Reverse mathematics, well-quasi-orders, and Noetherian spaces

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What is a well-quasi-order?

A quasi-order is a set Q with a reflexive, transitive, binary relation $\leq_Q$.

A quasi-order is like a partial order, except that you can have $x \leq_Q y$, $y \leq_Q x$, and $x \neq y$ all at the same time.

A quasi-order $(Q, \leq_Q)$ is a well-quasi-order (wqo) if it satisfies any of the following equivalent conditions.

- There are no infinite descending chains and no infinite antichains.
- If $(q_n)_{n \in \mathbb{N}}$ is a sequence from Q, then there are $n < m$ such that $q_n \leq_Q q_m$.
- Every linear extension is a well-order.
- For every $X \subseteq Q$, there is a finite $F \subseteq X$ such that $(\forall q \in X)(\exists r \in F)(r \leq_Q q)$.

Some famous (theorems about) well-quasi-orders

Kruskal's tree theorem: The set of finite trees ordered by homeomorphic embedding is a wqo.

Laver's theorem (confirming Fraïssé's conjecture): The set of countable linear orders ordered by embedding is a wqo.

The Robertson-Seymour theorem (i.e., the graph minor theorem): The set of finite, undirected graphs ordered by the graph minor relation is a wqo.

What is a Noetherian space?

A topological space is Noetherian if it satisfies any of the following equivalent conditions.

- Every subspace is compact.
- Every increasing sequence of open sets stabilizes: if

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$$

  then there is an N such that $(\forall n > N)(G_n = G_N)$.
- Every decreasing sequence of closed sets stabilizes: if

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$$

  then there is an N such that $(\forall n > N)(F_n = F_N)$.

Where the name comes from:
The Zariski topology on the spectrum of a Noetherian ring is Noetherian.

What do Noetherian spaces have to do with wqo's?

Let $(Q, \leq_Q)$ be a quasi-order. For $E \subseteq Q$, let

$$E{\uparrow} = \{q \in Q : (\exists p \in E)(p \leq_Q q)\}$$
$$E{\downarrow} = \{q \in Q : (\exists p \in E)(q \leq_Q p)\}.$$

We consider two topologies on Q:

- The **open sets** of the Alexandroff topology ($\mathcal{A}(Q)$) are those of the form $E{\uparrow}$ for $E \subseteq Q$.
- The **basic open sets** of the Upper topology ($\mathcal{U}(Q)$) are those of the form $Q \setminus (E{\downarrow})$ for $E \subseteq Q$ finite.

Why these two topologies?
In a topological space, say $x \preceq y$ if every open set that contains x also contains y. $\mathcal{A}(Q)$ and $\mathcal{U}(Q)$ are the finest and coarsest topologies on Q such that $\preceq$ is $\leq_Q$.

What do Noetherian spaces have to do with wqo's?

Proposition

Let Q be a quasi-order. Then Q is a wqo if and only if $\mathcal{A}(Q)$ is Noetherian.

It is more interesting to consider quasi-orders and topologies on the subsets of a quasi-order Q:

- $\mathcal{P}(Q)$ is the power set of Q. $\mathcal{P}_{\mathrm{f}}(Q)$ is the set of all finite subsets of Q.
- For $A, B \subseteq Q$:
  - $A \leq_Q^\flat B \Leftrightarrow A \subseteq B{\downarrow}$.
  - $A \leq_Q^\sharp B \Leftrightarrow B \subseteq A{\uparrow}$.
- $\mathcal{P}^\flat(Q)$ denotes $(\mathcal{P}(Q), \leq_Q^\flat)$. $\mathcal{P}_{\mathrm{f}}^\flat(Q)$ denotes $(\mathcal{P}_{\mathrm{f}}(Q), \leq_Q^\flat)$. Similarly with $\sharp$ in place of $\flat$.
- $\mathcal{P}^\flat(Q)$, $\mathcal{P}^\sharp(Q)$, $\mathcal{P}_{\mathrm{f}}^\flat(Q)$, and $\mathcal{P}_{\mathrm{f}}^\sharp(Q)$ are all quasi-orders.

