The left side of Cichoń's Diagram

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Cichoń's Diagram

Main result

Theorem

Let $\kappa_{an} < \kappa_{cn} < \kappa < \kappa_{nm} < \kappa_{ct}$ *be regular uncountable cardinals with* $\kappa^{\aleph_0} = \kappa$, $\kappa_{ct}^{\aleph_0} = \kappa_{ct}$ *and* $2^\kappa \geq \kappa_{ct}$. *Assume* $\mathfrak{b} = \mathfrak{d} = \kappa$.
Then there is a ccc poset forcing

- $\mathrm{add}(\mathrm{null}) = \kappa_{an}$,
- $\mathrm{cov}(\mathrm{null}) = \kappa_{cm}$,
- $\mathfrak{b} = \kappa$,
- $\mathrm{non}(\mathrm{meager}) = \kappa_{nm}$,
- $\mathrm{cov}(\mathrm{meager}) = 2^{\aleph_0} = \kappa_{ct}$.

Joint work with Diego Mejía and Saharon Shelah.
(See our preprint for a slightly stronger version, available on arXiv.org soon.)

Iterands

Naive strategy:

- To ensure add(null) $\geq \kappa_{an}$, use small subposets of amoeba forcing.
- To ensure cov(null) $\geq \kappa_{cn}$, use small subposets of random forcing.
- To ensure $\mathfrak{b} \geq \kappa_b$, use small subposets of Hechler forcing.
- To ensure non(meager) $\geq \kappa_{nm}$, use small subposets of eventually different forcing $\mathbb{E}$.
  More precisely: In V^{P_α}, let $Q_\alpha := \mathbb{E} \cap V^{P'_\alpha}$, for some small $P'_\alpha \lessdot P_\alpha$.

To ensure the converse inequalities, use well-known preservation theorems (σ-centered, μ-centered, etc.)

Definition

Eventually different forcing $\mathbb{E}$ is the set of all conditions (s, φ), where $s \in \omega^{<\omega}$, φ is a slalom with domain $\omega \setminus \mathrm{dom}(s)$ of bounded width: $\exists w \in \omega\ \forall n: \ |\varphi(n)| \leq w$.
The condition (s, φ) forces that the generic function $g: \omega \to \omega$ extends s and avoids φ (i.e., $\forall n: g(n) \notin \varphi(n)$).

Ultrafilter limits

Definition

Let D be an ultrafilter on ω. For any sequence $\bar{A} = (A_k : k \in \omega)$ of subsets of ω we define $B := \lim_D \bar{A}$ by

$$\forall i \in \omega : \quad i \in B \;\Leftrightarrow\; \{k : i \in A_k\} \in D.$$

- Note that B may be empty. (E.g., $A_k = \{k\}$.)
- Even if all set A_k are finite, B may be infinite. (E.g., $A_k = \{0, \ldots, k\}$.)
- However: If $\exists w \in \omega : \forall k \; |A_k| \leq w$, then also $\lim \bar{A}$ is finite. (In fact, any uniform bound for the A_n will also be a bound for their limit.)

Ultrafilter limits in $\mathbb{E}$

Definition

Let $s \in \omega^{<\omega}$, $w \in \omega$.
A condition p obeys (s, w) if p is of the form (s, φ) with $|\varphi(n)| \leq w$ for all n.
A sequence $\bar{p} = (p_k : k \in \omega)$ of conditions in $\mathbb{E}$ is called uniform if they all obey the same (s, w).

Definition

Let $\bar{p} = (p_k : k \in \omega)$ be a uniform sequence of conditions, all obeying (s, w). Let $p_k = (s, \varphi_k)$. Let D be an ultrafilter on ω.
Then $q = \lim_D \bar{p}$ is defined as follows: $q = (s, \psi)$, where

$$\forall i \, \forall n : \; i \in \psi(n) \Leftrightarrow \{n : i \in \varphi_k(n)\} \in D,$$

i.e., ψ is the pointwise D-limit of $(\varphi_k : k \in \omega)$.

Note that q also obeys (s, w).

Why $\mathbb{E}$ does not destroy unbounded families

Lemma (Miller, TAMS 1981: Compactness)

Let D be a nonprincipal ultrafilter.
Then there is an $\mathbb{E}$-name $\underset{\sim}{D}^{+1}$ such that:

- $\Vdash_{\mathbb{E}} \underset{\sim}{D}^{+1} \supseteq D$ *is an ultrafilter.*
- *For all uniform sequences $\bar{p} = (p_k : k \in \omega)$ of conditions, their D-limit $\lim_D \bar{p}$ forces: $\{k : p_k \in G_{\mathbb{E}}\} \in D^{+1}$.*

(This says: Conditions from a uniform sequence are very compatible: infinitely many of them fit into the same generic filter.)

Proof.

