Generalized degree structures and large cardinals

Xianghui Shi
Beijing Normal University

IMS, Singapore
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1. Generalizing Turing degree

2. Higher Degree Theory
   - In L, $L[\mu]$
   - In $L[\bar{\mu}]$ and beyond
   - Degree structures under I_0
   - Remarks

Turing degrees

$(\mathscr{D}, \leq_T)$ is the primary subject of classical recursion theory.

- Given $A, B \subseteq \omega$, $A \leq_T B$ if there is a Turing machine that correctly computes the membership of A with oracle B.

  An equivalent set theoretical definition is that A is Δ_1 definable in the structure $(H(\omega), \in, B)$.

- Write $A \equiv_T B$ if $A \leq_T B$ and $B \leq_T A$. This is an equivalence relation, and the equivalent classes are the Turing degrees. Write $\underset{\sim}{A} =$ the degree of A, $\mathscr{D} = \{\underset{\sim}{A} \mid A \subseteq \omega\}$.

- A', the Turing jump of A, is the set corresponding to the halting problem (relativized to A). It is also the Σ_1-theory of the structure $(H(\omega), \in, A)$.

Generalizations of Turing degrees

Various variations/generalizations of Turing degrees have been studied.

- Turing reduction is the simplest form of definability reduction. The notion of Turing of degree can be naturally extended to those of higher level of definability reduction, such as arithmetic degrees, hyperarithmetic degrees, higher degrees of projective hierarchies, even constructible degrees, degrees induced by inner model operators, etc.

  — Descriptive Set Theory.

- One can also extend the notion of Turing degree on subsets of ω to subsets of large ordinals.

  — α-recursion theory.

Generalized degree notion

Let Γ be a reasonable fragment (or extension) of ZFC.

Definition

Let λ be an infinite cardinal. Fix a well-ordering $w : H(\lambda) \to \lambda$. For $a, b \subset \lambda$:

- Let $M[a]$ be the minimal Γ-model of the form $L_\alpha[w][a]$, $\alpha > \lambda$.[a]
- Let α_a denote the height of $M[a]$, call it a Γ-ordinal for a.
- $a \leq_\Gamma b$ if $M[a] \subseteq M[b]$. $a \equiv_\Gamma b$ if $a \leq_\Gamma b$ and $b \leq_\Gamma a$
- Write $\underset{\sim}{a}$ for the degree of a, the $\equiv_\Gamma$-equivalence class of a. Write $(\mathscr{D}_\Gamma^\lambda, \leq_\Gamma)$ as the degree poset.
- $J_\Gamma(a)$, the Γ-jump of G, is the subset of λ coding the structure $(M[a], \in, a)$.

[a] This requires $\mathrm{Con}(\Gamma)$.

An typical example: $\lambda = \omega$ and $\Gamma =$ KP theory.

In this case, for $a \subset \omega$,

- $\underset{\sim}{a}$ is the hyperarithmetic degree of a,
- $J_{\mathrm{KP}}(a)$ is Δ_1-equivalent to the hyper-jump of a, which is the complete Σ_1-theory of $L_{\omega_1^a}[a]$. In particular, $J_{\mathrm{KP}}(\varnothing) = \mathcal{O}$.

For this talk, let $\Gamma =$ Z, i.e. ZF $-$ Replacement.[1]
We use Z-degrees to illustrate the main idea.

[1] This theory suffices for our later covering argument.

Higher Degree Theory

- α-degrees were studied only in L. Our recent work on generalized degree notions reveals some interesting connections between large cardinals and degree structures at uncountable cardinals, in particular, strong limit singular cardinals of countable cofinality.

- We shall present a new type of generalized degree structure in the core model of a certain large cardinal, via which we would like to propose a new research program, Higher Degree Theory, to reopen the study of generalized degree structures, in particular, focusing on the connection between large cardinals and the complexity of degree structures.