Noetherian spaces as topological generalizations of wqo's

Theorem (Erdős and Rado)

If Q is a wqo, then $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ is a wqo.

However, if Q is a wqo, then $\mathcal{P}^{\flat}(Q)$, $\mathcal{P}^{\sharp}(Q)$, and $\mathcal{P}_{\mathrm{f}}^{\sharp}(Q)$ are **not** necessarily wqo's.

Nevertheless, moving from Q to $\mathcal{P}(Q)$ preserves well-foundedness in a topological sense:

Theorem (Goubault-Larrecq)

If Q is a wqo, then $\mathcal{U}(\mathcal{P}^{\flat}(Q))$, $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$, $\mathcal{U}(\mathcal{P}^{\sharp}(Q))$, and $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ are Noetherian.

The strength of Goubault-Larrecq's theorem

We analyzed the logical strength of Goubault-Larrecq's theorem. Here is our theorem:

Theorem (F H M S VdM)

The following are equivalent over RCA_0*.*

(i) ACA_0.

(ii) *If* Q *is a wqo, then* $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ *is Noetherian.*

(iii) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ *is Noetherian.*

(iv) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ *is Noetherian.*

(v) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}^{\flat}(Q))$ *is Noetherian.*

(vi) *If* Q *is a wqo, then* $\mathcal{U}(\mathcal{P}^{\sharp}(Q))$ *is Noetherian.*

Why bother?

Why bother determining the logical strength of Goubault-Larrecq's theorem?

- It contributes to the reverse mathematics of wqo theory and strengthens/generalizes a few previous results.
- It is an example of something that can be done with non-metric topologies in second-order arithmetic.

Reverse mathematics refresher: RCA_0 and ACA_0

Reverse mathematics is a program designed to answer the question *How strong is my theorem relative to some pre-specified base theory?*

The **typical situation** in reverse mathematics is:

- Consider two sentences φ and ψ in the language of second-order arithmetic (often expressing two well-known theorems).
- Does $\mathsf{RCA}_0 \vdash \varphi \to \psi$?

RCA_0 is a system that says that sets computable from existing sets exist. Formally, Δ^0_1 comprehension. Warning: induction is allowed, but only for Σ^0_1 formulas.

ACA_0 is a system that says that every arithmetical formula defines a set (plus induction for arithmetical formulas).

Intuition: ACA_0 can earn an **undergraduate degree in mathematics**.

Wqo's in second-order arithmetic

There are several characterizations of wqo. In RCA_0, we use the **no bad sequences** definition.

Definition (RCA_0)

A quasi-order Q is a wqo if for every sequence $(q_n)_{n \in \mathbb{N}}$ from Q there are $n < m$ such that $q_n \leq_Q q_m$.

A sequence $(q_n)_{n \in \mathbb{N}}$ such that $\forall n \forall m (n < m \rightarrow q_n \not\leq_Q q_m)$ is called a bad sequence.

A quasi-order Q is a wqo if and only if there are no bad sequences.

Inequivalence of definitions

Warning!
The equivalent characterizations of wqo are **not** equivalent over RCA_0.

For example:

- Let $\mathrm{wqo}(Q)$ be the statement "Q has no bad sequences."
- Let $\mathrm{wqoAnti}(Q)$ be the statement "Q has no infinite descending chains and no infinite antichains."

Theorem (M & Simpson/Cholak, M & Solomon/F)

- $\mathsf{RCA}_0 \vdash (\forall \text{ quasi-orders } Q)(\mathrm{wqo}(Q) \to \mathrm{wqoAnti}(Q))$.
- $\mathsf{RCA}_0 \vdash \mathsf{CAC} \to (\forall \text{ quasi-orders } Q)(\mathrm{wqoAnti}(Q) \to \mathrm{wqo}(Q))$.
- $\mathsf{RCA}_0 \vdash (\forall \text{ quasi-orders } Q)(\mathrm{wqoAnti}(Q) \to \mathrm{wqo}(Q)) \to \mathsf{CAC}$.

Countable second-countable spaces in RCA_0

François Dorais developed a very nice framework for working with countable second-countable spaces in RCA_0.

