Proof of SCH from reflection principles without scales

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Singular Cardinal Hypothesis

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SCH $\equiv$ $2^\lambda = \lambda^+$ for all singular strong limit cardinal λ.

Large cardinals imply SCH above them:

Theorem (Solovay 1971)

If κ is a strongly compact cardinal, then SCH holds above κ.

Strong forcing axioms also imply SCH:

Theorem (1 Foreman-Magidor-Shelah 1988, 2 Viale 2006)

1. Martin's Maximum (MM) implies SCH.
2. Proper Forcing Axiom (PFA) implies SCH.

Reflection principles

Weak Reflection Principle

For a cardinal $\mu \geq \omega_2$,

$\mathrm{WRP}(\mu) \equiv$ for any stationary $X \subseteq [\mu]^\omega$ there is $R \in [\mu]^{\omega_1}$ such that $\omega_1 \subseteq R$ and $X \cap [R]^\omega$ is stationary in $[R]^\omega$.

$\mathrm{WRP} \equiv \mathrm{WRP}(\mu)$ holds for all cardinals $\mu \geq \omega_2$.

- WRP follows from MM. (Foreman-Magidor-Shelah)
- WRP has interesting consequences. For example:
  - $2^\omega \leq \omega_2$ (Todorčević)
  - Chang's Conjecture (Foreman-Magidor-Shelah)
  - NS_{ω_1} is presaturated. (Feng-Magidor)

Fodor-type Reflection Principle (Fuchino, Soukup, Usuba et al. 2010)

For a regular cardinal $\mu \geq \omega_2$,

FRP(μ) $\equiv$ for any stat. $S \subseteq \mu \cap \mathrm{Cof}(\omega)$ and any $\langle b_\alpha \mid \alpha \in S \rangle$ with $b_\alpha \in [\alpha]^\omega$ there is $\gamma \in \mu \cap \mathrm{Cof}(\omega_1)$ such that for any function g on $S \cap \gamma$ with $g(\alpha) \in b_\alpha$ there is β with $g^{-1}[\{\beta\}]$ stationary in γ.

FRP $\equiv$ FRP(μ) holds for all regular cardinals $\mu \geq \omega_2$.

- FRP follows from MM. (Fuchino, Soukup, Usuba et al.)
- FRP has many equivalent reflection principles in terms of topology, graph theory, etc. For example:
  - For every locally countably compact space X, if X is non-metrizable, then there is a non-metrizable subspace of X of size ω_1.
  - For every infinite graph G, if the coloring number of G is uncountable, then there is $G' \subseteq G$ of size ω_1 whose coloring number is uncountable.

  (Fuchino, Soukup, S., Usuba et al.)

Reflection principles and SCH

Theorem (1 Shelah 2004, 2 Fuchino-Rinot 2011)

1. WRP implies SCH.
2. FRP implies SCH.

Their proofs use the following fact:

Fact (Shelah)

Assume SCH fails. Then there is a singular cardinal of cofinality ω at which a better scale exists.

In fact the following are proved in the proofs of the above theorem:

Theorem (1 Shelah 2004, 2 Fuchino-Rinot 2011)

If λ is a singular cardinal of cofinality ω at which a better scale exists, then

1. $\mathrm{WRP}(\lambda^+)$ fails,
2. $\mathrm{FRP}(\lambda^+)$ fails.

The fact about better scales is a deep theorem in PCF theory.
It is quite useful, but its proof is long and complicated...

Question

Can we deduce SCH from reflection principles directly without using better scales?

Another motivation: TP, ITP and SCH

Weiß introduced a tree property $\mathrm{TP}(\kappa, \lambda)$ and an ineffability property $\mathrm{ITP}(\kappa, \lambda)$ on $\mathcal{P}_\kappa(\lambda)$, which characterize the strongly compactness and the supercompactness for inaccessible κ:

- κ is strongly compact iff κ is inaccessible, and $\mathrm{TP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$.
- κ is supercompact iff κ is inaccessible, and $\mathrm{ITP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$.

$\mathrm{TP}(\kappa)$ and $\mathrm{ITP}(\kappa)$ denote that $\mathrm{TP}(\kappa, \lambda)$ and $\mathrm{ITP}(\kappa, \lambda)$ hold for all $\lambda \geq \kappa$, resp.