We investigate degree structures in canonical inner models.

- Not much of degree structures (at uncountable cardinals) can be determined by ZFC alone.
- Forcing can create all kind of "untamed" degree structures. We would like to have a theory that is robust under forcing.
- Fine structure models provide more "controlled" settings.

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- Forcing can create all kind of "untamed" degree structures. We would like to have a theory that is robust under forcing.
- Fine structure models provide more "controlled" settings.

To differentiate these degree structures, we test them with a list of degree theoretical questions.

A list of questions

1. (Post Problem). Are there incomparable degrees, i.e.
$$\neg(\underset{\sim}{a} \leq \underset{\sim}{b}) \wedge \neg(\underset{\sim}{b} \leq \underset{\sim}{a})?$$
2. (Minimal Cover). Given $\underset{\sim}{a}$, is there a $\underset{\sim}{b}$ minimal w.r.t. $\underset{\sim}{a}$, i.e.
$$\underset{\sim}{a} < \underset{\sim}{b} \wedge \neg\exists\underset{\sim}{c}(\underset{\sim}{a} < \underset{\sim}{c} < \underset{\sim}{b})?$$
3. (Posner-Robinson). Is it true for co-λ many $x \subset \lambda$ that
$$(\exists G)[\underset{\sim}{x} \oplus \underset{\sim}{G} \equiv_{\mathsf{Z}} \underset{\sim}{J_{\mathsf{Z}}(G)}]?$$
4. (Degree Determinacy). Is $\mathrm{Det}_\lambda(\mathsf{Z}\text{-}\mathrm{Deg})$ true?
Here $\mathrm{Det}_\lambda(\mathsf{Z}\text{-}\mathrm{Deg})$: *Every Z-degree invariant subset of $\mathscr{P}(\lambda)$ either contains a cone or is disjoint from a cone.*[2]

[2] A set $A \subset \mathscr{P}(\lambda)$ is Z-degree invariant if $\forall a \in A\,(\underset{\sim}{a} \subset A)$. A cone is a set of the form $C_a = \{b \mid a \leq b\}$.

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For $(\mathscr{D}, \leq_T)$, the answers to 1-3 are Yes. Degree determinacy is false in ZFC, but true under ZF + DC + AD.

[2] A set $A \subset \mathscr{P}(\lambda)$ is Z-degree invariant if $\forall a \in A\,(\underset{\sim}{a} \subset A)$. A cone is a set of the form $C_a = \{b \mid a \leq b\}$.

Three cases

- λ is regular. *Not very interesting*.

  In L-like models, such λ has the property $\lambda^{<\lambda} = \lambda$. Most recursion theoretic constructions at ω (like priority argument, recursion theoretic forcing) can be generalized to such λ.

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  In L-like models, such λ has the property $\lambda^{<\lambda} = \lambda$. Most recursion theoretic constructions at ω (like priority argument, recursion theoretic forcing) can be generalized to such λ.

- $\mathrm{cf}(\lambda) > \omega$, e.g. $\lambda = \aleph_{\omega_1}$. *Nothing interesting left.*

Theorem (Sy Friedman, 81) ($V = L$)

$\aleph_{\omega_1}$-degrees are well-ordered above any singularizing degree.[a]

[a] A degree that computes an ω_1-sequence cofinal in $\aleph_{\omega_1}$.

The key in Friedman's proof is the analysis of *stationary subsets* of $\mathrm{cf}(\lambda)$. His argument works for most reasonable definability degree notions.

Corollary (V = a fine structure extender model)

Z-degrees at singular cardinals of uncountable cofinality are well-ordered above any singularizing degree.

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This closes the case $\mathrm{cf}(\lambda) > \omega$.

