# Reflection Principles in terms of winning strategy of certain infinite games


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**Sets and Computations**

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# Two Classical Reflection Principles (1/3)



## Fleissner's Axiom R (AR)

(AR): For any set $X$, stationary $\mathcal{S} \subseteq [X]^{\aleph_0}$ and an $\omega_1$-club $\mathcal{U} \subseteq [X]^{\leq\aleph_1}$, there is $U \in \mathcal{U}$ s.t.

$$\mathcal{S} \cap [U]^{\aleph_0} \text{ is stationary in } [U]^{\aleph_0}.$$

- $\mathcal{U} \subseteq [X]^{\leq\aleph_1}$ is $\omega_1$-club if
  - $\mathcal{U}$ is cofinal in $[X]^{\leq\aleph_1}$ w.r.t. $\subseteq$; and
  - $\mathcal{U}$ is closed w.r.t. union of increasing chain of length $\omega_1$.

Many "mathematical" reflection principles (down to $\aleph_1$) are known to be consequences of AR. For example:

(A) A locally countably compact space $X$ is metrizable if all subspaces of $X$ of size $\leq \aleph_1$ are metrizable (Z. Balogh, 2002).

(B) A graph $G$ has countable coloring number if all subgraphs of $G$ of size $\leq \aleph_1$ have countable coloring number (W. Fleissner, 1986).

(C) A Boolean algebra $B$ is openly generated if club many subalgebras of $B$ of size $\leq \aleph_1$ are openly generated (F., 1994).

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# Two Classical Reflection Principles (2/3)

## Fleissner's Axiom R (AR)

- $\mathrm{MA}^+(\sigma\text{-closed})$ implies AR.
- AR implies the total failure of square principle.

**(A reformulation of AR as the reflection principle down to an internally cofinal model):**

For any $\lambda \geq \aleph_2$ and stationary $\mathcal{S} \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$, there is an internally cofinal $M \prec \langle \mathcal{H}(\lambda), \in, \sqsubseteq \rangle$ of size $\aleph_1$, s.t. $\mathcal{S} \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

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# Two Classical Reflection Principles (3/3)



## Rado Conjecture (RC)

**(RC):** For any tree $T$, if all $T' \in [T]^{\leq\aleph_1}$ are special, then $T$ is special.

- $T$ is special if there are $T_i \subseteq T$, $i \in \omega$ s.t. $T = \bigcup_{i\in\omega} T_i$ and each $T_i$ is pairwise incomparable (antichain).

- (Todorcevic) If $\kappa$ is strongly compact then $V^{Col(\omega_1,<\kappa)} \models$ RC.
- RC implies also the total failure of the square principle.

- (Todorcevic) RC implies $\neg MA_{\aleph_1}$.
- AR is compatible with $MA_{\aleph_1}$ (it even follows from MM).

- In spite of the fact above, AR and RC have many common consequences.

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# A principle in terms of games



- For a regular $\kappa$, let ${}^{\kappa\downarrow}\kappa$ be the set $\{f \in {}^{\kappa}\kappa : f \text{ is regressive}\}$.
- The game $G^{\downarrow}_{\omega}(\kappa)$ for Players $I$ and $II$ is defined as follows:
  A match in $G^{\downarrow}_{\omega}(\kappa)$ is a sequence of the form:

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|---|---|---|---|---|---|---|---|---|---|
| $I$ | $f_0 \in {}^{\kappa\downarrow}\kappa$ | | $f_1 \in {}^{\kappa\downarrow}\kappa$ | | $\cdots$ | $f_n \in {}^{\kappa\downarrow}\kappa$ | | $\cdots$ |
| $II$ | | $\delta_0 \in \kappa$ | | $\delta_1 \in \kappa$ | $\cdots$ | | $\delta_n \in \kappa$ | $\cdots$ |

$(n < \omega)$

$II$ wins in a match of $G^{\downarrow}_{\omega}(\kappa)$ as above if

$\{\alpha \in E^{\kappa}_{\omega_1} : f_n(\alpha) < \sup\{\delta_i : i \in \omega\} \text{ for all } n \in \omega\}$ is unbounded.

**$(G_0)$:** The Player II has a winning strategy in the game $G^{\downarrow}_{\omega}(\kappa)$ for all $\kappa > \aleph_1$.

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# A characterization of $G_0$



- In $G_\omega^\downarrow(\kappa)$, player I can take his moves so that they enumerate regressive Skolem functions. This idea leads to the equivalence of $G_0$ to the following:

  Let $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubseteq \rangle$ where $\theta$ is sufficiently large regular cardinal and $\sqsubseteq$ a well-ordering of $\mathcal{H}(\theta)$.