Definition (RCA_0)

A base for a topology on a set X consists of a sequence $\mathcal{U} = (U_i)_{i \in I}$ of subsets of X and a function $k\colon X \times I \times I \to I$ such that

- if $x \in X$, then $x \in U_i$ for some $i \in I$;
- if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

Definition (RCA_0)

A countable second-countable space is a triple $(X, \mathcal{U}, k)$ where $\mathcal{U} = (U_i)_{i \in I}$ and $k\colon X \times I \times I \to I$ form a base for a topology on X.

Effectively open sets and effectively closed sets

Let $(X, \mathcal{U}, k)$ be a countable second-countable space, where $\mathcal{U} = (U_i)_{i \in I}$.

Every $h \colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ codes ...

- the effectively open set $G_h = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in h(n)} U_i$ and
- the effectively closed set $F_h = \bigcap_{n \in \mathbb{N}} \bigcap_{i \in h(n)} X \setminus U_i$.

RCA_0 need not prove that G_h and F_h actually exist as **sets**. So

- $x \in G_h$ abbreviates the formula $(\exists n)(\exists i \in h(n))(x \in U_i)$, and
- $x \in F_h$ abbreviates the formula $(\forall n)(\forall i \in h(n))(x \notin U_i)$.

Moreover, every $g \colon \mathbb{N} \times \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ codes ...

- a sequence of effectively open sets $(G_n)_{n \in \mathbb{N}}$, where each G_n is $G_{g(n,\cdot)} = \bigcup_{m \in \mathbb{N}} \bigcup_{i \in g(n,m)} U_i$ and
- a sequence of effectively closed sets $(F_n)_{n \in \mathbb{N}}$, where each F_n is $F_{g(n,\cdot)} = \bigcap_{m \in \mathbb{N}} \bigcap_{i \in g(n,m)} X \setminus U_i$.

Subspaces and compactness

Let $(X, \mathcal{U}, k)$ be a countable second-countable space, where $\mathcal{U} = (U_i)_{i \in I}$.

Definition (RCA_0)

If $X' \subseteq X$, then the corresponding subspace $(X', \mathcal{U}', k')$ is defined by $U_i' = U_i \cap X'$ for all $i \in I$ and $k' = k \restriction (X' \times I \times I)$.

Definition (RCA_0)

$(X, \mathcal{U}, k)$ is compact if for every $h \colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ such that $X = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in h(n)} U_i$, there is an $N \in \mathbb{N}$ such that $X = \bigcup_{n < N} \bigcup_{i \in h(n)} U_i$.

Noetherian countable second-countable spaces

The equivalent characterizations of Noetherian **are** equivalent in RCA_0.

Proposition (RCA_0; F H M S VdM)

For a countable second-countable space, the following are equivalent.

(i) *Every effectively open set is compact.*

(ii) *For every effectively open set G_h, there is an $N \in \mathbb{N}$ such that $G_h = \bigcup_{n<N} \bigcup_{i \in h(n)} U_i$.*

(iii) *Every subspace is compact.*

(iv) *For every sequence $(G_n)_{n \in \mathbb{N}}$ of effectively open sets such that $\forall n(G_n \subseteq G_{n+1})$, there is an N such that $(\forall n > N)(G_n = G_N)$.*

(v) *For every sequence $(F_n)_{n \in \mathbb{N}}$ of effectively closed sets such that $\forall n(F_n \supseteq F_{n+1})$, there is an N such that $(\forall n > N)(F_n = F_N)$.*

$\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ and $\mathcal{P}_{\mathrm{f}}^{\sharp}(Q)$ in RCA_0

If Q is a countable quasi-order, then $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ and $\mathcal{P}_{\mathrm{f}}^{\sharp}(Q)$ are also countable quasi-orders.

Thus $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ and $\mathcal{P}_{\mathrm{f}}^{\sharp}(Q)$ fit nicely into Dorais's framework, so we discuss these cases first.

Definition (RCA_0)

Let Q be a quasi-order.