- $\mathrm{cf}(\lambda) = \omega$, e.g. $\lambda = \aleph_\omega$. *Where the fun is.*

Pictures in L

Both Sy Friedman and Woodin observed independently that

Theorem

$(V = L)$ If $\mathrm{cf}(\lambda) = \omega$, then the Z-degrees at λ are well-ordered above any singularizing degree. In particular, the Z-degrees at $\aleph_\omega$ are well-ordered.

So in L, one can find only one type of degree structures at singulars of countable cofinality: well-ordered after the least singularizing degee.

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So in L, one can find only one type of degree structures at singulars of countable cofinality: well-ordered after the least singularizing degee.

A key ingredient of the argument is

Covering Lemma for L. (Jensen, 74)

Assume $\neg\exists 0^\sharp$. Then every set $x \subset$ Ord is covered by a $y \in L$, with $|y| = |x| + \omega_1$.

Proof

- Suppose $a \subset \lambda$, $a \geq_Z d$, and d singularizes λ. Then a computes a "cutoff" function.
- Work in $M[a]$. Identify subset $x \subset \lambda$ as a member of $[\lambda]^\omega$.
- $M[a]$ has no sharps, by Covering, $\exists b \in L^{M[a]} \cap \mathscr{P}(\lambda)$ s.t. $a \subset b \wedge |b| \leq \omega_1$. Then
$$\frac{a}{b} \sim \frac{z}{\omega_1}, \text{ for some } z \subset \omega_1.$$
- $M[a]$ and $L^{M[a]}$ have the same $\mathscr{P}(\omega_1)$. Thus $a \in L^{M[a]}$. In other word, $M[a] = L_{\alpha_a} \trianglelefteq L$.
- Z-degrees at λ are well-ordered above $\underset{\sim}{d}$. $\dashv$

Answers to the list:

Post Problem	No.
Minimal Cover	Yes. "No" for > 1 minimal covers.
Posner-Robinson	No.
Degree Determinacy	No.

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Remark

- For inner models between L and $L[\mu]$, the minimal inner model for one measurable cardinal, the same argument applies, as their Covering Lemmas are of the same form. (e.g. $L[0^\sharp]$)
- A little wrinkle in $L[\mu]$, due to the different form of the covering lemma for $L[\mu]$. The argument for L can be adapted to yield the same picture – well-ordered above some point – at every singular cardinal of countable cofinality.

Pictures in $L[\mu]$

Let κ be the measurable, λ strong limit and $\mathrm{cf}(\lambda) = \omega$.

Reorganize $L[\mu]$ as $L[E]$, by Steel's construction, using partial measures. The point is the acceptability condition, i.e. $\forall \gamma < \alpha$,

$$(L_{\alpha+1}[E] - L_\alpha[E]) \cap \mathscr{P}(\gamma) \neq \varnothing \quad \Rightarrow \quad L_\alpha[E] \models |\alpha| = \gamma.$$

Two cases:

- $\lambda > \kappa$. Argue as in L.
- $\lambda < \kappa$. Fix $a \subset \lambda$ above the least $L_{\alpha+1}[E]$ that singularizes λ. $M[a]$ contains no $0^\dagger$. The most $K^{M[a]}$, the core model for $M[a]$, could be is either $L[\mu']$ or there is no measurable.
  - If no measurable, then $M[a] = K^{M[a]}$, by Covering as before. By Comparison, $M[a] \trianglelefteq K = L[E]$.
  - If $K^{M[a]} = L[\mu']$, then there are two cases.

Pictures in $L[\mu]$

Covering Lemma for $L[\mu]$. (Dodd-Jensen, 82)

Assume $\neg\exists 0^\dagger$, but there is an inner model $L[\mu]$. Let $\kappa = \mathrm{crit}(\mu)$. Then for every set $x \subset \mathrm{Ord}$, one of the following holds:

1. Every set $x \subset \mathrm{Ord}$ is covered by a $y \in L[\mu]$, with $|y| = |x| + \omega_1$.
2. $\exists C$, Prikry generic over $L[\mu]$, s.t. every set $x \subset \mathrm{Ord}$ is covered by a $y \in L[\mu][C]$, with $|y| = |x| + \omega_1$. Such C is unique up to finite difference.

