Master conditions of towers

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Definition (Shelah)

Let S be a stationary subset of κ which consists of limit ordinals. We say that a sequence $\vec{C} = \langle C_\alpha \mid \alpha \in S \rangle$ is a **tail** (resp. **full**) **club guessing sequence on** S if and only if

1. for each $\alpha \in S$, C_α is an unbounded subset of α and
2. for every club set D of κ, there exists an $\alpha \in S$ such that $C_\alpha \subset^* D$(resp. $C_\alpha \subset D$).

When $S = \mathrm{Lim}(\kappa)$, we simply say that $\vec{C}$ is a TCG(resp. FCG)-sequence on κ.

Fact

1. *$\diamondsuit(S)$ implies the existence of FCG-sequence on S.*
2. *Let μ and κ be regular cardinals such that $\mu^+ < \kappa$ and S a stationary subset of $\kappa \cap Cof(\mu)$. Then there exists a FCG-sequence on S.*

Definition (Shelah)

Let $\vec{C} = \langle C_\alpha \mid \alpha \in \kappa \rangle$ be a TCG-sequence on κ. We define the **tail club guessing** (resp. **full club guessing**) **ideal** $\mathrm{TCG}(\vec{C})$ ($\mathrm{FCG}(\vec{C})$) associated with $\vec{C}$ as the ideal on κ generated by the sets of the form $\{\delta \in \kappa \cap \mathrm{Lim} \mid C_\delta \not\subset^* D\}$ ($\{\delta \in \kappa \cap \mathrm{Lim} \mid C_\delta \not\subset D\}$) for some club subset D of κ.

Any TCG ideal or FCG ideal contains the nonstationary ideal over κ. Any TCG ideal is κ-complete and normal. FCG ideal can never be normal.

Definition

Let I be an ideal on κ. We define an equivalence class $=_I$ on $P(\kappa)$ by $X =_I Y$ iff $(X \triangle Y) \in I$.
Let $P(\kappa)/I = \{[X]_I \mid X \subset \kappa \text{ and } X \notin I\}$, which is ordered by $[X]_I \leq [Y]_I$ iff $X \setminus Y \in I$.

Definition

1. An ideal I on κ is **precipitous** iff for every generic filter $G \subseteq P(\kappa)/I$, the ultrapower $Ult(V, U_G)$ is well-founded, where $U_G \subset P(\kappa)$ is generated from G.
2. An ideal I on κ is **saturated** iff $P(\kappa)/I$ satisfies κ^+-c.c.

If κ-complete ideals I on κ is saturated, then I is also precipitous.

Fact

1. *Con(there exists one measurable cardinal)* $\iff$ *Con(nonstationary ideal on* ω_1 *is precipitous).(Jech-Magidor-Mitchell-Prikry)*
2. *Con(there exists one Woodin cardinal)* $\iff$ *Con(nonstationary ideal on* ω_1 *is saturated).(Shelah, Woodin)*
3. *Con(there exists one measurable cardinal)* $\iff$ *Con(there exists a precipitous TCG ideal on* ω_1*).(Ishiu)*
4. *Con(there exists one Woodin cardinal)* $\iff$ *Con(there exists a saturated TCG ideal on* ω_1*).(Ishiu)*

Remark about these models:
(1) In this model, no TCG ideal can be precipitous.
(2) In this model, it is unknown whether there exists a TCG sequence on ω_1.
(3) In this model, no restriction of NS_{ω_1} can be precipitous.
(4) In this model, NS_{ω_1} is not saturated.

Fact

Con(ZF+AD)$\Longrightarrow$ Con(there is a TCG sequence $\vec{C}$+NS_{ω_1} and $TCG(\vec{C})$ are both saturated.)(Woodin)

Question

What is the consistency strength of the following statement: NS_{ω_1} and $TCG(\vec{C})$ are both precipitous?

Lower bound: one measurable cardinal.
Upper bound: omega many Woodin cardinal.