- A base for the Alexandroff topology on Q is given by $\mathcal{U} = (U_q)_{q \in Q}$, where $U_q = q\uparrow$ for each $q \in Q$, and $k(q, p, r) = q$.
- A base for the upper topology on Q is given by $\mathcal{V} = (V_{\mathbf{i}})_{\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)}$, where $V_{\mathbf{i}} = Q \setminus (\mathbf{i}\downarrow)$ for each $\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)$, and $\ell(q, \mathbf{i}, \mathbf{j}) = \mathbf{i} \cup \mathbf{j}$.

$\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ in ACA_0

Proposition (RCA_0; F H M S VdM)

Let Q be a quasi-order.

- *If $\mathcal{A}(Q)$ Noetherian, then $\mathcal{U}(Q)$ Noetherian.*
- *Q is a wqo if and only if $\mathcal{A}(Q)$ is Noetherian.*

Theorem (M)

The statement "if Q is a wqo, then $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ is a wqo" is equivalent to ACA_0 over $\mathsf{RCA}_0 + \mathsf{RT}^2_2$.

(We improved this theorem by removing RT^2_2.)

So ACA_0 proves that if Q is a wqo, then $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ and $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ are Noetherian.

$\mathsf{ACA}_0 \vdash Q$ a wqo $\rightarrow \mathcal{U}(\mathcal{P}_\mathrm{f}^\sharp(Q))$ Noetherian

We need to prove the statement "if Q is a wqo, then $\mathcal{U}(\mathcal{P}_\mathrm{f}^\sharp(Q))$ is Noetherian" in ACA_0.

We prove the contrapositive. Let $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ be a sequence of effectively closed sets that does **not** stabilize. We want to build a **bad sequence** in Q.

Topology reminder!
Basic closed sets have the form $\{\mathbf{e}_0, \dots \mathbf{e}_{n-1}\}\!\downarrow^\sharp$, where $\mathbf{e}_i \in \mathcal{P}_\mathrm{f}(Q)$.

Key observation!
If $\{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}\!\downarrow^\sharp \supseteq F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ does not stabilize, then

$$F_0 \cap \mathbf{e}_i\!\downarrow^\sharp \supseteq F_1 \cap \mathbf{e}_i\!\downarrow^\sharp \supseteq F_2 \cap \mathbf{e}_i\!\downarrow^\sharp \supseteq \dots$$

does not stabilize for some $i < n$. (As $\{\mathbf{e}_0, \dots, \mathbf{e}_{n-1}\}\!\downarrow^\sharp = \bigcup_{i<n} \mathbf{e}_i\!\downarrow^\sharp$.)

Building a bad sequence

Suppose we have a finite bad sequence $(q_k)_{k<m}$ from Q and a sequence $(\mathbf{e}_k)_{k<m}$ from $\mathcal{P}_{\mathrm{f}}(Q)$ such that

- $q_k \in \mathbf{e}_k$ and
- $F'_0 \supseteq F'_1 \supseteq F'_2 \supseteq \dots$ does not stabilize, where $F'_n = F_n \cap \bigcap_{k<m} \mathbf{e}_k{\downarrow}^\sharp$.

Now ...

- There are $i < j$ and $\mathbf{p}$ such that $\mathbf{p} \in F'_i \setminus F'_j$.
- Note $(\forall k < m)(\mathbf{p} \leq^\sharp_Q \mathbf{e}_k)$.
- There is a basic closed set $E{\downarrow}^\sharp$ such that $E{\downarrow}^\sharp \supseteq F'_j$ but $\mathbf{p} \notin E{\downarrow}^\sharp$.
- By the **key observation** there is an $\mathbf{e}_m \in E$ such that $F'_0 \cap \mathbf{e}_m{\downarrow}^\sharp \supseteq F'_1 \cap \mathbf{e}_m{\downarrow}^\sharp \supseteq F'_2 \cap \mathbf{e}_m{\downarrow}^\sharp \supseteq \dots$ does not stabilize.
- Choose $q_m \in \mathbf{e}_m \setminus \mathbf{p}{\uparrow}$.
- If $q_k \leq_Q q_m$, then $q_m \in \mathbf{e}_k{\uparrow} \subseteq \mathbf{p}{\uparrow}$, a contradiction.