Case 1. $M[a] \models V = L[\mu']$, argue as in L.

Case 2. Note that $\lambda < \kappa' = \mathrm{crit}(\mu')$, and $C \subset \kappa'$ adds no new bounded subsets of κ'. It must be Case 1, as the covering set y is in $M[a]$.

By Comparison, $M[a] \trianglelefteq K = L[E]$.

Pictures in $L[\bar{\mu}]$

- A new picture starts to emerge in the canonical model for ω many measurable cardinals, $L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ and each μ_n is a measure on κ_n and $\kappa_n < \kappa_{n+1}$, $n < \omega$.

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- The case $\lambda \neq \kappa_\omega$ is argued as in $L[\mu]$, Zermelo degrees at λ is well ordered above any singularizing degree. A new degree structure appears at $\lambda = \kappa_\omega$.

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- The case $\lambda \neq \kappa_\omega$ is argued as in $L[\mu]$, Zermelo degrees at λ is well ordered above any singularizing degree. A new degree structure appears at $\lambda = \kappa_\omega$.
- The key is the Covering. The Covering for $L[\bar{\mu}]$ is similar to that of $L[\mu]$, except that C in case 2 now is a system of indiscernibles $C = \langle C_n : n < \omega \rangle$ with the properties:
  1. Each $C_n \subset \kappa_n$ is either finite or a Prikry sequence;
  2. C as a whole is a uniform system of indiscernibles, i.e.

$$(\forall \bar{x} \in L[\bar{\mu}])\,(\forall n < \omega)(x_n \in \mu_n) \Rightarrow |\bigcup\{C_n \setminus x_n \mid n < \omega\}| < \omega.$$

  Let f_C be such that $f_C(n) = |C_n|$, $n < \omega$.

Covering Lemma for $L[\bar{\mu}]$

Fix an $f : \omega \to \omega \cup \{\omega\}$ with infinite support.

Lemma (Covering Lemma for $L[\bar{\mu}]$)

Assume the sharp of $L[\bar{\mu}]$ does not exist and there is an inner model containing ω measurable cardinals. Let $L[\bar{\mu}]$ be such that $\lambda = \sup_{n<\omega} \kappa_n$ is as small as possible, where each $\kappa_n = \mathrm{crit}(\mu_n)$. Then one of the following two statements holds:

1. For every set x of ordinals there is a set $y \in L[\bar{\mu}]$ with $x \subseteq y$ and $|y| = |x| + \omega_1$.
2. There is a $(\mathbb{P}^f_{\bar{\mu}}, L[\bar{\mu}])$-generic system of indiscernibles $C \subseteq \lambda$ such that $f_C = f$ and for every set $x \subset \mathrm{Ord}$ there is a set $y \in L[\bar{\mu}, C]$ such that $x \subseteq y$ and $|y| = |x| + \omega_1$. Furthermore, the system C is unique up to finite differences.

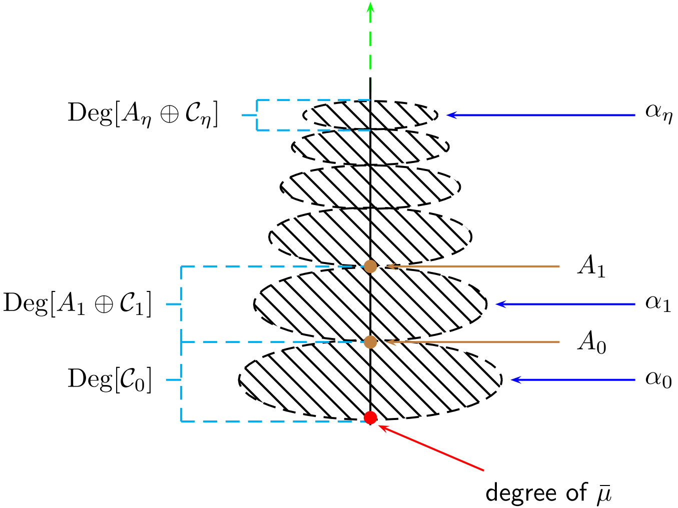
For our next theorem, only the case $\forall n\, f(n) = 1$ is needed.

