Partial homogeneity of projective Fraisse limits and homogeneity of the pseudo-arc

Sławomir Solecki

University of Illinois at Urbana–Champaign

Research supported by NSF grant DMS-1266189

April 2015

Most of the work is joint with Todor Tsankov.

Outline of Topics

1. The pseudo-arc and projective Fraïssé limits
2. Projective types
3. Homogeneity for points with minimal types
4. The transfer theorem
5. Questions and extensions

The pseudo-arc and projective Fraïssé limits

The pseudo-arc

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

$\mathcal{K}([0,1]^2)$ = compact subsets of $[0,1]^2$ with the Vietoris topology

$\mathcal{K}([0,1]^2)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^2)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudo-arc**.

$\mathcal{K}([0,1]^3)$ = compact subsets of $[0,1]^3$ with the Vietoris topology

$\mathcal{K}([0,1]^3)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^3)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudo-arc**.

$\mathcal{K}([0,1]^4)$ = compact subsets of $[0,1]^4$ with the Vietoris topology

$\mathcal{K}([0,1]^4)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^4)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudo-arc**.

$\mathcal{K}([0,1]^n)$ = compact subsets of $[0,1]^n$ with the Vietoris topology, $n \geq 2$

$\mathcal{K}([0,1]^n)$ is compact, $n \geq 2$

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^n)$, $n \geq 2$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudo-arc**.

$\mathcal{K}([0,1]^\omega)$ = compact subsets of $[0,1]^\omega$ with the Vietoris topology

$\mathcal{K}([0,1]^\omega)$ is compact

$\mathcal{C}$ = all connected sets in $\mathcal{K}([0,1]^\omega)$

$\mathcal{C}$ is compact

There exists a (unique up to homeomorphism) $P \in \mathcal{C}$ such that

$$\{P' \in \mathcal{C} \colon P' \text{ homeomorphic to } P\}$$

is a dense G_δ in $\mathcal{C}$.

This P is called the **pseudo-arc**.

Continuum = compact and connected

Continuum = compact and connected

The pseudo-arc is a **hereditarily indecomposable** continuum

Continuum = compact and connected

The pseudo-arc is a **hereditarily indecomposable** continuum, that is, if $C_1, C_2 \subseteq P$ are continua with $C_1 \cap C_2 \neq \emptyset$, then $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

Continuum = compact and connected

The pseudo-arc is a **hereditarily indecomposable** continuum, that is, if $C_1, C_2 \subseteq P$ are continua with $C_1 \cap C_2 \neq \emptyset$, then $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

The pseudo-arc was discovered by **Knaster** in 1922.

Projective Fraïssé limits

$\mathcal{F}$ a family of finite structures

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ is called a **projective Fraïssé family** if it
Joint Epimorphism Property and **Projective Amalgamation Property**.

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ is called a **projective Fraïssé family** if it
Joint Epimorphism Property and **Projective Amalgamation Property**.

Irwin–S.: If $\mathcal{F}$ is a projective Fraïssé family, then there exists a unique projective limit

$$\mathbb{F} = \varprojlim \mathcal{F}$$

that is projectively universal and projectively homogeneous.

$\mathcal{F}$ a family of finite structures

Consider $\mathcal{F}$ with **epimorphisms** between structures.

$\mathcal{F}$ is called a **projective Fraïssé family** if it
Joint Epimorphism Property and **Projective Amalgamation Property**.

Irwin–S.: If $\mathcal{F}$ is a projective Fraïssé family, then there exists a unique projective limit

$$\mathbb{F} = \varprojlim \mathcal{F}$$

that is projectively universal and projectively homogeneous.

$\mathbb{F}$ also has **Projective Extension Property**.

