Computability theory and uncountable structures

Noah Schweber

Sets and Computations, April 17 2015

Joint with Greg Igusa, Julia Knight and Antonio Montalbán

Table of Contents

What phenomena occur across every generic extension

Global behavior tends to be independent of the generic: if $\mathbb{P}$ is reasonably homogeneous, then the theory of $V[G]$ does not depend on G.
On the other hand, individual sets in generic extensions must vary wildly:

Theorem (Solovay)

Suppose G_0, G_1 are two mutually generics. Then $V[G_0] \cap V[G_1] = V$.

Proof. Take $\nu[G_0] \in (V[G_0] \cap V[G_1]) - V$ of minimal rank.
Then $\nu[G_0] \subset V$.
If $\nu[G_0] = \mu[G_1]$, then this is forced by some $(p, q) \in \mathbb{P}^2$.
The set $\{x \in V : \exists r \leq p(r \Vdash x \in \nu)\}$ is in V, and must equal $\nu[G_0]$. $\square$

Generically presentable structures

Solovay: if a set is in every generic extension by some forcing, it exists already.

Definition

A *generically presentable structure up to* $\cong$ is a pair $(\nu, \mathbb{P})$ such that

$$\Vdash_{\mathbb{P}} \text{“}\nu \text{ is a structure with domain } \omega\text{”} \quad \text{and} \quad \Vdash_{\mathbb{P}^2} \text{“}\nu[G_0] \cong \nu[G_1]\text{”}.$$

A *copy* of $(\nu, \mathbb{P})$ is a $\mathcal{A} \in V$ with $\Vdash_{\mathbb{P}}$ “$\mathcal{A} \cong \nu$”. (Maybe $dom(\mathcal{A}) \neq \omega$.)

Recently and independently introduced by Kaplan and Shelah.

Question

If $(\nu, \mathbb{P})$ is a generically presentable structure, what hypotheses ensure that it has a copy in V?

Looking at the forcing: positive results

Looking at the forcing: positive results

Theorem (Knight, Montalbán, S.)

1. *If $\mathcal{A}$ is generically presentable by a forcing not making ω_2 countable, then $\mathcal{A}$ has a copy in V;*
2. *If $\mathcal{A}$ is generically presentable by a forcing not making ω_1 countable, then that copy is countable.*

Looking at the forcing: positive results

Theorem (Knight, Montalbán, S.)

1. *If $\mathcal{A}$ is generically presentable by a forcing not making ω_2 countable, then $\mathcal{A}$ has a copy in V;*
2. *If $\mathcal{A}$ is generically presentable by a forcing not making ω_1 countable, then that copy is countable.*

Independently proved by Kaplan and Shelah.

Proof. For (1), the age of the Morleyization of $\mathcal{A}$ lives in V; by Delhomme-Pouzet-Sagi-Sauer, Fraisse limits of ages of size $\aleph_1$ exist. For (2), Scott sentence is in $\mathcal{L}^V_{\omega_1\omega}$ since $\omega_1^V = \omega_1^{V[G]}$, and existence of countable models is absolute. □

Counterexamples to Vaught's conjecture

Corollary (Harrington)

Any counterexample to Vaught's conjecture has models of size $\aleph_1$ with Scott rank arbitrarily high below ω_2.

Independently by Larson, and by Baldwin/S. Friedman/Koerwien/Laskowski.

Proof. Given $\alpha < \omega_2$, collapse ω_1 to ω, get $B \models T$ with $sr(B) > \alpha$. If B not generically presentable, can get perfect set of such models. So B is generically presentable, hence has a copy in V since ω_2^V is still uncountable. □

Remark

Hjorth showed that counterexamples need not have models of size $\aleph_2$.

Looking at the structure: positive results

Theorem (Knight, Montalbán, S.)

If $\mathcal{A}$ is generically presentable and rigid *(no nontrivial automorphisms), then $\mathcal{A}$ has a copy in V.*

Independently by Paul Larson.

Proof uses amalgamation argument — unique way to amalgamate is even better than lots of ways to amalgamate. Given $p \in \mathbb{P}$ presenting $\mathcal{A}$, look at portion $\mathcal{A}_p$ of structure built by p; can glue these together in unique way, so inside V. $\square$

Looking at the structure: positive results

Theorem (Knight, Montalbán, S.)