Theorem

Assume $V = L[\bar{\mu}]$, where $\bar{\mu} = \langle \mu_n : n < \omega \rangle$ is a sequence of measures s.t. $\kappa_\omega = \sup \operatorname{crit}(\mu_n)$ is least possible. Suppose $\lambda > \omega$ and $\operatorname{cf}(\lambda) = \omega$.

1. If $\lambda \neq \kappa_\omega$, then the Z-degrees at λ are wellordered above any singularizing degree;
2. If $\lambda = \kappa_\omega$, consider Z-degrees at λ above $\bar{\mu}$, viewing $\bar{\mu}$ as a subset of λ. For $\eta < \lambda^+$, let β_η denote the η-th Z-ordinal (for $\varnothing$), and $B =_{\mathsf{def}} \{\beta_\eta \mid \eta < \lambda^+ \wedge \beta_\eta > \lim_{\xi<\eta} \beta_\xi\}$. Then
   1. $B = \{\alpha_a \mid a \subset \lambda\}$. Define $\underset{\sim}{a} \preccurlyeq \underset{\sim}{b} \Leftrightarrow \alpha_a \leq \alpha_b$, for $a, b \subset \lambda$. Then $\preccurlyeq$ prewellorders the Z-degrees at λ above $\bar{\mu}$.
   2. For $\eta < \lambda^+$, let $\alpha_\eta =$ the η-th member of B, let A_η be a subset of λ that codes the sequence $\langle \alpha_\xi : \xi < \eta \rangle$, and $\mathcal{C}_\eta$ be the set of all $(\mathbb{P}_{\bar{\mu}}, L_{\alpha_\eta}[\bar{\mu}])$-generic sequences. Then the Z-degrees at λ (above $\bar{\mu}$) with Z-ordinal α_η are exactly the degrees induced by

$$A_\eta \oplus \mathcal{C}_\eta = \{(A_\eta, C) \mid C \in \mathcal{C}_\eta \cup \{\varnothing\}\}.$$


$\mathrm{Deg}[A_\eta \oplus \mathcal{C}_\eta]$
α_η
A_1
$\mathrm{Deg}[A_1 \oplus \mathcal{C}_1]$
α_1
A_0
$\mathrm{Deg}[\mathcal{C}_0]$
α_0
degree of $\bar{\mu}$

Proof of 2.

- Fix an $a \subset \lambda = \kappa_\omega$, the real coding the theory of $L[\bar{\mu}]$ is not in $M(a)$, so one can apply the Covering for $L[\bar{\mu}]$ within $M[a]$.
  1. a is covered by a set $y \in (L[\bar{\mu}])^{M[a]}$ with $|y| = |a| + \aleph_1$,
  2. a is covered by a set $y \in (L[\bar{\mu}][C_a])^{M[a]}$ with $|y| = |a| + \aleph_1$, where C_a is $\mathbb{P}_{\bar{\mu}}$-generic over $L_{\alpha_a}[\bar{\mu}]$. Such C_a is "unique".

- Case 1: $M[a] = (L[\bar{\mu}])^{M[a]} = L_{\alpha_a}[\bar{\mu}]$,
  Case 2: $M[a] = (L[\bar{\mu}][C_a])^{M[a]} = L_{\alpha_a}[\bar{\mu}, C_a]$, for some C_a.