Let $\underset{\sim}{A}_{\bar{p}} := \{k : p_k \in G_{\mathbb{E}}\}$ if $\lim_D \bar{p} \in G_{\mathbb{E}}$, and $\underset{\sim}{A}_{\bar{p}} := \omega$ otherwise.
Show that the family of all such $A_{\bar{p}}$ has the finite intersection property. □

Why $\mathbb{E}$ does not destroy unbounded families, continued

Theorem (Miller 1981)

Let $\bar{f} := (f_i : i < \kappa)$ be a strongly unbounded sequence in ω^ω.
(I.e., $(f_i : i \in S)$ unbounded for all $S \in [\kappa]^\kappa$.)
Then $\bar{f}$ is still unbounded in $V^{\mathbb{E}}$.

Proof.

Assume $\underset{\sim}{g}$ is a bound for $\bar{f}$. Fix $D \in V$, $D^{+1} \in V^{\mathbb{E}}$ as in the theorem.
Find $(n_i : i \in \kappa)$ and $(p_i : i \in \kappa)$ such that $p_i \Vdash \forall n \geq n_i : f_i(n) \leq \underset{\sim}{g}(n)$.
For some $S \in [\kappa]^\kappa$ we get that $(n_i : i \in S)$ is constant, say with value 0, and that $(p_i : i \in S)$ is uniform. (Same stem, bounded width.)

Wlog $\{f_i(0) : i \in S\}$ is unbounded. Thin out to a uniform sequence $\bar{p} = (p_{i_k} : k \in \omega)$ such that $(f_{i_k}(0) : k \in \omega)$ is strictly increasing.
Let $q := \lim_D \bar{p}$. Then q forces that $\underset{\sim}{g}(0)$ bounds "almost all" $f_{i_k}(0)$. □

What can go wrong

Let $\mathbb{E}' \subseteq \mathbb{E}$ be a small subforcing. To show that $\mathbb{E}'$ does not destroy any unbounded family, we would like to have:

Lemma (Wishful thinking)

Whenever $\bar{p} = (p_k : k \in \omega)$ is a uniform sequence of conditions in $\mathbb{E}'$, then there is a name D^{+1} of an ultrafilter extending an ultrafilter D in the ground model such that $\lim_D \bar{p} \Vdash \{k : p_k \in G_{\mathbb{E}'}\} \in D^{+1}$.

This MAY NOT WORK for certain $\mathbb{E}'$, because $\lim_D \bar{p}$ may be in $\mathbb{E} \setminus \mathbb{E}'$.
We will have to choose $\mathbb{E}'$ appropriately, see below.
Note that this CANNOT WORK for all $\mathbb{E}'$, because:

Theorem (Pawlikowski 1992)

There may be (nice) subposets $\mathbb{E}' \subseteq \mathbb{E}$ which add a dominating real.
For example, if $\mathbb{I}$ is the "infinitely often equal" forcing,
then $\mathbb{I}$ forces that $\mathbb{E} \cap V$ adds a dominating real.

Setup

Let $\delta < (2^\kappa)^+$, and let $\delta = S \cup E$ be a partition into two unbounded sets. Let $\bar{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ be a finite support iteration with FS limit P_δ, of length $\delta < (2^\kappa)^+$ with FS limit P_δ, where:

- *For all $\alpha \in S$ the forcing Q_α is forced to have universe $\lambda_\alpha < \kappa$ ("small" forcing);*
- *For all $\alpha \in E$ the forcing Q_α is forced to be of the form $Q_\alpha = \mathbb{E} \cap V^{P'_\alpha}$ for an appropriate (see below) $P'_\alpha \lessdot P_\alpha$. of size $< \kappa_{nm}$.*

Main Goal

Such iterations will not destroy any strongly unbounded family.

Without loss of generality we will only consider the dense subset of all conditions p which have a "shadow" $s = \text{shadow}(p)$ such that

- s is a finite partial function with $\text{dom}(s) = \text{dom}(p)$.
- For all $\alpha \in \text{supp}(p) \cap S$: $s(\alpha) \in \lambda_\alpha$ and $p{\restriction}\alpha \Vdash p(\alpha) = s(\alpha)$.
- For all $\alpha \in \text{supp}(p) \cap E$: $s(\alpha) \in \omega \times \omega^{<\omega}$ and $p{\restriction}\alpha \Vdash p(\alpha) \| s(\alpha)$. (i.e., $s(\alpha) \in V$ determines the stem and the width of $p(\alpha)$).

Let $\delta \leq 2^\kappa$ (or even $\delta < (2^\kappa)^+$), $\delta = S \cup E$. We consider an iteration $\bar{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ where we use small forcings $\alpha \in S$, and subposets of $\mathbb{E}$ for $\alpha \in E$.

Subgoal

Whenever $\bar{p} = (p_k : k \in \omega)$ is a sufficiently nice family of conditions, then there exists a sequence $\bar{D} = (D^\alpha : \alpha \leq \delta)$ of ultrafilter names (D^α a P_α-name) such that:

- *Some kind of $\bar{D}$-limit of $\bar{p}$ is defined, and:*
- $(\lim_{\bar{D}} \bar{p}) \Vdash$ *almost all* p_k *(with respect to D_δ) are in G_{P_δ}.* *("infinitely many" is good enough)*

To get from the subgoal to the goal is left as an exercise.
What does "sufficiently nice" mean?