If $\mathcal{A}$ is generically presentable and rigid *(no nontrivial automorphisms), then $\mathcal{A}$ has a copy in V.*

Independently by Paul Larson.

Proof uses amalgamation argument — unique way to amalgamate is even better than lots of ways to amalgamate. Given $p \in \mathbb{P}$ presenting $\mathcal{A}$, look at portion $\mathcal{A}_p$ of structure built by p; can glue these together in unique way, so inside V. $\square$

Theorem (Zapletal, unpublished)

Generically presentable trees have copies.

Kaplan-Shelah, following Zapletal: study when generically presentable linear orders, models of superstable theories, already exist.

Looking at the forcing: negative results

Looking at the forcing: negative results

Theorem (Knight, Montalbán, S.)

If forcing with $\mathbb{P}$ makes ω_2 countable, then there is a structure $\mathcal{A}$, generically presentable by $\mathbb{P}$, which has no copy in the ground model.

Looking at the forcing: negative results

Theorem (Knight, Montalbán, S.)

If forcing with $\mathbb{P}$ makes ω_2 countable, then there is a structure $\mathcal{A}$, generically presentable by $\mathbb{P}$, which has no copy in the ground model.

Independently by Kaplan-Shelah

Uses construction of Laskowski and Shelah, and later Hjorth: theory with predicate U and no atomic models if $|U| = \aleph_2$, but countable atomic model in which U is set of indiscernibles. In generic extension, we can attach $(\omega_2, <)$ to indiscernibles of this model; resulting structure has a copy after making ω_2 countable but has no copy in ground model.

Aside: generically presentable cardinalities

Definition

A *generically presentable cardinality* is a pair $(\nu, \mathbb{P})$ where ν is a $\mathbb{P}$-name and $\Vdash_{\mathbb{P}\times\mathbb{P}} \nu[G_0] \equiv \nu[G_1]$. $(\nu, \mathbb{P})$ *has no copy in V* if for no $A \in V$ do we have $\Vdash_{\mathbb{P}} A \equiv \nu$.

Question

Is it consistent with ZF that there are generically presentable cardinalities with no copies?

Note that forcing over ZF-models can add new cardinalities (Ex: Truss ????)

Question (Zapletal)

Is it consistent with ZF that there is a generically presentable cardinality $(\nu, \mathbb{P})$ with no copy in V, such that ν is a name for a set of reals?

Classical computable structure theory

We study the complexity of a structure by looking at its *copies*: for a countable structure $\mathcal{S}$, a *copy* of $\mathcal{S}$ is a structure S with domain ω isomorphic to $\mathcal{S}$.

Throughout, structures have finite signature.

If $\mathcal{A}$ is a countable structure, a *copy* of $\mathcal{A}$ is a structure which is isomorphic to $\mathcal{A}$ and has domain ω; we identify copies with the reals coding them.

Definition (Muchnik reducibility)

If $\mathcal{A}$, $\mathcal{B}$ are structures, $\mathcal{A}$ is *Muchnik reducible to* $\mathcal{B}$ if (nonuniformly) every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$; we write $\mathcal{A} \leq_w \mathcal{B}$.

Classical computable structure theory

We study the complexity of a structure by looking at its *copies*: for a countable structure $\mathcal{S}$, a *copy* of $\mathcal{S}$ is a structure S with domain ω isomorphic to $\mathcal{S}$.

Throughout, structures have finite signature.

If $\mathcal{A}$ is a countable structure, a *copy* of $\mathcal{A}$ is a structure which is isomorphic to $\mathcal{A}$ and has domain ω; we identify copies with the reals coding them.

Definition (Muchnik reducibility)

If $\mathcal{A}$, $\mathcal{B}$ are structures, $\mathcal{A}$ is *Muchnik reducible to* $\mathcal{B}$ if (nonuniformly) every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$; we write $\mathcal{A} \leq_w \mathcal{B}$.