The connection

$\mathcal{P}$ = all finite, linear, reflexive graphs with graph relation R

$\mathcal{P}$ = all finite, linear, reflexive graphs with graph relation R

Irwin–S.: $\mathcal{P}$ is a projective Fraïssé family.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

$R^{\mathbb{P}}$ is a compact equivalence relation on $\mathbb{P}$, whose equivalence classes have at most 2 elements each.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

$R^{\mathbb{P}}$ is a compact equivalence relation on $\mathbb{P}$, whose equivalence classes have at most 2 elements each.

Irwin–S.: $\mathbb{P}/R^{\mathbb{P}}$ is the pseudo-arc.

Let $\mathbb{P} = \varprojlim \mathcal{P}$ be the projective Fraïssé limit of $\mathcal{P}$ with relation $R^{\mathbb{P}}$.

$R^{\mathbb{P}}$ is a compact equivalence relation on $\mathbb{P}$, whose equivalence classes have at most 2 elements each.

Irwin–S.: $\mathbb{P}/R^{\mathbb{P}}$ is the pseudo-arc.

There is a natural continuous homomorphism

$$\mathrm{Aut}(\mathbb{P}) \to \mathrm{Homeo}(\mathbb{P}/R^{\mathbb{P}})$$

with dense range.

Bing: The pseudo-arc is homogeneous

Bing: The pseudo-arc is homogeneous, that is, for any $x, y \in P$, there exists $f \in \mathrm{Homeo}(P)$ such that $f(x) = y$.

Projective types

M is a **structure** if

M is a **structure** if

- M is a compact, 0-dimensional, second countable space,
- R^M is a closed binary relation on M,
- each continuous function $M \to X$, with X finite, factors through an epimorphism $M \to A$ for some $A \in \mathcal{P}$.

Let $f\colon M \to X$ be continuous, with X finite.

Let $f\colon M \to X$ be continuous, with X finite. So f is a projective tuple.

Let $f\colon M \to X$ be continuous, with X finite. So f is a projective tuple.
Let $p \in M$.

Let $f\colon M \to X$ be continuous, with X finite. So f is a projective tuple. Let $p \in M$.

Define

$$t_{(M,p)}(f) = \{f(K)\colon p \in K \subseteq M,\ K \text{ structure}\}.$$

Let $f\colon M \to X$ be continuous, with X finite. So f is a projective tuple. Let $p \in M$.

Define

$$t_{(M,p)}(f) = \{f(K)\colon p \in K \subseteq M,\ K \text{ structure}\}.$$

$t_{(M,p)}(f)$ is a family of subsets of the finite set X.

X finite set, $x \in X$

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by $\subseteq$ and with $\{x\} \in c$.

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by $\subseteq$ and with $\{x\} \in c$.

t is a **double chain at** x if $t = c_1 \oplus c_2 = \{I \cup J \colon I \in c_1,\ J \in c_2\}$ for some c_1, c_2 chains at x.

X finite set, $x \in X$

c is a **chain at** x if c is a maximal family of subsets of X linearly ordered by $\subseteq$ and with $\{x\} \in c$.

t is a **double chain at** x if $t = c_1 \oplus c_2 = \{I \cup J \colon I \in c_1,\ J \in c_2\}$ for some c_1, c_2 chains at x.

Lemma

Let M be a structure, $p \in M$, and $f : M \to X$ be continuous with X finite. Then $t_{(M,p)}(f)$ is a double chain at $f(p)$.

$t_{(M,p)}(f)$ is called:

$t_{(M,p)}(f)$ is called:

almost minimal if $t_{(M,p)}(f) = c_1 \oplus c_2 = c_1 \cup c_2$, for some chains c_1 and c_2 at $f(p)$;

$t_{(M,p)}(f)$ is called:

almost minimal if $t_{(M,p)}(f) = c_1 \oplus c_2 = c_1 \cup c_2$, for some chains c_1 and c_2 at $f(p)$;

minimal if $t_{(M,p)}(f)$ is a chain at $f(p)$.