Theorem

Con(there exists one measurable cardinal) $\iff$ *Con(there is a TCG sequence* $\vec{C}$ *+* NS_{ω_1} *and* $TCG(\vec{C})$ *are precipitous).*

It suffices to fulfil the following goals at the same time:

1. Force $TCG(\vec{C})$ to be of the form $MS_{\omega_1} \restriction S$, where S is a stationary subset of ω_1.
2. Force NS_{ω_1} to be precipitous.

Master conditions argument and precipitousness of NS_{ω_1}

Aim: Build a model from a measurable cardinal in which NS_{ω_1} is precipitous.

1. Let κ be a measurable cardinal. Let $j : V \to M$ be the ultrapower map.
2. Levy-collapse κ to ω_1. Then, there is a precipitous ideal on ω_1.
3. Keep shooting clubs by countable conditions so that the ideal remains precipitous but becomes NS_{ω_1}.

Key point: Construct $(j''N, j(P))$-generic condition for the lifting of iterated shooting club forcing, where N is a κ size elementary submodel of $H(\theta)$. (So called "master condition argument").

Definition

Suppose $j : V \to M$. Suppose P is a forcing poset in V. Assume G is P generic over V. We say a condition $p \in j(P)$ is a **master condition** (for j and G) if for any $q \in G$, $p < j(q)$.

Goal 1: the forcing

Definition (Foreman-Komjath)

We say that a sequence $\vec{C} = \langle C_\alpha \mid \alpha \in \omega_1 \rangle$ is a **strong club guessing (SCG) sequence on** S if and only if

1. for each $\alpha \in \kappa$, C_α is an unbounded subset of α and
2. for every club set D of ω_1, there exists an club E such that for all $\alpha \in E \cap S$, $C_\alpha \subset^* D$.

$\vec{C}$ is a SCG sequence on S if and only if $TCG(\vec{C}) = NS \restriction S$.

Fact

1. *In the generic extension by $Add(\kappa, \kappa^+)$, there is no strong club guessing sequence on any stationary subset of κ.*
2. *For a give stationary subset S of κ, it is possible to force a strong club guessing sequence on S.(Foreman-Komjath)*
3. *In L, for every regular, non-ineffable cardinal κ, there is a strong club guessing sequence on $\kappa \cap Sing$.(Ishiu)*

Definition

An ordinal ξ is **indecomposable** iff for every ordinal $\alpha, \beta < \xi, \alpha + \beta < \xi$.

Definition

We say a TCG sequence $\langle C_\alpha \mid \alpha \in S \rangle$ **has order type** ξ iff for club many $\alpha \in S$, the order type of C_α is ξ.

It is consistent that a TCG sequence has no order type.

Theorem (Foreman-Komjath)

Assume GCH, there is a cardinal preserving forcing P forces the existence of SCG sequence $\vec{C}$ on S, where S is a stationary subset of ω_1.

The variant of Foreman-Komjath forcing

A three step iteration $P_{SCG} = P_0 * P_1 * P_2$:

P_0: $Add(\omega_1, 1)$. Let S be the derived generic subset of ω_1. Both S and $\omega_1 \setminus S$ are stationary.

P_1: $P(\vec{C}, S)$, the forcing notion to shoot a measure one set though S:
$p \in P(\vec{C}, S)$ iff p is a closed bounded subset of ω_1 such that for every $\alpha \in p \cap \mathrm{Lim}$, if $C_\alpha \subset^* p$, then $\delta \in S$. $P(\vec{C}, S)$ is ordered by end-extension.