The following indicates that ω_2 is similar as a supercompact cardinal under PFA:

Theorem (Weiß 2010)

PFA implies $\mathrm{ITP}(\omega_2)$.

The similar holds for WRP:

Theorem (S.-Veličković 2011)

$\mathrm{WRP} + \mathrm{MA}_{\omega_1}$ implies $\mathrm{ITP}(\omega_2)$.

Open Problem

Does $\mathrm{ITP}(\omega_2)$ (or $\mathrm{TP}(\omega_2)$) imply SCH?

To prove that $\mathrm{ITP}(\omega_2)$ implies SCH, we cannot use better scales:

Theorem (Magidor)

PFA is consistent with the existence of better scales at all singular cardinals. (Hence so is $\mathrm{ITP}(\omega_2)$.)

Thus if we could prove that $\mathrm{ITP}(\omega_2)$ implies SCH, then we would have a proof of SCH from $\mathrm{WRP} + \mathrm{MA}_{\omega_1}$ without using better scales.

Question

Can we deduce SCH from WRP without using better scales?

Answer

Answer

Yes, both for WRP and FRP.

The fact about better scales can be replaced with some simple arguments. Moreover, as for WRP, the rest of the proof becomes simpler.

Proof of SCH from FRP without scales

We prove $\neg$SCH $\Rightarrow \neg$FRP.

FRP(μ) $\equiv$ for any stat. $S \subseteq \mu \cap \mathrm{Cof}(\omega)$ and any $\langle b_\alpha \mid \alpha \in S\rangle$ with $b_\alpha \in [\alpha]^\omega$ there is $\gamma \in \mu \cap \mathrm{Cof}(\omega_1)$ such that for any function g on $S \cap \gamma$ with $g(\alpha) \in b_\alpha$ there is β with $g^{-1}[\{\beta\}]$ stationary in γ.

Lemma 1

Let λ be a cardinal. If there is a sequence $\langle b_\alpha \mid \alpha < \lambda^+\rangle$ in $[\lambda]^\omega$ such that

($\star$) for any $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$ there are a club $c \subseteq \gamma$ and $\langle e_\alpha \mid \alpha \in c\rangle$ such that $e_\alpha \subseteq b_\alpha$ is finite and $\langle b_\alpha \setminus e_\alpha \mid \alpha \in c\rangle$ is pairwise disjoint,

then FRP(λ^+) fails.

Proof:
We claim that if $\langle b_\alpha \mid \alpha < \lambda^+\rangle$ satisfies ($\star$), then its restriction to $S = (\lambda^+ \cap \mathrm{Cof}(\omega)) \setminus \lambda$ witnesses $\neg$FRP(λ^+).

Suppose $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$. Let g be a function on c such that $g(\alpha) \in b_\alpha \setminus e_\alpha$. Then g is 1-1 on c. □

Assume $\neg$SCH. Let λ be the least singular strong limit cardinal with $2^\lambda > \lambda^+$. We show there is $\langle b_\alpha \mid \alpha < \lambda^+ \rangle$ satisfying $(\star)$.

Note that $\mathrm{cof}(\lambda) = \omega$ by Silver's Theorem. So $\lambda^\omega = 2^\lambda > \lambda^+$.

Lemma 2

For any $\mathcal{A} \subseteq [\lambda]^{<\lambda}$ of size $\leq \lambda^+$ there is $b \in [\lambda]^\omega$ s.t. $b \cap A$ is finite for any $A \in \mathcal{A}$.

Proof:

- Take a bijection $f : [\lambda]^{<\omega} \to \lambda$. W.m.a. each $A \in \mathcal{A}$ is closed under f^{-1}.
- We can take $b' \in [\lambda]^\omega$ s.t. $b' \not\subseteq A$ for any $A \in \mathcal{A}$.
  (Otherwise, $\lambda^\omega = |[\lambda]^\omega| = |\bigcup_{A \in \mathcal{A}} [A]^\omega| \leq \lambda^+$.)
- Let $b' = \{\beta'_n \mid n < \omega\}$, and for each n let $\beta_n := f(\{\beta'_m \mid m < n\})$.
  Note that if $A \in \mathcal{A}$, and $\beta'_m \notin A$, then $\beta_n \notin A$ for all $n > m$.
  So $b = \{\beta_n \mid n < \omega\}$ is as desired. $\square$

We show that there is $\langle b_\alpha \mid \alpha < \lambda^+ \rangle$ satisfying $(\star)$.

