On Normal Numbers

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Émile Borel (1909): Normal Numbers

Definition

Let ξ be a real number.

- ξ is *simply normal to base b* iff in its base-b expansion, $(\xi)_b$, each digit appears with limiting frequency equal to $1/b$.
- ξ is *normal to base b* iff in $(\xi)_b$ every finite pattern of numbers occurs with limiting frequency equal to the expected value $1/b^{\ell}$, where ℓ is the pattern length.
- ξ is *absolutely normal* iff it is normal to every base b.

Normality

If the sequence of digits in $(\xi)_b$ were chosen independently at random, then the simple normality of ξ in base b would be a special case of the Law of Large Numbers.

Theorem (Borel 1909)

Almost all real numbers are absolutely normal.

Problem

Give one example of an absolutely normal number.

It is not known whether any (or all) of the familiar irrational numbers are absolutely normal: π, e, $\frac{1+\sqrt{5}}{2}$

Conjecture (Borel 1950)

Irrational algebraic numbers are absolutely normal.

Examples

First constructions of absolutely normal numbers by Lebesgue and Sierpiński, independently, 1917.

Theorem (Champernowne 1933)

$0.123456789101112131415161718192021222324\ldots$ *is normal to base ten.*

An elementary but intricate counting argument shows that Champernowne's number is normal to base 10, but it is not known whether it is absolutely normal.

A computable example

Theorem (Turing ∼1938 (see Becher, Figueira and Picchi 2007))

There is a computable absolutely normal number.

Other computable instances Schmidt 1961/1962, Becher and Figueira 2002.

Absolutely normal numbers in just above quadratic time

Theorem (Becher, Heiber and Slaman 2013)

Suppose $f : \mathbb{N} \to \mathbb{N}$ *is a computable non-decreasing unbounded function. There is an algorithm to compute an absolutely normal number* ξ *such that, for any base* b*, the algorithm outputs the first* n *digits in* $(\xi)_b$ *after* $O(f(n)\, n^2)$ *elementary operations.*

Lutz and Mayordomo (2013) and Figueira and Nies (2013) have another argument for an absolutely normal number in polynomial time, based on polynomial-time martingales.

Discrepancy

Let $\{b^n\xi\}$ denote the fractional part of $b^n\xi$.

Theorem (Wall 1949)

A real number ξ is normal to base b iff the sequence $(\{b^n\xi\} : 0 \le n < \infty)$ is uniformly distributed in $[0,1]$: for every $0 \le u < v \le 1$,

$$\lim_{N\to\infty} \frac{\#\{n : 1 \le n \le N, u \le \{b^n\xi\} < v\}}{N} = (v-u).$$

Definition

Let N be a positive integer. Let $\xi_1, \dots, \xi_N$ be real numbers in $[0,1]$. The *discrepancy* of $\xi_1, \dots, \xi_N$ is

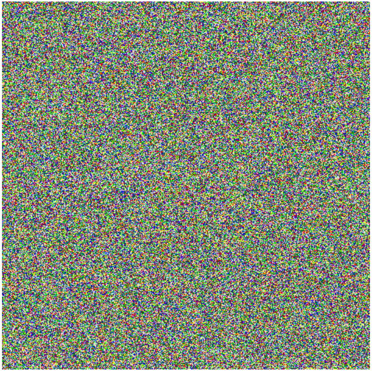
$$D(\xi_1, \dots, \xi_N) = \sup_{0\le u<v\le 1} \left| \frac{\#\{n : 1 \le n \le N, u \le \xi_n < v\}}{N} - (v-u) \right|.$$

A real number ξ is normal to base b iff $\lim_{N\to\infty} D(\{b^n\xi\} : 0 \le n \le N) = 0$.