The reversal

The reversal is based on the construction of a recursive linear order L

- of type $\omega + \omega^*$
- such that every infinite subset of the ω^* part computes $0'$.

Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. Call an $n \in \mathbb{N}$ true if $(\forall k > n)(f(n) < f(k))$.

If T is an infinite set of true numbers, then $\mathrm{ran}(f) \leq_{\mathrm{T}} T \oplus f$.

Build L so that the ω^* part consists of exactly the true numbers.

How to build L

At the beginning of stage $n+1$, L consists of

- a potential ω part of numbers already witnessed to be false and
- a potential ω^* part of numbers that might be true.

$$\underbrace{\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet}_{\text{false \#'s}} \qquad \underbrace{\bullet \quad \bullet \quad \bullet \quad \bullet}_{\text{true \#'s}}$$

Stage $n+1$ witnesses that some of the most recently added (i.e., least in L) true numbers are actually false. Put $n+1$ immediately above these points.

$$\underbrace{\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet}_{\text{old false \#'s}} \qquad \underbrace{\bullet \quad \bullet}_{\substack{\text{new} \\ \text{false \#'s}}} \qquad \underbrace{\overset{n+1}{\bullet} \qquad \bullet \quad \bullet}_{\text{true \#'s}}$$

Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injection. We need to use

$$Q \text{ a wqo} \to \mathcal{U}(\mathcal{P}^\sharp_\mathrm{f}(Q)) \text{ Noetherian}$$

to show that $\mathrm{ran}(f)$ exists.

In fact, we use the contrapositive

$$\mathcal{U}(\mathcal{P}^\sharp_\mathrm{f}(Q)) \textbf{ not } \text{Noetherian} \to Q \textbf{ not } \text{a wqo}.$$

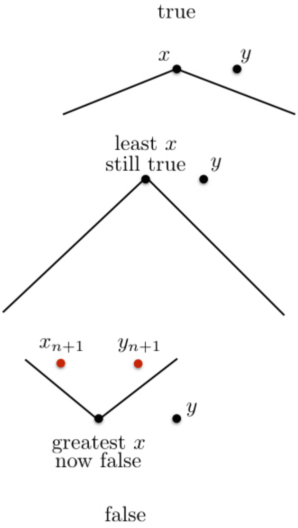
The game is to construct an f-recursive quasi-order Q such that

- there is a f-recursive non-stabilizing decreasing sequence of closed sets in $\mathcal{U}(\mathcal{P}^\sharp_\mathrm{f}(Q))$, but
- every bad sequence from $\mathcal{P}^\sharp_\mathrm{f}(Q)$ helps compute $\mathrm{ran}(f)$.

Building Q

Q consists of elements x_n and y_n for $n \in \mathbb{N}$.

- The x_n's will be ordered like L from the previous slides.
- If n is true, then almost every element will be below x_n.
- If n is false, then almost every element will be above x_n.
- The x_n's help with computing $\mathrm{ran}(f)$ from f and a bad sequence.
- The y_n's help with computing a decreasing sequence of closed sets.



Computing $\operatorname{ran}(f)$ from f and a bad sequence

Suppose $(q_i)_{i \in \mathbb{N}}$ is a bad sequence from Q.

Claim:
The number n is true if and only if $\exists i(q_i \leq_Q x_n)$.

- $(\Rightarrow)$ If n is true, then almost everything is below x_n.
- $(\Leftarrow)$ If n is false, then almost everything is above x_n. So if $q_i \leq_Q x_n$, then for every sufficiently large j $q_i \leq_Q x_n \leq_Q q_j$.

So the set of true numbers is Π^0_1 in f and Σ^0_1 in $(q_i)_{i \in \mathbb{N}}$ and f.

Thus the set of true numbers and $\operatorname{ran}(f)$ exist by Δ^0_1 comprehension.

Computing a non-stabilizing decreasing sequence

(Just some of the ideas because the pictures involved exceed my abilities in graphic design.)

We want to compute a non-stabilizing decreasing sequence

$$F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$$

of effectively closed sets in $\mathcal{U}(\mathcal{P}_\mathrm{f}^\sharp(Q))$.