- In both cases, α_a is a β_η for some $\eta < \lambda^+$. By the minimality of $M[a]$, $\beta_\eta > \lim_{\xi<\eta} \alpha_\xi$. Thus $B \supseteq \{\alpha_a \mid a \subset \lambda\}$.

- For $\eta < \lambda^+$, as β_η is the least Zermelo ordinal above $\lim_{\xi<\eta} \beta_\xi = \lim_{\xi<\eta} \alpha_\xi$, $M[A_\eta] = L_{\beta_\eta}[\bar{\mu}]$. This proves 2-1.

- But then Case 1 gives $M[a] = M[A_\eta]$, for some η; and Case 2 gives $M[a] = M[A_\eta, C_a]$, for some η. This proves 2-2. $\dashv$

Moreover,

Theorem

Assume $V = L[\bar{\mu}]$, and $\bar{\mu}, \bar{\kappa}, \lambda$ be as before. The following are definable over the degree structure $(\mathscr{D}^{\lambda}_{\mathsf{Z}}, <_{\mathsf{Z}})$:

1. $\mathcal{I} = \{\underset{\sim}{A}_\eta \mid A_\eta \subset \lambda$ codes $\langle \alpha_i : i < \eta \rangle, \eta < \lambda^+\}$.
2. $\mathcal{R} = \{(\underset{\sim}{a}, \underset{\sim}{b}) \mid a, b \subset \lambda, \alpha_a = \alpha_b\}$

[3] In fact, one can embed $([\omega]^\omega, \subseteq^*)$. It's unknown if one can have 2^λ many.

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Assume $V = L[\bar\mu]$, and $\bar\mu, \bar\kappa, \lambda$ be as before. The following are definable over the degree structure $(\mathscr{D}^{\lambda}_{\mathsf{Z}}, <_{\mathsf{Z}})$:

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Answers to the list. (at $\lambda = \sup_n \kappa_n$)

Post Problem	Yes. $\exists\, 2^\omega$ many pairwise incomp. degrees.[3]
Minimal Cover	No.
Posner-Robinson	No.
Degree Determinacy	No.

[3] In fact, one can embed $([\omega]^\omega, \subseteq^*)$. It's unknown if one can have 2^λ many.

Two more properties

Corollary ($V = L[\bar{\mu}]$)

1. There are infinite descending chains of Z-degrees at λ.
2. There is no infinite sequence $\langle \underset{\sim}{a}_i : i < \omega \rangle$ above the degree of $\bar{\mu}$ such that $J_{\mathsf{Z}}(\underset{\sim}{a}_{i+1}) \leq_{\mathsf{Z}} \underset{\sim}{a}_i$.

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- (1) implies that $(\mathscr{D}_{\mathsf{Z}}^{\lambda}, \leq_{\mathsf{Z}})$ is illfounded.
- For $(\mathscr{D}, \leq_T)$, both types of infinite descending sequences of Turing degrees exist, i.e. (1) is true, (2) is false.
- Both are false for the aforementioned wellordered degree structures.

Picture in $L[\mathcal{U}]$

Theorem (Yang)

Assume $\bar{\kappa} = \langle \kappa_n : n < \omega \rangle$ is a sequence of measurable cardinals s.t. each κ_{n+1} carries κ_n many normal measures. Let $\lambda = \sup_n \kappa_n$. Then there is a minimal Z-degree cover for $\underset{\sim}{W}$, where $W \subset \lambda$ codes relevant information, in particular, the matrix $\mathcal{U}$ of measures.

- One can find this structure in Mitchell's model for $o(\kappa) = \kappa$.

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- One can find this structure in Mitchell's model for $o(\kappa) = \kappa$.

ANSWERS TO THE LIST:

Post Problem	Yes. $\exists\, 2^\lambda$ many pairwise incomp. degrees.
Minimal Cover	Yes. $\exists\, 2^\lambda$ many minimal covers for $\underset{\sim}{W}$.
Posner-Robinson	very likely to be "No".
Degree Determinacy	very likely to be "No".