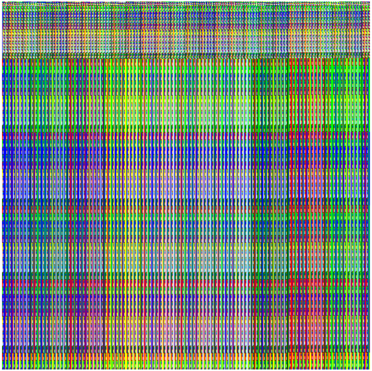
The output of the effective algorithm in base 10

Programmed by Martin Epszteyn

0.4031290542003809132371428380827059102765116777624189775110896366...



First 250000 digits output by the algorithm
Plotted in 500x500 pixels, 10 colors



First 250000 digits of Champernowne
Plotted in 500x500 pixels, 10 colors

Normality to Different Bases

There is one readily-identified connection between normality to one base and normality to another.

Definition

For natural numbers b_1 and b_2 greater than 0, we say that b_1 and b_2 are *multiplicatively dependent* if they have a common power.

Theorem (Maxfield 1953)

If b_1 and b_2 are multiplicatively dependent bases, then, for any real ξ, ξ is normal to base b_1 iff it is normal to base b_2.

Multiplicative independence

Theorem (Schmidt 1961/62)

Let R be a subset of the natural numbers greater than or equal to 2 which is closed under multiplicative dependence. There is a real ξ such that ξ is normal to every base in R and not normal to any integer base in the complement of R.

Normal numbers and Weyl's criterion

Theorem (Weyl's Criterion)

A sequence $(\xi_n : n \geq 1)$ is uniformly distributed modulo one iff for every complex-valued 1-periodic continuous function f,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\xi_n) = \int_0^1 f(x)dx.$$

That is, iff for every non-zero integer t, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i t \xi_n} = 0$

Thus, ξ is normal to base b iff for every non-zero t

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i t b^n \xi} = 0.$$

Schmidt's argument rested upon subtle estimates of such harmonic sums.

A Logician's Analysis of Independence Between Bases.

Let $\mathcal{S}$ be the set of minimal representatives of the multiplicative dependence classes.

Theorem (Becher and Slaman 2013)

Let R be a Π^0_3 subset of $\mathcal{S}$. There is a real ξ such that ξ is normal to every base in R and not normal to any of the other elements of $\mathcal{S}$. Furthermore, ξ is uniformly computable in the Π^0_3 formula which defines R.

An index set calculation:

Theorem (Becher and Slaman 2013)

The set of real numbers that are normal to at least one base is Σ^0_4-complete.

A fixed point:

Theorem (Becher and Slaman 2013)

For any Π^0_3 formula φ there is a computable real ξ such that for all b in $\mathcal{S}$, ξ is normal to base b iff $\varphi(\xi, b)$ is true.

Simple Normality

ξ is *simply normal to base b* iff each digit appears with limiting frequency equal to $1/b$ in the base-b expansion of ξ.

Necessary Conditions:

Theorem

For any base b and real number ξ, the following hold.

- *For any positive integers k and m, if ξ is simply normal to base b^{km} then ξ is simply normal to base b^m.*
- *(Long 1957) If there are infinitely many positive integers m such that ξ is simply normal to base b^m, then ξ is simply normal to all powers of b.*

Simple Normality

ξ is *simply normal to base b* iff each digit appears with limiting frequency equal to $1/b$ in the base-b expansion of ξ.

Necessary and Sufficient Conditions:

Theorem (Becher, Bugeaud and Slaman 2013)

Let M be a set of natural numbers greater than or equal to 2 such that the following necessary conditions hold.

- *For any b and positive integers k and m, if $b^{km} \in M$ then then $b^m \in M$.*
- *For any b, if there are infinitely many positive integers m such that $b^m \in M$, then all powers of b belong to M.*

There is a real number ξ such that for every base b, ξ is simply normal to base b iff $b \in M$.

Thanks to Mark Haiman for providing a needed result in combinatorial number theory.

A Comment on Hausdorff Dimension

In the above, we exhibit a Cantor-like construction of a fractal such that its uniform measure concentrates on reals of the desired simple-normality type. By inspection of the construction, there are such fractals with Hausdorff dimension arbitrarily close to one.