P_2: An ω_2-length, countable support iteration. For any $X \in TCG(\vec{C})$, shoot a club through $\omega_1 \setminus (S \cup X)$.

generic condition for towers and TCG-preserving

Definition (Shelah)

Let P be a forcing notion. For a set N, a condition $q \in P$ is said to be (N, P)-**generic** if and only if for every P-name $\dot{\alpha}$ for an ordinal lying in N, $q \Vdash \dot{\alpha} \in N$. A condition $q \in P$ is said to be (N, P)-**semigeneric** if and only if for every P-name $\dot{\alpha}$ for a countable ordinal lying in N, $q \Vdash \dot{\alpha} \in N$.

Definition (Shelah)

We say that P is **proper** if and only if for some sufficiently large regular cardinal λ, for every countable elementary substructure N of $H(\lambda), \in\rangle$ with $P \in N$ and $p \in P \cap N$, there exists a $q < p$ which is (N, P)-generic.

Definition

A sequence $\langle N_\alpha : \alpha < \eta \rangle$ is called a **tower** if and only if

1. for every limit $\alpha < \eta$, $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$, and
2. for every $\alpha < \eta$, $\langle N_\beta : \beta \leq \alpha \rangle \in N_{\alpha+1}$.

Definition (Miyamoto)

For a TCG sequence $\vec{C}$. A forcing notion P is said to be $\vec{C}$-**semiproper** if and only if for some sufficiently large regular cardinal λ, there exists a structure A expanding $\langle H(\lambda), \in \rangle$ such that for every tower $\langle N_\alpha : \alpha < \eta \rangle$ of countable elementary substructures of A with $\vec{C}, P \in N_0$, if $p \in N_0 \cap P$, $\delta = \omega_1 \cap \bigcup_{\alpha<\eta} N_\alpha$, $\delta_0 = \omega_1 \cap N_0$, and $C_\delta \setminus \delta_0 = \{\omega_1 \cap N_\alpha : \alpha < \eta\}$, then there exists a $q < p$ such that q is (N_α, P)-semigeneric for every $\alpha < \eta$.

Any $\vec{C}$-semiproper forcing forces that $\vec{C}$ remains a TCG sequence.

Fact

P_{SCG} is ω_2-c.c, proper and $\vec{C}$-semiproper. Moreover, P_{SCG} is ξ-proper if $\vec{C}$ is of order type ξ.

Combing Goal 1 and Goal 2

The first attempt:

1. Let κ be a measurable cardinal. Let $j : V \to M$ be the ultrapower map.
2. Levy-collapse κ to ω_1. Then, there is a precipitous ideal on ω_1.
3. Add a TCG sequence $\vec{C}$ and a stationary set S.
4. Keep shooting clubs by countable conditions so that the ideal becomes NS_{ω_1} and $\vec{C}$ becomes a FCG-sequence on S.

Obstacle: For TCG preservation of P_{SCG}, we need to find $(\vec{N}, P)$-semigeneric condition q for any tower $\vec{N}$. On the other hand, due to the nature of the forcing for NS_{ω_1} being precipitous (note that this forcing kills so many stationary subsets of ω_1), we can only find (N, P)-generic condition for stationary many N, which are derived from a "unique" "master model" $j''N$ with N a κ-size elementary substructure.

Solution: We construct $(\vec{N}, P)$-semigeneric condition for only "stationary" many towers $\vec{N}$, which are derived from a "unique" tower of "master model".

Tower of master model(for TCG sequences of type ω)