$(\star)$ For any $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$ there are a club $c \subseteq \gamma$ and $\langle e_\alpha \mid \alpha \in c \rangle$ such that $e_\alpha \subseteq b_\alpha$ is finite and $\langle b_\alpha \setminus e_\alpha \mid \alpha \in c \rangle$ is pairwise disjoint.

- For each $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$, fix a club $c_\gamma \subseteq \gamma$ of order-type ω_1.
- By induction on $\alpha < \lambda^+$ take b_α.
  Suppose $\alpha < \lambda^+$ and b_β has been taken for each $\beta < \alpha$.
  - For each $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$ let $A^\alpha_\gamma := \bigcup \{ b_\beta \mid \beta \in c_\gamma \cap \alpha \}$.
  - By Lemma 2 let b_α be s.t. $b_\alpha \cap A^\alpha_\gamma$ is finite for every $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$.
- $\langle b_\alpha \mid \alpha < \lambda^+ \rangle$ is as desired:

  Suppose $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$. For each $\alpha \in c_\gamma$ let $e_\alpha := b_\alpha \cap A^\alpha_\gamma$.
  Then e_α is finite, and $\langle b_\alpha \setminus e_\alpha \mid \alpha \in c_\gamma \rangle$ is pairwise disjoint. □

Proof of SCH from WRP without scales

Assume $\neg$SCH. Let λ be the least singular strong limit cardinal with $2^\lambda > \lambda^+$.
Note that $\mathrm{cof}(\lambda) = \omega$.

We claim that WRP(λ^+) fails.
In this talk we only construct a stationary $X \subseteq [\lambda^+]^\omega$ such that
$X \cap [R]^\omega$ is non-stationary for any $R \in [\lambda^+]^{\omega_1}$ with $\omega_1 \subseteq R$ and $\mathrm{cof}(\sup(R)) = \omega_1$.

This X can be shrunken (without using scales) so that
$X \cap [R]^\omega$ is non-stationary for any $R \in [\lambda^+]^\omega$ with $\omega_1 \subseteq R$,
which witnesses $\neg$WRP(λ^+).

Lemma 3

For any $\mathcal{A} \subseteq [\lambda^+]^{<\lambda}$ of size $\leq \lambda^+$ there is $b \in [\lambda^+]^\omega$ s.t. $b \cap A$ is finite for any $A \in \mathcal{A}$.

Proof: The same as Lemma 2. □

Lemma 4

For any $\mathcal{A} \subseteq [\lambda^+]^{<\lambda}$ of size $\leq \lambda^+$ and any partition $\langle I_\alpha \mid \alpha < \lambda^+ \rangle$ of λ^+, there is $b \in [\lambda^+]^\omega$ such that for any $A \in \mathcal{A}$ we have $I_\alpha \cap A = \emptyset$ for all but finitely many $\alpha \in b$.

Proof:

- By increasing each $A \in \mathcal{A}$ if necessary, we may assume that for each $A \in \mathcal{A}$ if $I_\alpha \cap A \neq \emptyset$, then $\alpha \in A$.
- By Lemma 3 take $b \in [\lambda^+]^\omega$ s.t. $b \cap A$ is finite for all $A \in \mathcal{A}$. Then b is as desired. □

Lemma 5

For any $\mathcal{A} \subseteq [\lambda^+]^{<\lambda}$ of size $\leq \lambda^+$ there are stationary many $x \in [\lambda^+]^\omega$ such that $x \cap A$ is bounded in $\sup(x)$ for any $A \in \mathcal{A}$.

This can be proved using Lemma 4 and a game introduced by Veličković.
For a function $F : [\lambda^+]^{<\omega} \to \lambda^+$ let $G(F)$ be the following game of length ω:

I	J_0	J_1	$\cdots$	J_n	$\cdots$
II	β_0	β_1	$\cdots$	β_n	$\cdots$

- At the n-th stage, I chooses a bounded interval $J_n \subseteq \lambda^+$ with $\beta_{n-1} \leq \min J_n$, and then II chooses $\beta_n < \lambda^+$.
- I wins iff $\mathrm{cl}_F(\{\min J_n \mid n < \omega\}) \subseteq \bigcup_{n\in\omega} J_n$.

Fact (Veličković)

For any $F : [\lambda^+]^{<\omega} \to \lambda^+$, I has a winning strategy for $G(F)$.