Irrationality Exponents

Definition (originating with Liouville 1855)

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

- When z is large and $0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$, then p/q is a good approximation to ξ when seen in the scale of $1/q$.
- The irrationality exponent of ξ is a indicator for how well ξ can be approximated by rational numbers (a linear version of Kolmogorov complexity).

Context

- Liouville numbers are those with infinite irrationality exponent.
- Almost all real numbers have irrationality exponent equal to 2.
- (Roth 1955) Irrational algebraic numbers have irrationality exponent equal to 2.

We will compute real numbers so as to control their irrationality exponents in combination with other properties.

The Jarník-Besicovitch Theorem

Theorem (Jarník 1929 and Besicovitch 1934)

For every real number a greater than or equal to 2, the set of numbers with irrationality exponent equal to a has Hausdorff dimension $2/a$.

By direct application of the definitions, the Hausdorff dimension of the set of numbers with irrationality exponent a is less than or equal to $2/a$. The other inequality comes from an early application of fractal geometry.

Jarník's Fractal

For each real number a greater than 2, Jarník gave a Cantor-like construction of a fractal J contained in $[0, 1]$ of Hausdorff dimension $2/a$ such that the uniform measure ν on J satisfies the following:

- The set of numbers with irrationality exponent greater than or equal to a has ν-measure equal to 1.
- For all b greater than a, the set of numbers with irrationality exponent greater than or equal to b has ν-measure equal to 0.

Computing Real Numbers with Specified Irrationality Exponent

Theorem (Becher, Bugeaud and Slaman)

A real number a is the irrationality exponent of a recursive real number iff a is right recursively enumerable relative to $0'$.

Further into Normality

Definition

Suppose μ is a measure on $\mathbb{R}$. The *Fourier Transform* $\hat{\mu}$ of μ is given by

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{2\pi i t\xi} d\mu(\xi)$$

Theorem (Davenport, Erdős and LeVeque 1963)

If μ is a measure on $\mathbb{R}$ such that $\hat{\mu}$ vanishes at ∞ sufficiently quickly, then μ-almost every real number is absolutely normal.

Absolutely Normal Liouville Numbers

- (Kaufman 1981) For any real number $a > 2$, there is a measure ν on the Jarník fractal for a such that the Fourier transform of ν vanishes at infinity. The measure ν is a smooth version of the uniform measure.
- (Bluhm 2000) There is a measure ν supported by the Liouville numbers such that the Fourier transform of ν vanishes at infinity.
- (Bugeaud 2002) There is an absolutely normal Liouville number.

Computing Absolutely Normal Liouville Numbers

Theorem (Becher, Heiber and Slaman 2013)

There is a computable absolutely normal Liouville number.

Computing Numbers with Specified Irrationality Exponent and Pattern of Normality

Theorem (Becher, Bugeaud and Slaman)

For every for every real number $a \geq 2$ and every set of bases M satisfying the conditions for simple normality, there is a real number ξ that has irrationality exponent a and is simply normal to exactly the bases in M.

One considers a mix between the Jarník and Simple-Normality fractals. The uniform measure on the mixed fractal is supported by real numbers of the desired type. Further, such a ξ can be computed from a and M.

Irrationality Exponents and Effective Hausdorff Dimension

Work in Progress

Definition

The effective Hausdorff dimension of a real number ξ is the limit infimum of the rational numbers r such that for infinitely many n the sequence of the first n digits in the binary expansion of ξ has Kolmogorov complexity less than or equal to $r \times n$.

If the irrationality exponent of ξ is equal to a, then ξ has effective Hausdorff dimension less than or equal to $2/a$.

Theorem (Becher, Reimann and Slaman)

For each recursive $a \geq 2$, there is a real number ξ such that the irrationality exponent of ξ is a and the effective Hausdorff dimension of ξ is $2/a$.

We conjecture that for every $a \geq 2$, every b in $[0, 2/a]$ can be the effective Hausdorff dimension of some ξ with irrationality exponent a.