Configuration:
Let $\langle M_k, j_{kl} \mid k \leq l \leq \omega + 1 \rangle$ be the direct system of iterated ultrapower derived from j.
Let $\theta > \kappa$ be a sufficiently large regular cardinal. Let $\mathcal{M}$ be the structure $\langle H(\theta), \in, \mathbb{P}_{\omega_2}, \lhd \rangle$, where $\lhd$ is a well-ordering of $H(\theta)$. Consider $j_{0\omega}(\mathcal{M})$. Let M_i be the Skolem Hull of κ_i in $j_{0\omega}(\mathcal{M})$. Note that $M_i = j''_{i_\omega} N_i$, where N_i is the Skolem Hull of κ_i in $j_{0,i}(\mathcal{M})$.
As all maximal antichains in $Col(\omega, < \kappa_\omega) \cap M_i$ are in M_i, a canonical argument shows that for any G^ω which is $Col(\omega, < j_{0\omega})$ generic over M_ω, $M_i[G^\omega] \prec M_\omega[G^\omega]$. Going to $G^{\omega+1}$, which is $Col(\omega, < j_{0\omega+1})$ generic over M_ω, $M_\omega[G^\omega] \prec M_{\omega+1}[G^{\omega+1}]$. It is thus clear that all $M_i[G^{\omega+1}] = M_i[G^\omega]$ form a tower.
By Mathias property of Prikry forcing, this tower is generic over M_ω and can be embedded into $M_{\omega+1}[G^{\omega+1}]$.

Proposition

For any forcing P in $V[G]$, if there exists a $(\langle M_i[G^{\omega+1}] \mid i < \omega \rangle, j_{\omega_1}(P))$-generic condition in $M[G^\omega]$, then in $V[G]$, there is a stationary tower $\vec{N}$ such that the $(\vec{N}, P)$-generic condition exists. (Here $j_{\omega_1} : V[G] \to M[G^\omega]$ is the lifting of the original j_ω.)

Construction:

1) Levy-Collapse κ to ω_1.

2) Add a TCG sequence $\vec{C}$ of order type ω.

3) Add a stationary, so-stationary set $S \subset A_\xi$. Define two distinct ideal I_0 and I_1 on ω_1 base on the value of $j(S)(\kappa)$.

4) Keep shooting clubs so that the TCG ideal becomes $NS_{\omega_1} \restriction S$.

5) Keep shooting clubs so that $\vec{C}$ becomes a SCG-sequence on S.

6) Verify that I_0 and I_1 will be lifted to precipitous ideals.

Key point:
The ideals I_0 and I_1 correspond to κ_i (for $i < \omega$) and κ_ω respectively. During the construction, we will keeping searching the correspond generic condition to apply the proposition. As a result, the master condition through I_0 guarantee the forcing will be ω_2-c.c and has $< \omega_1$-closed subset. The master condition through I_1 guarantee the forcing preserves TCG. $I_0 \cap I_1$ will be NS_{ω_1} in the final model.

Theorem

Con(there exists one measurable cardinal) $\iff$ *Con(there is a TCG sequence* $\vec{C}$ *of order type* $\omega{+}NS_{\omega_1}$ *and* $TCG(\vec{C})$ *are precipitous).*

Now we turn to the case when order types are greater than ω. The key issue here is we will need to deal with tower of height greater than ω in a essential way. In fact, the existence of TCG sequence of different order type can be used to separate $FA(\alpha - proper)$ for distinct $\alpha < \omega_1$.

Tower of master model(for TCG sequence of types greater than ω)

Definition

Let κ be a measurable cardinal. If U_0 and U_1 are normal measures on κ, let $U_0 \lhd U_1$ if and only if $U_0 \in Ult(V, U_1)$. The relation $\lhd$ is called **Mitchell order**.

The Mitchell order is well-founded.

Definition

If U is a normal measure on κ, let $o(U)$, the order of U, denote the rank of U in $\lhd$. Let $o(\kappa)$, the order of κ, denote the height of $\lhd$.

In particular, if $o(\kappa)$ is a successor ordinal $\xi + 1$, then there is a $\lhd$-increasing sequence of normal measures of length ξ.