- $\mathcal{A}$ is computably presentable $\implies$ $\mathcal{A} \leq_w \mathcal{B}$
- For $X \subseteq \mathcal{A}$ finite, the substructure generated by X is $\leq_w \mathcal{A}$
- $\mathcal{L}, \mathcal{L}_0, \mathcal{L}_1$ linear orders $\implies$ $\mathcal{L} \leq_w \mathcal{L}_0 + 1 + \mathcal{L} + 1 + \mathcal{L}_1$
- If $\mathcal{L} \prec \hat{\mathcal{L}}$ are linear orders, need not have $\mathcal{L} \leq_w \hat{\mathcal{L}}$ (Harrison order)

Uncountable computable structure theory

For $\mathcal{A}$ uncountable, $\mathcal{A}$ has no copies whatsoever, so $\leq_w$ is not useful.
There are many ways one might generalize computability structure theory to uncountable settings.
Today: want notion which agrees with $\leq_w$ on countable structures, and is generally not contingent on set-theoretic axioms.

Uncountable computable structure theory

For $\mathcal{A}$ uncountable, $\mathcal{A}$ has no copies whatsoever, so $\leq_w$ is not useful. There are many ways one might generalize computability structure theory to uncountable settings.
Today: want notion which agrees with $\leq_w$ on countable structures, and is generally not contingent on set-theoretic axioms.

We ask, "what would the complexity of $\mathcal{A}$ be **if** $\mathcal{A}$ **were countable**?"

Definition (Generic Muchnik reducibility (S.))

For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, we write $\mathcal{A} \leq_w^* \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in every generic extension of the universe in which both are countable.

Can similarly study other computability-theoretic reductions between uncountable structures

Absoluteness

Definition (Generic Muchnik reducibility (S.))

For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, $\mathcal{A} \leq_w^* \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in every generic extension in which both are countable.

Absoluteness

Definition (Generic Muchnik reducibility (S.))

For $\mathcal{A}, \mathcal{B}$ structures of arbitrary cardinality, $\mathcal{A} \leq_w^* \mathcal{B}$ if $\mathcal{A} \leq_w \mathcal{B}$ in every generic extension in which both are countable.

Theorem (Shoenfield Absoluteness)

If φ is Π^1_2 with parameters from $\mathbb{R}$, we have:

$$V[G] \models \varphi \iff V \models \varphi.$$

Corollary

- *We can replace "every generic extension" by "some generic extension" in definition of generic Muchnik reducibility.*
- *For $\mathcal{A}, \mathcal{B}$ countable, $\mathcal{A} \leq_w^* \mathcal{B} \iff \mathcal{A} \leq_w \mathcal{B}$.*

Back to generic presentability

Theorem (Knight, Montalbán, S.)

If $\mathcal{A}$ is generically presentable, and is generically Muchnik reducible to a structure $\mathcal{B} \in V$ with $|\mathcal{B}| \leq \aleph_1$, then $\mathcal{A}$ has a copy in V.

Proof.
In $V[G]$ with $\omega = |\omega_1^V| < |\omega_2^V|$, let $B \cong \mathcal{B}$ with domain ω.
Let $V[G][H]$ be further extension in which $\mathcal{A}$ has a copy.
$\exists e$ such that $V[G][H] \models$ "$\Phi_e^B \cong \mathcal{A}$".
In $V[G]$, Φ_e^B satisfies Scott sentence of $\mathcal{A}$.
Existence of countable models of $\mathcal{L}_{\omega_1\omega}$-sentences is absolute.
So $\mathcal{A}$ has a copy in $V[G]$, and hence in V.

Proposition (Knight, Montalban, S.)

Counterexample to "Shoenfield for structures" is $\leq_w^* (\omega_2, <)$.

Proof. Theory of Laskowski-Shelah has computable atomic model. □

Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

$\mathcal{C} = (\mathbb{C}; +, \times)$, $\mathcal{W} = (\omega, \mathcal{P}(\omega); Succ, \in)$, $\mathcal{R} = (\mathbb{R}; +, \times)$

Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

$\mathcal{C} = (\mathbb{C}; +, \times)$, $\mathcal{W} = (\omega, \mathcal{P}(\omega); Succ, \in)$, $\mathcal{R} = (\mathbb{R}; +, \times)$

Observation

$\mathcal{C}$ is "generically computably presentable:" $\mathcal{C}$ has a computable copy in every generic extension in which it is countable.

Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

$$\mathcal{C} = (\mathbb{C}; +, \times), \qquad \mathcal{W} = (\omega, \mathcal{P}(\omega); Succ, \in), \qquad \mathcal{R} = (\mathbb{R}; +, \times)$$

Observation

$\mathcal{C}$ is "generically computably presentable:" $\mathcal{C}$ has a computable copy in every generic extension in which it is countable.

Observation

Every countable structure is generically Muchnik reducible to $\mathcal{W}$ and to $\mathcal{R}$.

Some examples, I/III: Real and complex numbers

Consider the following uncountable structures:

$\mathcal{C} = (\mathbb{C}; +, \times)$, $\mathcal{W} = (\omega, \mathcal{P}(\omega); Succ, \in)$, $\mathcal{R} = (\mathbb{R}; +, \times)$

Observation

$\mathcal{C}$ is "generically computably presentable:" $\mathcal{C}$ has a computable copy in every generic extension in which it is countable.

Observation

Every countable structure is generically Muchnik reducible to $\mathcal{W}$ and to $\mathcal{R}$.

Theorem (Igusa, Knight)

$\mathcal{W}$ is strictly less complicated than $\mathcal{R}$: $\mathcal{W} <^*_w \mathcal{R}$.

Independently by Downey, Greenberg, J. Miller (unpublished).

Some examples, II/III: ω_1

Some examples, II/III: ω_1

Proposition

For $\mathcal{A}$ countable:

$$\mathcal{A} \leq_w^* (\omega_1, <) \iff \exists \text{ countable ordinal } \alpha \text{ with } \mathcal{A} \leq_w^* (\alpha, <).$$

Proof. Suppose $\mathcal{A} \leq_w^* (\omega_1, <)$. Let $V[G]$ be generic extension in which ω_1 is countable. Then we have

$$V[G] \models \text{“}\mathcal{A} \leq_w (\alpha, <) \text{ for some countable ordinal } \alpha.\text{”}$$

This is a Σ^1_2 sentence with a real parameter (since $\mathcal{A}$ is countable), so already true in V. □

Question

What families of countable structures are captured by some single uncountable structure?

Some examples, III/III: ω_1 and $\mathbb{R}$

Proposition (Richter)

If a real X is computable in every copy of a linear order $\mathcal{L}$, then X is computable.

Corollary

$(\omega_1, <) \not\geq_w^* \mathcal{W}$.

Proposition (Ash, Knight)

If X' computes a copy of a linear order $\mathcal{L}$, then X computes a copy of $\omega \cdot \mathcal{L}$.

Corollary

$(\omega_1, <) <_w^* \mathcal{W}$.

Versions of the reals

$$\mathcal{W} = (\omega \sqcup 2^{\omega}; Succ, \in), \quad \mathcal{B} = (\omega \sqcup \omega^{\omega}; Succ, \circ)$$

$$\mathcal{R} = (\mathbb{R}; +, \times)$$

$$\mathcal{R}^* = \ \omega_1\text{-saturated real closed field realizing all types in } V$$

$$\mathcal{R}_f = (\mathbb{R}; +, \times, f), \quad \mathcal{R}^+ = (\mathbb{R}; +)$$

Versions of the reals

$$\mathcal{W} = (\omega \sqcup 2^\omega; Succ, \in), \quad \mathcal{B} = (\omega \sqcup \omega^\omega; Succ, \circ)$$

$$\mathcal{R} = (\mathbb{R}; +, \times)$$

$$\mathcal{R}^* = \ \omega_1\text{-saturated real closed field realizing all types in } V$$

$$\mathcal{R}_f = (\mathbb{R}; +, \times, f), \quad \mathcal{R}^+ = (\mathbb{R}; +)$$

There seem to be two levels of complexity:

$$\mathcal{W} \equiv^*_w \mathcal{B} <^*_w \mathcal{R}^+ \equiv^*_w \mathcal{R} \equiv^*_w \mathcal{R}_f \quad (f \text{ analytic})$$

Simple reductions

Proposition (Igusa, Knight)

$\mathcal{R}^* \geq^*_w \mathcal{B} \geq^*_w \mathcal{W}$.