We define a sequence of basic closed sets $(E_s{\downarrow}^\sharp)_{s \in \mathbb{N}}$, where each E_s is a finite subset of $\mathcal{P}_\mathrm{f}(Q)$.

Then F_s will be $\bigcap_{t \leq s} E_s{\downarrow}^\sharp$.

The E_s's

E_s contains elements of the form $\mathbf{a}_t$ and $\mathbf{b}_t$ for $t \leq s$.

For every s:

- $\mathbf{a}_s, \mathbf{b}_s \in E_s$ and
- $x_s \in \mathbf{a}_s$ and $y_s \in \mathbf{b}_s$.
- $\bigcup E_s$ is an antichain in Q.
- $\mathbf{a}_s \notin (E_s \setminus \{\mathbf{a}_s\})\downarrow^\sharp$ and $\mathbf{b}_s \notin (E_s \setminus \{\mathbf{b}_s\})\downarrow^\sharp$.

Start with $\mathbf{a}_0 = \{x_0\}$, $\mathbf{b}_0 = \{y_0\}$, and $E_0 = \{\mathbf{a}_0, \mathbf{b}_0\}$.

How to define $\mathbf{a}_{s+1}$, $\mathbf{b}_{s+1}$ and E_{s+1} depends on whether or not a number became false at stage $s+1$.

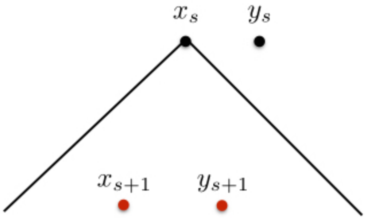
No new numbers false at stage $s+1$

If no new numbers became false at stage $s+1$ then ...

- x_{s+1} and y_{s+1} went immediately below x_s.
- Update $\mathbf{a}_s$ to $\mathbf{a}_{s+1}$ by replacing x_s with x_{s+1}.
- Update $\mathbf{b}_s$ to $\mathbf{b}_{s+1}$ by replacing y_s with y_{s+1}.
- Update E_s to E_{s+1} by replacing $\mathbf{a}_s$ with $\mathbf{a}_{s+1}$ and by adding $\mathbf{b}_{s+1}$.

The effect is that we shrank F_s by eliminating $\mathbf{a}_s$.

- $\mathbf{a}_s \in F_s$ but
- $\mathbf{a}_s \notin E_{s+1}{\downarrow}^\sharp \supseteq F_{s+1}$ because $\mathbf{a}_s{\uparrow}$ does not contain x_{s+1} or y_{s+1}.



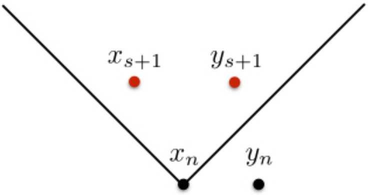
Some new numbers false at stage $s+1$

If n is the least number (x_n greatest in Q) witnessed false at stage $s+1$ then ...

- x_{s+1} and y_{s+1} went immediately above x_n.
- Let $\mathbf{a}_{s+1} = \mathbf{b}_n \cup \{x_{s+1}\}$ and let $\mathbf{b}_{s+1} = \mathbf{b}_n \cup \{y_{s+1}\}$.
- Let $E_{s+1} = (E_n \setminus \{\mathbf{a}_n, \mathbf{b}_n\}) \cup \{\mathbf{a}_{s+1}, \mathbf{b}_{s+1}\}$.

The effect is that we shrank F_s by eliminating $\mathbf{b}_n$.

- $\mathbf{b}_n \in F_s$ but
- $\mathbf{b}_s \notin E_{s+1}{\downarrow}^\sharp \supseteq F_{s+1}$ because $\mathbf{b}_n{\uparrow}$ does not contain x_{s+1} or y_{s+1}.



Summary of the $\mathcal{P}_{\mathrm{f}}(Q)$ cases

Proving the reversal for the $\flat$ case is similar. Thus

Theorem (F H M S VdM)

The following are equivalent over RCA_0.

(i) ACA_0.