Lemma 5

For any $\mathcal{A} \subseteq [\lambda^+]^{<\lambda}$ of size $\leq \lambda^+$ there are stationary many $x \in [\lambda^+]^\omega$ such that $x \cap A$ is bounded in $\sup(x)$ for any $A \in \mathcal{A}$.

Proof:

- Take an arbitrary $F : [\lambda^+]^{<\omega} \to \lambda^+$. It suffices to find $x \in [\lambda^+]^\omega$ which is closed under F and s.t. $x \cap A$ is bounded in $\sup(x)$ for any $A \in \mathcal{A}$.
- Let τ be a winning strategy of I for $G(F)$.
- Let C be the set of all ordinals $< \lambda^+$ closed under τ. Then C is club. Let $\langle \beta_\alpha \mid \alpha < \lambda^+ \rangle$ be the increasing enumeration of $C \cup \{0\}$. Let $I_\alpha := [\beta_\alpha, \beta_{\alpha+1})$. Then $\langle I_\alpha \mid \alpha < \lambda^+ \rangle$ is a partition of λ^+.
- By Lemma 4 take an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ such that for any $A \in \mathcal{A}$ we have $I_{\alpha_n} \cap A = \emptyset$ for all but finitely many n.
- Let $J_n := \tau(\langle \beta_{\alpha_m} \mid m < n \rangle)$. Then $J_0 \subseteq I_0$, and $J_n \subseteq I_{\alpha_{n-1}}$ because $\beta_{\alpha_{n-1}+1}$ is closed under τ. So for any $A \in \mathcal{A}$ we have $J_n \cap A = \emptyset$ for all but finitely many n.
- Let $x := \mathrm{cl}_F(\{\min J_n \mid n < \omega\})$. Then x is as desired: $x \cap A$ is bounded in $\sup(x)$ for any $A \in \mathcal{A}$ because $x \subseteq \bigcup_{n<\omega} J_n$. □

Now we construct a stationary $X \subseteq [\lambda^+]^\omega$ such that $X \cap [R]^\omega$ is non-stationary for any $R \in [\lambda^+]^{\omega_1}$ with $\omega_1 \subseteq R$ and $\mathrm{cof}(\sup(R)) = \omega_1$:

- For each $\gamma \in \lambda^+ \cap \mathrm{Cof}(\omega_1)$ take a partition $\langle A^\gamma_n \mid n < \omega \rangle$ of γ such that $|A^\gamma_n| < \lambda$.
- Let X be the set of all $x \in [\lambda^+]^\omega$ such that $x \cap A^\gamma_n$ is bounded in $\sup(x)$ for all γ and n. Then X is stationary by Lemma 5.
- We claim that X is non-reflecting:
  - Suppose $R \in [\lambda^+]^{\omega_1}$ and $\mathrm{cof}(\sup(R)) = \omega_1$. Let $\gamma := \sup(R)$.
  - There is n with $R \cap A^\gamma_n$ is unbounded in γ.
    Then there are club many $y \in [R]^\omega$ s.t. $y \cap A^\gamma_n$ is unbounded in $\sup(y)$.
    So $X \cap [R]^\omega$ is non-stationary.

□

Semi-stationary reflection

Let W be a set $\supseteq \omega_1$. $X \subseteq [W]^\omega$ is *semi-stationary* if the set

$$\{y \in [W]^\omega \mid \exists x \in X\, [\, y \supseteq x \;\&\; y \cap \omega_1 = x \cap \omega_1 \,]\}$$

is stationary in $[W]^\omega$.

Semi-Stationary Reflection

For a cardinal $\mu \geq \omega_2$,

$\mathrm{SSR}(\mu) \equiv$ for any semi-stationary $X \subseteq [\mu]^\omega$ there is $R \in [\mu]^{\omega_1}$ such that $\omega_1 \subseteq R$ and $X \cap [R]^\omega$ is semi-stationary in $[R]^\omega$.

$\mathrm{SSR} \equiv \mathrm{SSR}(\mu)$ holds for all cardinals $\mu \geq \omega_2$.

- SSR follows from WRP.
- SSR is equivalent to each of the following:
  - Every ω_1-stationary set preserving poset is semi-proper. (Shelah)
  - CC^{**} (Doebler-Schindler)

Theorem (S.-Veličković)

SSR implies SCH.

Original proof uses better scales. But this can be also proved without scales.