Definition

A family $(p_i : i \in I)$ of conditions is called a uniform Δ-system, if:

- The supports $(\text{supp}(p_i) : i \in I)$ form a Δ-system.
- The shadows agree on the root.

Let $\delta \leq 2^\kappa$ (or even $\delta < (2^\kappa)^+$), $\delta = S \cup E$. We consider an iteration $\bar{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ where we use small forcings $\alpha \in S$, and subposets of $\mathbb{E}$ for $\alpha \in E$.

Definition

Let $\bar{P}$ be an iteration as above. A guardrail for $\bar{P}$ is a "shadow with full support", that is: a sequence $(h(\alpha) : \alpha < \delta)$ where each $h(\alpha)$ is

- if $\alpha \in S$: an ordinal $< \lambda_\alpha$, i.e., a standard name for a condition in Q_α.
- if $\alpha \in \delta \setminus S$: a pair (s, w) with $s \in \omega^{<\omega}$, $w \in \omega$.

A condition $p \in P_\delta$ is compatible with h if h extends the shadow of p.

Note:

- Shadows have finite support. ("basic neighborhood in product space")
- Guardrails have full support. ("element of product space")
- For every guardrail h, the set $P_h := \{p \in P_\delta : \text{shadow}(p) \subseteq h\}$ is centered.

Let $\delta \leq 2^\kappa$ (or even $\delta < (2^\kappa)^+$), $\delta = S \cup E$. We consider an iteration $\bar{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$ where we use small forcings $\alpha \in S$, and subposets of $\mathbb{E}$ for $\alpha \in E$.

Definition

A family $(p_i : i \in I)$ of conditions is called a uniform Δ-system, if:

- The supports $(\mathrm{supp}(p_i) : i \in I)$ form a Δ-system.
- The shadows agree on the root.

Each countable uniform Δ-system $\bar{p}$ defines a basic open set in the σ-box product topology

Lemma (Engelking-Karłowicz 1965)

Assume $\delta < (2^\kappa)^+$, $\kappa = \kappa^{\aleph_0}$. Then there is a family $(h_\epsilon : \epsilon < \kappa)$ of guardrails such that for every countable uniform Δ-system $\bar{p} = (p_k : k \in \omega)$ there is some ϵ such that h_ϵ witnesses the uniformity of $\bar{p}$.

The σ-box product is κ-separable.

Recall that we have a "dense" family $(h_\epsilon : \epsilon < \kappa)$ of guardrails. Every countable uniform Δ-system follows one of these guardrails.
Fix $\epsilon < \kappa$ and consider h_ϵ.

Subsubgoal

Whenever $\bar{p} = (p_k : k \in \omega)$ is a Δ-system of conditions, all compatible with h_ϵ, then there exists a sequence $\bar{D} = \bar{D}^\epsilon = (D^\epsilon_\alpha : \alpha \leq \delta)$ of ultrafilter names (D^ϵ_α a P_α-name) such that: $\lim_{\bar{D}^\epsilon} \bar{p}$ is defined, and:

- $(\lim_{\bar{D}^\epsilon} \bar{p}) \Vdash$ *almost all p_k (with respect to D^ϵ_δ) are in G_{P_δ}.*

It is easy to achieve this goal. The support of $q := \lim \bar{p}$ will be the root Δ of $\bar{p}$. For $\alpha \in E \cap \Delta$ we let $q(\alpha) = \lim_{D_\alpha}(p_k(\alpha) : k \in \omega)$.
We have to make sure that $q(\alpha)$ will be in Q_α, which was defined as $\mathbb{E} \cap V^{P'_\alpha}$ for appropriate $P'_\alpha \lessdot P_\alpha$ of size $< \kappa_{nm}$.
When defining $\bar{P} = (P_\alpha, Q_\alpha : \alpha < \delta)$, we will also define a sequence $(D^\epsilon_\alpha : \alpha < \delta)$. For $\alpha \in E$, we have to ensure that $D^\epsilon_\alpha \cap V^{P'_\alpha}$ is an element of $V^{P'_\alpha}$.
Since there are only κ many ϵ, we can do this for all ϵ.

Summary

Given a strongly unbounded family $\bar{F}$ of size κ and $\kappa \leq \kappa_{nm} \leq \kappa_{ct} \leq 2^\kappa$, we can construct a finite support iteration $(P_\alpha, Q_\alpha : \alpha < \kappa_{ct})$, mixing

- arbitrary ccc forcings Q_α of size $< \kappa$
- forcings Q_α of the form $\mathbb{E} \cap V^{P'_\alpha}$ for "sufficiently closed" $P'_\alpha \lessdot P_\alpha$ of size $< \kappa_{nm}$

which will preserve the unboundedness of $\bar{F}$.