For the descending chain questions, Yes to the first one, unknown for the second.

Picture from I_0

Definition

$I_0(\lambda)$ is the following assertion: There exists an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\mathrm{crit}(j) < \lambda$.

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Theorem

Assume ZFC $+ I_0(\lambda)$. Then

1. For almost all (co-λ many) $X \subset \lambda$,
$$(\exists G \subset \lambda\ )\,[\underset{\sim}{x} \oplus \underset{\sim}{G} \equiv_{\mathsf{Z}} \underset{\sim}{J_{\mathsf{Z}}(G)}].$$
2. Suppose in V_λ, $\kappa_0 =_{\mathsf{def}} \mathrm{crit}(j)$ is supercompact, and its supercompactness is indestructible by κ_0-directed posets.[a]. Then
$$L(V_{\lambda+1}) \models \neg\mathrm{Det}_\lambda(\mathsf{Z}\text{-}\mathrm{Deg}).$$

[a] Let us call this $I_0^*(\lambda)$. $\exists\lambda I_0^*(\lambda)$ is equiconsistent with $\exists\lambda I_0(\lambda)$

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Posner-Robinson	Yes.
Degree Determinacy	almost "No".

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Degree Determinacy	almost "No".

If replace $I_0(\lambda)$ by $I_0^*(\lambda)$, then the last one is "No".

A table

	$\mathscr{D}_T^\omega$	$(\mathscr{D}_{\mathsf{Z}}^{\aleph_\omega})^L$	$(\mathscr{D}_{\geq_{\mathsf{Z}}\bar{\mu}}^\lambda)^{L[\bar{\mu}]}$	$(\mathscr{D}_{\geq_{\mathsf{Z}}\mathcal{U}}^\lambda)^{L[\mathcal{U}]}$	$I_0^*(\lambda)$
Post	✓	×	✓	✓	✓
Min-Cov	✓	-	×	✓	✓
P–R	✓	×	×	?	✓
Deg-Det	$*$	×	×	?	×

$*$: independent of ZF.

?: unknown.

-: ✓ for one minimal cover, × for multiple minimal covers.

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	$\mathscr{D}^{\omega}_{T}$	$(\mathscr{D}^{\aleph_\omega}_{\mathsf{Z}})^L$	$(\mathscr{D}^{\lambda}_{\geq_{\mathsf{Z}}\bar{\mu}})^{L[\bar{\mu}]}$	$(\mathscr{D}^{\lambda}_{\geq_{\mathsf{Z}}\mathcal{U}})^{L[\mathcal{U}]}$	$I^*_0(\lambda)$
Post	✓	×	✓	✓	✓
Min-Cov	✓	-	×	✓	✓
P–R	✓	×	×	?	✓
Deg-Det	∗	×	×	?	×
WF	✓	×	✓	✓	?
Des-chain-2	✓	×	×	?	?

∗: independent of ZF.

?: unknown.

-: ✓ for one minimal cover, × for multiple minimal covers.

Remark

- The diversities of the types of degree structures at singular cardinals of countable cofinality reflects the strength of large cardinals carried in the model.
- Among (fine structure) inner models, the "richness" of the degree structures seems correlated to the location of λ in these inner models, rather than to the strength of a particular inner model.

An example

Example

Assume ZFC + GCH and there is a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$ plus a measurable cardinal $\kappa' > \kappa$.

By results of Woodin and Gitik, with a small forcing, one can arrange that in the generic extension

- $\kappa = \aleph_\omega$,
- GCH is true below $\aleph_\omega$,
- $2^{\aleph_\omega} = \aleph_{\omega+2}$
- κ' is measurable.

As every degree has only $\aleph_\omega$ many predecessors in the degree partial ordering, the Zermelo degree at $\aleph_\omega$ cannot be well ordered in the generic extension, in contrast to the picture in $L[\mu]$.

Thank you!