Lemma (A, important)

$t_{(M,p)}(f)$ is minimal if and only if

Lemma (A, important)

$t_{(M,p)}(f)$ is minimal if and only if there are $A \in \mathcal{P}$, an epimorphism $h\colon M \to A$, and $g\colon A \to X$ such that

Lemma (A, important)

$t_{(M,p)}(f)$ is minimal if and only if there are $A \in \mathcal{P}$, an epimorphism $h\colon M \to A$, and $g\colon A \to X$ such that

(i) $f = g \circ h$;

(ii) $h(p)$ *is an endpoint of* A.

Homogeneity for points with minimal types

Lemma

Let $p \in \mathbb{P}$, $f : \mathbb{P} \to X$ continuous, X finite.

Lemma

Let $p \in \mathbb{P}$, $f : \mathbb{P} \to X$ *continuous,* X *finite.*
Then $t_{(\mathbb{P},p)}(f)$ *is almost minimal.*

$p \in \mathbb{P}$ **has minimal types** if $t_{(M,p)}(f)$ is minimal for each continuous $f : \mathbb{P} \to X$ with X finite.

$p \in \mathbb{P}$ **has minimal types** if $t_{(M,p)}(f)$ is minimal for each continuous $f : \mathbb{P} \to X$ with X finite.

Theorem (S.–Tsankov)

Let $p, q \in \mathbb{P}$. Assume that $R^{\mathbb{P}}(p) = \{p\}$ and $R^{\mathbb{P}}(q) = \{q\}$ and p and q have minimal types.

$p \in \mathbb{P}$ **has minimal types** if $t_{(M,p)}(f)$ is minimal for each continuous $f : \mathbb{P} \to X$ with X finite.

Theorem (S.–Tsankov)

Let $p, q \in \mathbb{P}$. Assume that $R^{\mathbb{P}}(p) = \{p\}$ and $R^{\mathbb{P}}(q) = \{q\}$ and p and q have minimal types. Then there exists $f \in \mathrm{Aut}(\mathbb{P})$ such that $f(p) = q$.

Proof uses the important Lemma A and the following strong Projective Extension Property.

Proof uses the important Lemma A and the following strong Projective Extension Property.

Lemma (B, important)

Given:

Proof uses the important Lemma A and the following strong Projective Extension Property.

Lemma (B, important)

Given: $p \in \mathbb{P}$ *with* $R^{\mathbb{P}}(p) = \{p\}$, $A, B \in \mathcal{P}$, $a \in A$ *an endpoint,* $b \in B$;

Proof uses the important Lemma A and the following strong Projective Extension Property.

Lemma (B, important)

Given: $p \in \mathbb{P}$ *with* $R^{\mathbb{P}}(p) = \{p\}$, $A, B \in \mathcal{P}$, $a \in A$ *an endpoint,* $b \in B$;
$f\colon \mathbb{P} \to A$ *an epimorphism,* $g\colon B \to A$ *an epimorphism,*

Proof uses the important Lemma A and the following strong Projective Extension Property.

Lemma (B, important)

Given: $p \in \mathbb{P}$ *with* $R^{\mathbb{P}}(p) = \{p\}$, $A, B \in \mathcal{P}$, $a \in A$ *an endpoint,* $b \in B$;
$f : \mathbb{P} \to A$ *an epimorphism,* $g : B \to A$ *an epimorphism,*
$f(p) = a$, $g(b) = a$.

Proof uses the important Lemma A and the following strong Projective Extension Property.

Lemma (B, important)

Given: $p \in \mathbb{P}$ *with* $R^{\mathbb{P}}(p) = \{p\}$, $A, B \in \mathcal{P}$, $a \in A$ *an endpoint,* $b \in B$;
$f\colon \mathbb{P} \to A$ *an epimorphism,* $g\colon B \to A$ *an epimorphism,*
$f(p) = a$, $g(b) = a$.
Conclusion: there exists an epimorphism $h\colon \mathbb{P} \to B$ *such that* $h(p) = b$.