Proposition (Igusa, Knight)

$\mathcal{W} \equiv^*_w \mathcal{R}^*$.

Simple reductions

Proposition (Igusa, Knight)

$\mathcal{R}^* \geq^*_w \mathcal{B} \geq^*_w \mathcal{W}$.

Proposition (Igusa, Knight)

$\mathcal{W} \equiv^*_w \mathcal{R}^*$.

Theorem (Macintyre-Marker)

If S is a Scott set and $T \in S$ is a consistent theory, any enumeration of S computes the complete diagram of a recursively saturated model of T realizing exactly the types in S.

Proof of Prop. After collapse, $\mathcal{W}$ is still a Scott set, and each ground real — including $Th(\mathcal{R})$ — appears in $\mathcal{W}$; apply Macintyre-Marker. □

$\mathcal{R}^* <^*_w \mathcal{R}$, I/II

Definition

If K is a real closed field:

- K is *Archimedean* if every element of K is below some $q \in \mathbb{Q}$.
- The *residue field* $Res(K)$ of K be the quotient of the finite elements by the infinitesimal elements.
- A *residue field section* of K is a subfield of K isomorphic to $Res(K)$.
- $FT(K)$ ("finite transcendental") is the set of finite elements not infinitesimally close to an algebraic element.

$\mathcal{R}^* <_w^* \mathcal{R}$, I/II

Definition

If K is a real closed field:

- K is *Archimedean* if every element of K is below some $q \in \mathbb{Q}$.
- The *residue field* $Res(K)$ of K be the quotient of the finite elements by the infinitesimal elements.
- A *residue field section* of K is a subfield of K isomorphic to $Res(K)$.
- $FT(K)$ ("finite transcendental") is the set of finite elements not infinitesimally close to an algebraic element.

Lemma

If K is a real closed field with domain ω, then:

- *If $Res(K)$ has a $\Sigma_2^0(K)$ copy, then $FT(K)$ is $\Sigma_2^0(K)$ . . .*
- *. . . And so K has a residue field section which is $\Sigma_2^0(K)$.*

$\mathcal{R}^* <^*_w \mathcal{R}$, II/II

Lemma

If K is a real closed field with domain ω, then:

- *If $Res(K)$ has a $\Sigma^0_2(K)$ copy, then $FT(K)$ is $\Sigma^0_2(K)$ . . .*
- *. . . And so K has a residue field section which is $\Sigma^0_2(K)$.*

$\mathcal{R}^* <_w^* \mathcal{R}$, II/II

Lemma

If K is a real closed field with domain ω, then:

- *If $Res(K)$ has a $\Sigma^0_2(K)$ copy, then $FT(K)$ is $\Sigma^0_2(K)$ . . .*
- *. . . And so K has a residue field section which is $\Sigma^0_2(K)$.*

Theorem (Igusa, Knight)

- *(Reduction) If $Res(K) \leq^*_w K$, then $FT(K)$ is Σ^c_2-definable in K.*
- *(Undefinability) If K is a recursively saturated real closed field, the set $FT(K)$ is not Σ^c_2-definable in K.*

Since "recursively saturated" is absolute, this gives:

Theorem (Igusa, Knight.)

$\mathcal{R}^* <^*_w \mathcal{R}$.

Expansions of $\mathcal{R}$, I/II

What happens if we add expressive power to $\mathbb{R}$?

Definition

A function $f: \mathbb{R} \to \mathbb{R}$ is *trivial* if, in any $V[G]$ where $\mathbb{R}$ is countable, we have: Any copy $\mathcal{A}$ of $\mathcal{R} = (\mathbb{R}^V; +, \times)$ with domain ω computes a copy $\mathcal{B}$ of $\mathcal{R}_f = (\mathbb{R}^V; +, \times, f)$ with $\mathcal{B} \upharpoonright \{+, \times\} = \mathcal{A}$.

This is stronger than $\mathcal{R} \equiv_w^* \mathcal{R}_f$.

Expansions of $\mathcal{R}$, I/II

What happens if we add expressive power to $\mathbb{R}$?