(ii) *If Q is a wqo, then $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is Noetherian.*

(iii) *If Q is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is Noetherian.*

(iv) *If Q is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ is Noetherian.*

Notice that this also shows that "if Q is a wqo, then $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ is a wqo" implies ACA_0 over RCA_0 (the original reversal used RT_2^2).

(Can also eliminate RT_2^2 by showing that "if Q is a wqo, then $\mathcal{P}_{\mathrm{f}}^{\flat}(Q)$ is a wqo" implies that the product of two wqo's is a wqo.)

Setting up the uncountable case

We make the following definition to deal with uncountable second-countable spaces.

Definition (RCA_0)

A second-countable space is coded by a set $I \subseteq \mathbb{N}$ and formulas $\varphi(X)$, $\Psi_=(X, Y)$, and $\Psi_\in(X, n)$ such that the following properties hold.

- If $\varphi(X)$, then $\Psi_\in(X, i)$ for some $i \in I$.
- If $\varphi(X)$, $\Psi_\in(X, i)$, and $\Psi_\in(X, j)$ for some $i, j \in I$, then there is a $k \in I$ such that $\Psi_\in(X, k)$ and $\forall Y[\Psi_\in(Y, k) \to (\Psi_\in(Y, i) \wedge \Psi_\in(Y, j))]$.
- If $\varphi(X)$, $\varphi(Y)$, $\Psi_\in(X, i)$ for an $i \in I$, and $\Psi_=(X, Y)$, then $\Psi_\in(Y, i)$.

Setting up the uncountable case

The idea is that

- $\varphi(X)$ means that X codes a point;
- $\Psi_{\in}(X, i)$ means that the point coded by X is in the i^{th} open set;
- $\Psi_{=}(X, Y)$ means X and Y code the same point.

Important example:
The usual coding of complete separable metric spaces in RCA_0 fits in this framework.

Effectively open sets and effectively closed sets

Effectively open sets and effectively closed sets are coded as they were in the countable case.

Every $h\colon \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ codes ...

- $G_h = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in h(n)} \{X : \varphi(X) \wedge \Psi_{\in}(X, i)\}$;
- $F_h = \bigcap_{n\in\mathbb{N}} \bigcap_{i\in h(n)} \{X : \varphi(X) \wedge \neg\Psi_{\in}(X, i)\}$.

Again,

- $X \in G_h$ abbreviates $\varphi(X) \wedge (\exists n)(\exists i \in h(n))\Psi_{\in}(X, i)$, and
- $X \in F_h$ abbreviates $\varphi(X) \wedge (\forall n)(\forall i \in h(n))(\neg\Psi_{\in}(X, i))$.

Functions $g\colon \mathbb{N} \times \mathbb{N} \to \mathcal{P}_{\mathrm{f}}(I)$ code sequences of effectively open sets and sequences of effectively closed sets.

Noetherian second-countable spaces

One gives a definition of compactness as in the countable case and proves that the equivalent characterizations of Noetherian **are** equivalent in RCA_0.

Proposition (RCA_0; F H M S VdM)

For a second-countable space, the following are equivalent.

(i) *Every effectively open set is compact.*

(ii) *For every effectively open set G_h, there is an $N \in \mathbb{N}$ such that $\forall X(X \in G_h \leftrightarrow (\exists n < N)(\exists i \in h(n))\Psi_\in(X, i))$.*

(iii) *For every sequence $(G_n)_{n\in\mathbb{N}}$ of effectively open sets such that $\forall n(G_n \subseteq G_{n+1})$ there is an N such that $(\forall n > N)(G_n = G_N)$.*

(iv) *For every sequence $(F_n)_{n\in\mathbb{N}}$ of effectively closed sets such that $\forall n(F_n \supseteq F_{n+1})$ there is an N such that $(\forall n > N)(F_n = F_N)$.*

$\mathcal{U}(\mathcal{P}^\flat(Q))$ and $\mathcal{U}(\mathcal{P}^\sharp(Q))$

Let Q be a quasi-order. Let $I = \mathcal{P}_{\mathrm{f}}(Q)$.