Mitchell's complete iteration

Definition (Mitchell)

Suppose that U_0 is a coherent sequence in V and that κ_0 is measurable there. The iterated ultrapower $j : V \to M$ to construct **a complete system C of indiscernibles** for $\kappa = j(\kappa_0)$ is the limit of the embeddings $j_\alpha : V \to M_\alpha$ defined as follows by recursion on α: Suppose that j_α has been defined and let $U_\alpha = j_\alpha(U_0)$, $\kappa_\alpha = j_\alpha(\kappa_0)$, and let C_α be the indiscernibles constructed by j_α. If $C_\alpha(\kappa_\alpha, \beta)$ is cofinal in κ_α for each $\beta < o^{U_\alpha}(\kappa_\alpha)$ then we are done: j is set equal to j_α. Otherwise let β_α be the least $\beta < o(\kappa_\alpha)$ such that $C_\alpha(\kappa_\alpha, \beta)$ is bounded in κ_α, and let $j_{\alpha,\alpha+1} : M_\alpha \to M_{\alpha+1}$ be the ultrapower of M_α by $U_\alpha(\kappa_\alpha, \beta_\alpha)$. Then

$$C_{\alpha+1} \restriction (\kappa_\alpha + 1) = C_\alpha \restriction (\kappa_\alpha, \beta_\alpha),$$

$$C_{\alpha+1}(\kappa_{\alpha+1}, j_{\alpha,\alpha+1}(\beta)) = \begin{cases} C_\alpha(\kappa_\alpha, \beta) & \text{if } \beta \neq \beta_\alpha \\ C_\alpha(\kappa_\alpha, \beta) \cup \{\kappa_\alpha\} & \text{if } \beta = \beta_\alpha \end{cases}$$

and $C_\alpha(\nu, \beta) = \emptyset$ otherwise.

We will work in the Mitchell inner model $L[U_0]$, where U_0 is a coherent sequence.

In M, we can define the Magidor forcing along the coherent measure sequence $\langle j(U_0)(j(\kappa), \alpha) \mid \alpha < \xi \rangle$. It will change the cofinality of $j(\kappa)$ to ξ.

The following fact reveals the relation between the complete system and Magidor forcing:

Theorem (Mitchell)

The complete system C is Magidor (along $\langle j(U_0)(j(\kappa), \alpha) \mid \alpha < \xi \rangle$) generic over M.

Let M' be $Ult(M, j(U_0)(j(\kappa), \xi))$. Let $j' : V \to M'$. Magidor forcing along $\langle j(U_0)(j(\kappa), \alpha) \mid \alpha < \xi \rangle$ is in M' and of cardinality $j(\kappa)^+$. Thus it can be absorbed into $j'(Col(\omega, < \kappa)) = Col(\omega, < j'(\kappa))$.

Now suppose G is $Col(\omega, < \kappa)$-generic over V. We can define the lifting $j'_G : V[G] \to M'[\bar{G}]$ such that C is in $M'[\bar{G}]$.

Now in $M'[\bar{G}]$, we can define the tower $\langle N_\alpha \mid \alpha < \xi \rangle$ by let N_α be the Skolem Hull of κ_α in $j'_G(H(\theta)^{V[G]})$.

For any $\alpha < \xi$, we can ensure $\langle N_\beta \mid \beta < \alpha \rangle$ is in N_α.

Theorem

Let $\xi < \omega_1$ be an indecomposable ordinal greater than ω. Assume $V = L[\vec{U}]$, where $\vec{U}$ is a coherent sequence of measures witnessing κ is of Mitchell rank ξ. There is a forcing notion P such that in the generic extension V^P:

1. *There is a TCG sequence $\vec{C}$ of order type ξ.*
2. *There is a stationary subset S of ω_1 such that $\vec{C} \upharpoonright S$ is a SCG-sequence on S.*
3. *NS_{ω_1} is precipitous.*

It is unclear now whether this large cardinal assumption is necessary.

Question

Suppose

1. *there is a TCG sequence $\vec{C}$ of order type ξ,*
2. *there is a stationary subset S of ω_1 such that $\vec{C} \restriction S$ is a SCG-sequence on S,*
3. *and NS_{ω_1} is precipitous.*

Does there exist an inner model with a measurable cardinal with Mitchell rank ξ?

Question

What about regular cardinals $\geq \omega_2$?

Thank you for your attention!