsey

A naive approach: does multiple applications of RT^n_k imply the iterated versions of PH^n_k or FinRT^n_k?

Definition

- $m\mathrm{RT}^n_k$: for any finite sequence $\langle P_i : [\mathbb{N}]^n \to k \mid i < m \rangle$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is homogeneous for any P_i.
- $\mathrm{ItRT}^n_k \equiv \forall m \, m\mathrm{RT}^n_k$.

However, ItRT^n_k is just equivalent to $\mathrm{RT}^n_{<\infty}$, thus, it cannot prove ItPH^n_k in general.

Iterated infinite Ramsey

A naive approach: does multiple applications of RT^n_k imply the iterated versions of PH^n_k or FinRT^n_k?

Definition

- $m\mathrm{RT}^n_k$: for any finite sequence $\langle P_i : [\mathbb{N}]^n \to k \mid i < m\rangle$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is homogeneous for any P_i.
- $\mathrm{ItRT}^n_k \equiv \forall m\, m\mathrm{RT}^n_k$.

However, ItRT^n_k is just equivalent to $\mathrm{RT}^n_{<\infty}$, thus, it cannot prove ItPH^n_k in general.

Iterated infinite Ramsey

Next approach: does multiple applications of RT^{n+}_k imply the iterated versions of PH^n_k or FinRT^n_k?

Definition

- $m\mathrm{RT}^{n+}_k$: for any finite sequence of (n, k)-coloring families $\langle \mathcal{P}_i \mid i < m\rangle$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is homogeneous for any $\mathcal{P}_i$.
- $\mathrm{ItRT}^{n+}_k \equiv \forall m\, m\mathrm{RT}^{n+}_k$.

Note that one can easily show by induction (outside of the system) that RT^{n+}_k implies $m\mathrm{RT}^{n+}_k$ for any $m \in \omega$ over RCA_0.

Iterated infinite Ramsey

Next approach: does multiple applications of RT^{n+}_k imply the iterated versions of PH^n_k or FinRT^n_k?

Definition

- $m\mathrm{RT}^{n+}_k$: for any finite sequence of (n,k)-coloring families $\langle \mathcal{P}_i \mid i < m\rangle$, there exists an infinite set $H \subseteq \mathbb{N}$ such that H is homogeneous for any $\mathcal{P}_i$.
- $\mathrm{ItRT}^{n+}_k \equiv \forall m\, m\mathrm{RT}^{n+}_k$.

Note that one can easily show by induction (outside of the system) that RT^{n+}_k implies $m\mathrm{RT}^{n+}_k$ for any $m \in \omega$ over RCA_0.

Iterated infinite vs finitary Ramsey

This time, $m\mathrm{RT}^{n+}_k$ naturally implies $m\mathrm{PH}^n_k$, and in fact, we have the following.

Theorem

Over RCA_0*, we have*

$$\forall m(m\mathrm{RT}^{n+}_k \leftrightarrow m\mathrm{FinRT}^n_k).$$

We can lift up several known results for PH via this equivalence.

Iterated infinite vs finitary Ramsey

This time, $m\mathrm{RT}^{n+}_k$ naturally implies $m\mathrm{PH}^n_k$, and in fact, we have the following.

Theorem

Over RCA$_0$, *we have*

$$\forall m(m\mathrm{RT}^{n+}_k \leftrightarrow m\mathrm{FinRT}^n_k).$$

We can lift up several known results for PH via this equivalence.

Iterated infinite vs finitary Ramsey

Recall that $\mathrm{ItPH}^3_2 \Leftrightarrow \mathrm{PH} \Leftrightarrow \Sigma_1$-soundness of PA over $\mathrm{I}\Sigma^0_1$.

Theorem

For $n \geq 3$ and $k \geq 2$, the following are equivalent over RCA_0.

1. ItRT^{n+}_k.
2. $\mathrm{ItFinRT}^n_k$.
3. RT.
4. *A restricted version of Σ^1_1-soundness of ACA_0 (???).*

Conjecture

The following are equivalent over RCA_0.

1. $\mathrm{ItFinRT}^n_k$.
2. A restricted version of Σ^1_1-soundness of RT^{n+}_k.

Questions

The situation around RT^2_2 is left open.

Question

- Does RT^2_2 imply $m\mathrm{PH}^2_2$ for $m \in \omega$?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^{2+}_2 \equiv \mathrm{FinRT}^2_2 \leq \mathrm{RT}^2_2 + \mathrm{WKL}$.

- Which of the above is/are strict?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^2_{<\infty} \leq \mathrm{ItFinRT}^2_2 \leq \mathrm{ItFinRT}^2_{<\infty}$.

- Which of the above is/are strict?
- Especially, does $\mathrm{RT}^2_{<\infty}$ imply ItPH^2_2?

Questions

The situation around RT^2_2 is left open.

Question

- Does RT^2_2 imply $m\mathrm{PH}^2_2$ for $m \in \omega$?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^{2+}_2 \equiv \mathrm{FinRT}^2_2 \leq \mathrm{RT}^2_2 + \mathrm{WKL}$.

- Which of the above is/are strict?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^2_{<\infty} \leq \mathrm{ItFinRT}^2_2 \leq \mathrm{ItFinRT}^2_{<\infty}$.

- Which of the above is/are strict?
- Especially, does $\mathrm{RT}^2_{<\infty}$ imply ItPH^2_2?

Questions

The situation around RT^2_2 is left open.

Question

- Does RT^2_2 imply $m\mathrm{PH}^2_2$ for $m \in \omega$?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^{2+}_2 \equiv \mathrm{FinRT}^2_2 \leq \mathrm{RT}^2_2 + \mathrm{WKL}$.

- Which of the above is/are strict?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^2_{<\infty} \leq \mathrm{ItFinRT}^2_2 \leq \mathrm{ItFinRT}^2_{<\infty}$.

- Which of the above is/are strict?
- Especially, does $\mathrm{RT}^2_{<\infty}$ imply ItPH^2_2?

Questions

The situation around RT^2_2 is left open.

Question

- Does RT^2_2 imply $m\mathrm{PH}^2_2$ for $m \in \omega$?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^{2+}_2 \equiv \mathrm{FinRT}^2_2 \leq \mathrm{RT}^2_2 + \mathrm{WKL}$.

- Which of the above is/are strict?

Question

We have $\mathrm{RT}^2_2 \leq \mathrm{RT}^2_{<\infty} \leq \mathrm{ItFinRT}^2_2 \leq \mathrm{ItFinRT}^2_{<\infty}$.

- Which of the above is/are strict?
- Especially, does $\mathrm{RT}^2_{<\infty}$ imply ItPH^2_2?

References

- Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. Journal of Symbolic Logic, 66(1):1–55, 2001.
- Jaime Gaspar and Ulrich Kohlenbach. On Tao's " finitary " infinite pigeonhole principle. Journal of Symbolic Logic 75(1), 355–371, 2010.
- Stephen Flood. Reverse mathematics and a Ramsey-type König's Lemma. Journal of Symbolic Logic, 77(4):1272–1280, 2012.
- J. Paris and Leo A. Harrington. A mathematical incompleteness in Peano arithmetic. In Jon Barwise, editor, Handbook of Mathematical Logic, pages 1133–1142.
- J. B. Paris. Some independence results for Peano Arithmetic. Journal of Symbolic Logic, 43(4):725–731, 1978.
- Y. A strengthened version of Ramsey's theorem and its finite iteration, draft.