We code $\mathcal{U}(\mathcal{P}^\flat(Q))$ by

- $\varphi(X) \Leftrightarrow X \subseteq Q$;
- $\Psi_=(X, Y) \Leftrightarrow X = Y$;
- $\Psi_\in(X, \mathbf{i}) \Leftrightarrow \mathbf{i} \subseteq X{\downarrow}$.

We code $\mathcal{U}(\mathcal{P}^\sharp(Q))$ by

- $\varphi(X) \Leftrightarrow X \subseteq Q$;
- $\Psi_=(X, Y) \Leftrightarrow X = Y$;
- $\Psi_\in(X, \mathbf{i}) \Leftrightarrow \mathbf{i} \cap X{\uparrow} = \emptyset$.

For $\mathcal{U}(\mathcal{P}^\flat(Q))$, the idea is that $\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)$ codes the complement of the basic closed set $\{Q \setminus (q{\uparrow}) : q \in \mathbf{i}\}{\downarrow}^\flat$.

For $\mathcal{U}(\mathcal{P}^\sharp(Q))$, the idea is that $\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)$ codes the complement of the basic closed set $\{\{q\} : q \in \mathbf{i}\}{\downarrow}^\sharp$.

Are these the right topologies?

The basic closed sets of $\mathcal{U}(\mathcal{P}^\flat(Q))$ are supposed to be those of the form $\{E_0, \ldots, E_{n-1}\}{\downarrow}^\flat$ for $E_0, \ldots, E_{n-1} \subseteq Q$.

Unfortunately the statement

$$(\forall \text{ quasi-orders } Q)(\forall E \subseteq Q)[\{E\}{\downarrow}^\flat \text{ is effectively closed in } \mathcal{U}(\mathcal{P}^\sharp(Q))]$$

is equivalent to ACA_0 over RCA_0.

One may interpret this as the statement "$\mathcal{U}(\mathcal{P}^\flat(Q))$ is second-countable" being equivalent to ACA_0 over RCA_0.

However, RCA_0 does prove that $\{E\}{\downarrow}^\sharp$ is effectively closed in $\mathcal{U}(\mathcal{P}^\flat(Q))$ for every $E \subseteq Q$.

The relationship between $\mathcal{P}_{\mathrm{f}}(Q)$ and $\mathcal{P}(Q)$

$\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is in general **coarser** than the subspace topology on $\mathcal{P}_{\mathrm{f}}(Q)$ induced by $\mathcal{U}(\mathcal{P}^{\flat}(Q))$.

$\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ is **the same as** the subspace topology on $\mathcal{P}_{\mathrm{f}}(Q)$ induced by $\mathcal{U}(\mathcal{P}^{\sharp}(Q))$.

Theorem (RCA_0; F H M S VdM)

Let Q be a quasi-order.

(i) *If $\mathcal{U}(\mathcal{P}^{\flat}(Q))$ is Noetherian, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ is Noetherian.*

(ii) *If $\mathcal{U}(\mathcal{P}^{\sharp}(Q))$ is Noetherian, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ is Noetherian.*

This gives the reversals for the uncountable cases.

$\mathsf{ACA}_0 \vdash Q$ a wqo $\to \mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ and $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ Noetherian

Key observations:

- If F is effectively closed in $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(Q))$ and $A \subseteq Q$, then $A \in F \leftrightarrow \mathcal{P}_{\mathrm{f}}(A) \subseteq F$.
- If F is effectively closed in $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(Q))$ and $A \subseteq Q$, then $A \in F \leftrightarrow \mathcal{P}_{\mathrm{f}}(A) \cap F \neq \emptyset$.
- So in either case, two effectively closed sets are equal if and only if they agree on $\mathcal{P}_{\mathrm{f}}(Q)$.

Thus if $F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots$ is non-stabilizing, then

$$(F_0 \cap \mathcal{P}_{\mathrm{f}}(Q)) \supseteq (F_1 \cap \mathcal{P}_{\mathrm{f}}(Q)) \supseteq (F_2 \cap \mathcal{P}_{\mathrm{f}}(Q)) \supseteq \ldots$$

is non-stabilizing.

So we can give essentially the same proofs that we gave in the countable cases.

Thank you!

Thank you for coming to my talk!
Do you have a question about it?