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**(CC$^\downarrow$):** For any $\kappa > \aleph_1$ and countable $M \prec \mathcal{M}$ with $\kappa \in M$ and $\alpha \in \kappa$, there exists a countable $M^* \prec \mathcal{M}$ s.t. $M \prec M^*$, $\alpha^* \geq \alpha$ and $\mathrm{cf}(\alpha^*) = \omega_1$ for $\alpha^* = \inf(\kappa \cap M^* \setminus \sup(\kappa \cap M))$.

**Lemma 1**

$\mathsf{G}_0$ *is equivalent to* $\mathsf{CC}^\downarrow$.

## $\mathsf{G}_0$ follows from RC and AR



**Lemma 2**

RC *implies* $\mathsf{G}_0$.

Proof. Similarly to the proof of Chang's Conjeture from RC by Todorcevic. □

($\mathsf{RP}_{\mathsf{IC}}$) For any $\kappa \geq \aleph_2$ and stationary $\mathcal{S} \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, there is an internally club $M \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq \rangle$ of cardinality $\aleph_1$, s.t. $\mathcal{S} \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

- $M$ is internally club if $[M]^{\aleph_0} \cap M$ is club in $[M]^{\aleph_0}$ w.r.t. $\subseteq$.

**Lemma 3**

$\mathsf{RP}_{\mathsf{IC}}$ *implies* $\mathsf{G}_0$.

# $G_0$ follows from RC and AR

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# Fodor-type Reflection Principle (FRP) (1/2)



**(FRP):** For any regular $\kappa \geq \aleph_2$, stationary $S \subseteq E^{\kappa}_{\omega}$ and mapping $g : S \to [\kappa]^{\leq \aleph_0}$ there is $I \in [\kappa]^{\aleph_1}$ such that

- $\mathrm{cf}(\sup I) = \omega_1$;
- $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$;
- for any regressive $f : S \cap I \to \kappa$ s.t. $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $sup(I)$.

Theorem 4 ([F.,Sakai,Torres,Usuba])

$\mathsf{G}_0$ *implies* FRP.

Corollary 5

*Rado Conjecture implies* FRP.

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# Fodor-type Reflection Principle (FRP) (2/2)



Theorem 6 ([F.,Sakai,Soukup,Usuba], [F.,Rinot])

FRP *is equivaltent over* ZFC *to most of the reflection theorems previously known to hold under Axiom R including:*

(A) A locally countably compact space $X$ is metrizable if all subspaces of $X$ of size $\leq \aleph_1$ are metrizable.

(B) A graph $G$ has countable coloring number if all subgraphs of $G$ of size $\leq \aleph_1$ have countable coloring number.

(C) A Boolean algebra $B$ is openly generated if club many subalgebras of $B$ of size $\leq \aleph_1$ are openly generated.

Corollary 7

FRP *follows from Axiom R.*

Corollary 8

*(A), (B), (C) above are all consequences of Rado Conjecture.*

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# Another principle in terms of games (1/2)



- For a cardinal $\kappa > \aleph_1$. We call a function $f : [\kappa]^{\aleph_1} \to \kappa$ regressive if $f(a) \in a$ holds for all $a \in [\kappa]^{\aleph_1}$. Let

$$ {}^{[\kappa]^{\aleph_1}}{\downarrow}\kappa = \{f \in {}^{[\kappa]^{\aleph_1}}\kappa \,:\, f \text{ is regressive}\}. $$

A match in $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$ for Players $I$ and $II$ is a sequence of the form:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $I$ | $f_0 \in {}^{[\kappa]^{\aleph_1}}{\downarrow}\kappa$ | | $f_1 \in {}^{[\kappa]^{\aleph_1}}{\downarrow}\kappa$ | | $\cdots$ | $f_n \in {}^{[\kappa]^{\aleph_1}}{\downarrow}\kappa$ | | $\cdots$ |
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$(n < \omega)$

$II$ wins in a match in $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$ as above if

$$ \{a \in [\kappa]^{\aleph_1} \,:\, f_n(a) \in \bigcup\{d_i \,:\, i \in \omega\} \text{ for all } n \in \omega\} $$

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## Another principle in terms of games (2/2)



**Theorem 9 (([F.,Usuba]))**

$G^{\downarrow\downarrow}$ *is equivalent to the following reflection principle:*

*For any* $\kappa > \aleph_1$ *and stationary* $\mathcal{S} \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, *for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ *there is a* $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq \rangle$ *s.t.* $X \subseteq Z$ $|Z| = \aleph_1$ *and* $\mathcal{S} \cap Z$ *is stationary in* $[Z]^{\aleph_0}$.

**Corollary 10**

$\mathrm{RP}_{\mathrm{IC}} \rightarrow G^{\downarrow\downarrow} \rightarrow \mathrm{AR}.$

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