The transfer theorem

Aim: transfer partial homogeneity from $\mathbb{P}$ to full homogeneity of $\mathbb{P}/R^{\mathbb{P}}$.

Theorem (S.–Tsankov)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$*, there exists* $x \in \mathbb{P}/R^{\mathbb{P}}$ *and a homeomorphism* $\phi\colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ *such that*

Theorem (S.–Tsankov)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$*, there exists* $x \in \mathbb{P}/R^{\mathbb{P}}$ *and a homeomorphism* $\phi\colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ *such that*

(i) $x = p/R^{\mathbb{P}}$ *for some* $p \in \mathbb{P}$ *having minimal types and with* $R^{\mathbb{P}}(p) = \{p\}$*;*

Theorem (S.–Tsankov)

For each $y \in \mathbb{P}/R^{\mathbb{P}}$, *there exists* $x \in \mathbb{P}/R^{\mathbb{P}}$ *and a homeomorphism* $\phi\colon \mathbb{P}/R^{\mathbb{P}} \to \mathbb{P}/R^{\mathbb{P}}$ *such that*

(i) $x = p/R^{\mathbb{P}}$ *for some* $p \in \mathbb{P}$ *having minimal types and with* $R^{\mathbb{P}}(p) = \{p\}$;

(ii) $\phi(x) = y$.

From partial homogeneity of $\mathbb{P}$ and the above transfer theorem we get the following corollary.

From partial homogeneity of $\mathbb{P}$ and the above transfer theorem we get the following corollary.

Corollary (Bing)

The pseudo-arc is homogeneous.

A notion crucial in the proof of the transfer theorem

A notion crucial in the proof of the transfer theorem

Given $A \in \mathcal{P}$, define the pre-dual $\widehat{A} \in \mathcal{P}$ of A with a bijection

$$A \ni a \to \widehat{a} \text{ an edge in } \widehat{A}.$$

A notion crucial in the proof of the transfer theorem

Given $A \in \mathcal{P}$, define the pre-dual $\widehat{A} \in \mathcal{P}$ of A with a bijection

$$A \ni a \to \widehat{a} \text{ an edge in } \widehat{A}.$$

Weak commutation for epimorphisms

$$f : \mathbb{P} \to A,\ g : \mathbb{P} \to B \text{ and } h \colon \widehat{A} \to \widehat{B}:$$

A notion crucial in the proof of the transfer theorem

Given $A \in \mathcal{P}$, define the pre-dual $\widehat{A} \in \mathcal{P}$ of A with a bijection

$$A \ni a \to \widehat{a} \text{ an edge in } \widehat{A}.$$

Weak commutation for epimorphisms

$$f : \mathbb{P} \to A,\ g : \mathbb{P} \to B \text{ and } h \colon \widehat{A} \to \widehat{B}:$$

$$h[\widehat{f(p)}] \subseteq \widehat{g(p)} \ \text{ for each } p \in \mathbb{P}.$$

Questions

What is the complexity of the orbit equivalence relation of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$?

What is the complexity of the orbit equivalence relation of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$? Can orbits be characterized by types or sequences of types?

What is the complexity of the orbit equivalence relation of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$? Can orbits be characterized by types or sequences of types?

Is there a partial homogeneity of $\mathbb{P}$ for several points?

What is the complexity of the orbit equivalence relation of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$? Can orbits be characterized by types or sequences of types?

Is there a partial homogeneity of $\mathbb{P}$ for several points?

Can one prove a maximal homogeneity of $\mathbb{P}$?

What is the complexity of the orbit equivalence relation of the natural action of $\mathrm{Aut}(\mathbb{P})$ on $\mathbb{P}$? Can orbits be characterized by types or sequences of types?

Is there a partial homogeneity of $\mathbb{P}$ for several points?

Can one prove a maximal homogeneity of $\mathbb{P}$?

Formulate a logic in which $t_{(M,p)}(f)$ become types.

Panagiotopoulos–S.: Menger compacta as projective Fraïssé limits