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is *trivial* if, in any $V[G]$ where $\mathbb{R}$ is countable, we have: Any copy $\mathcal{A}$ of $\mathcal{R} = (\mathbb{R}^V; +, \times)$ with domain ω computes a copy $\mathcal{B}$ of $\mathcal{R}_f = (\mathbb{R}^V; +, \times, f)$ with $\mathcal{B} \upharpoonright \{+, \times\} = \mathcal{A}$.

This is stronger than $\mathcal{R} \equiv_w^* \mathcal{R}_f$.

Proposition

A function f is trivial iff it is piecewise algebraic.

Proof. Right-to-left is immediate. For left-to-right, build (in $V[G]$) a sufficiently generic copy of $\mathcal{R} = (\mathbb{R}; +, \times)$ by forcing with $\mathcal{R}^{<\omega}$. Can diagonalize against Φ_e unless "$f(x) = y$" determined by finitely many $\{+, \times\}$-atomic formulas. □

Interlude: *O*-minimality and bases

Definition

An ordered structure $\mathcal{A}$ is *o-minimal* if every definable subset of $\mathcal{A}$ is a union of finitely many intervals.

Theorem (Macintyre)

The structure $\mathcal{R}_{exp} = (\mathbb{R}; +, \times, exp)$ *is o-minimal*

Definition

A *basis* for $\mathcal{R}_{exp}$ is a set $U \subset \mathbb{R}$ such that

- Every real is definable over some tuple from U.
- No element of U is definable over any disjoint tuple from U.

A tuple is *independent* if it can be extended to a basis.

Expansions of $\mathcal{R}$, II/II

Definition

We let $IND_n(\mathcal{R}_{exp})$ be the set of independent n-tuples of $\mathbb{R}$.

Lemma

The sets $IND_n(\mathcal{R}_{exp})$ of independent n-tuples are $\Delta_2^{c,Th(\mathcal{R}_{exp})}$ in any copy of $\mathcal{R}$.

Proof. $\bar{a}$ is independent iff there is an assignment of open boxes around $\bar{a}$ to formulas such that the formula holds of $\bar{a}$ iff it holds in whole box. □

Theorem (Igusa, Knight, S.)

$\mathcal{R}_{exp}$ is generically Muchnik equivalent to $\mathcal{R}$.

Proof. We use lemma to get Δ_2^0-approximation to a basis for $\mathcal{R}_f$, and build the "term" model generated by this basis. □

Since $|\mathbb{R}^{\mathbb{R}}| > \mathbb{R}$, there are functions in V which add information.

Question

Is there a “reasonably definable” f which adds information?

Question

Is there a continuous function f adds information?

Conjecture (Igusa, Knight, S.)

Martin-Lof Brownian motion adds information.

Further versions of the reals

Theorem (Igusa, Knight, S.)

$\mathcal{R}_f \equiv_w^* \mathcal{R}$ for f analytic.

Uses Wilkie: $\mathcal{R}$ adjoined with analytic functions on compact intervals is o-minimal.

Theorem (Igusa, Knight, S.)

The field $(\mathbb{R}; +, \times)$ *is generically Muchnik reducible to the* group $(\mathbb{R}; +)$.

Theorem (Igusa, Knight, S.)

For $\langle a_i : i \in \omega \rangle \in \mathbb{R}^V$, the expansion $(\mathbb{R}; +, \times, a_0, a_1, ...)$ is generically Muchnik reducible to $\mathcal{R}$.

Proof. In each case, we show that the independence relation over the larger language is Σ_2^c in the smaller language.

Thanks!

- Baldwin, S.-D. Friedman, Koerwien, Laskowski. "Three red herrings around Vaught's conjecture." *submitted*, on Baldwin's webpage
- Hjorth. "A note on counterexamples to the Vaught conjecture." *Notre Dame Journal of Formal Logic 2007*
- Knight, Igusa. "Comparing different versions of the reals." *Submitted.*
- Knight, Igusa, S. *In preparation.*
- Kaplan, Shelah. "Forcing a countable structure to belong to the ground model." on arXiv
- Knight, Montalbán, S. "Computable structures in generic extensions." *submitted*, on arXiv
- Larson. "Scott processes." *On the arXiv.*
- Laskowski, Shelah. "On the existence of atomic models." *Journal of Symbolic Logic 1993*