The bounding number and the Ramsey calculus up to the continuum

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9. Summary

Notation

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For ordinals α, β, γ, the arrow notation

$$\gamma \to (\beta, \alpha)^2$$

denotes the statement that for all $f : [\gamma]^2 \to 2$ either there is $B \subseteq \gamma$ of order type β such that $f[B]^2 = 0$ or there is $A \subseteq \gamma$ of order type α such that $f[A]^2 = 1$.

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Theorem (Sierpiński 1933, Erdös-Rado 1956)

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Corollary

For $\gamma \leq 2^{\aleph_0}$,

$$\gamma \to (\gamma, \alpha)^2 \text{ } \textit{implies} \text{ } \alpha < \omega_1.$$

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Theorem (Shelah, 2009)

$\gamma \to (\gamma, \omega + 1)^2$ *when* $\gamma > 2^{\mathrm{cf}(\gamma)}$ *and* $\mathrm{cf}(\gamma) > \omega$.

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Theorem (T., 1983)

PFA implies $\omega_1 \to (\omega_1, \alpha)^2$ *for all* $\alpha < \omega_1$.

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Corollary (T., 1983, 1999)

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$$[\omega_1]^2 = K_0 \cup K_1 \cup \cdots \cup K_n$$

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Corollary (Baumgartner-Hajnal, 1973)

$$\omega_1 \to (\alpha)^2_k \text{ for all } \alpha < \omega_1 \text{ and } k < \omega.$$

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Remark

Suppose $f : [\gamma]^2 \to 2$ is a witness to $\gamma \not\to (\gamma,\ \omega : 2)^2$. Let $\mathcal{P}$ be the poset of all **finite** $C \subseteq \gamma$ such that such that $f[C]^2 = 0$. Then $\mathcal{P}$ is a ccc poset without a subset of size γ of pairwise compatible conditions.

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Recall that

$$\mathfrak{b} = \min\{|A| : A \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } (\forall b \in \mathbb{N}^{\mathbb{N}})(\exists a \in A) a \not\leq^* b\},$$

where $\leq^*$ is the ordering of eventual dominance in $\mathbb{N}^{\mathbb{N}}$.

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Assume $\mathfrak{b} = \omega_1$ and let X be any separable metric space of cardinality $\aleph_1$. Then for each $x \in X$ we can choose a sequence $H(x) = \{h_n(x) : n < \omega\}$ converging to x such that for every uncountable $Y \subseteq X$ there exist $x \neq y$ in Y such that $x \in H(y)$.

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Remark

Defining $[\omega_1]^2 = K_0 \cup K_1$ by letting $\{x, y\} \in K_1$ iff $x \in H(y)$ or $y \in H(x)$ we get $\omega_1 \not\rightarrow (\omega_1,\ \omega : 2)^2$.

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such that

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If $\mathfrak{b} = \omega_1$ the topology of any separable metric space of cardinality $\aleph_1$ can be refined to a locally countable locally compact topology that is hereditarily separable in all finite powers.

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Corollary (T., 1989, 2006)

If $\mathfrak{b} = \omega_1$ *then there is a scattered compactum* K *whose function space* $C(K)$ *contains no uncountable bi-orthogonal systems.*

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Remark

Prior to this Kunen(1984) and Shelah(1985) were able to construct such Banach spaces using CH and $\diamondsuit$, respectively.

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$\mathfrak{b} = \omega_3$ *does not imply* $\mathfrak{b} \not\to (\mathfrak{b},\ \omega : 2)^2$.

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What properties of a given carinal $\gamma \leq 2^{\aleph_0}$ *guarantee that*

$$\gamma \to (\gamma, \omega + 2)^2 \text{ or } \gamma \to (\gamma,\ \omega : 2)^2$$

holds? What about

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for all $\alpha < \omega_1$*? What if* $\gamma = \mathfrak{b}$*?*

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Proposition (Folklore?)

If γ *carries a* $\aleph_1$*-saturated normal ideal then* $\gamma \to (\gamma,\ \omega : 2)^2$.

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such that

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This fact fails for random reals, so one can ask the following.

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Problem

Assume PFA and let $\mathcal{R}$ be any measure algebra. Does $\mathcal{R}$ force $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$?

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MA plus $2^{\aleph_0} = \aleph_3$ *do not imply* $2^{\aleph_0} \not\to (2^{\aleph_0},\ \omega : 2)^2$.

Proposition

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Proof.

Start with $2^{\aleph_0} = 2^{\aleph_2} = \aleph_3$ and with a counterexample to $2^{\aleph_0} \to (2^{\aleph_0},\ \omega : 2)^2$ and force MA in the usual way. $\square$

Forcing $\gamma \not\to (\gamma,\ \omega : 2)^2$ versus forcing $\gamma \not\to (\gamma, \omega + 2)^2$

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No ccc poset can force $\gamma \not\to (\gamma,\ \omega : 2)^2$ *for an arbitrary regular cardinal* γ.

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If γ is measurable, the normal measure on γ generates an $\aleph_1$-saturated ideal in the ccc forcing extension and, as noted above, this implies $\gamma \to (\gamma,\ \omega : 2)^2$ $\square$

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Theorem (T., 1985)

For every regular uncountable cardinal γ there is a ccc poset of cardinality γ forcing $\gamma \not\to (\gamma, \omega + 2)^2$ and adding at least γ new reals.

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Corollary

$\gamma \to (\gamma,\ \omega : 2)^2$ *does not imply* $\gamma \to (\gamma, \omega + 2)^2$

Theorem (Baumgartner 1969, 1976)

Assuming GCH for every regular cardinal γ there is a cardinal preserving poset which forces $2^{\aleph_0} = \gamma^+$ and $2^{\aleph_0} \not\to (2^{\aleph_0},\ \omega : 2)^2$.

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Question

Can we have $2^{\aleph_0} \not\rightarrow (2^{\aleph_0},\ \omega : 2)^2$ with the continuum a successor of a singular cardinal?
How about $2^{\aleph_0} = \aleph_{\omega+1}$?

$\gamma \to (\gamma, \omega + 2)^2$ and the γ-Souslin Hypothesis

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Suppose $[\gamma]^2 = K_0 \cup K_1$ is a witness to $\gamma \not\to (\gamma, \omega : 2)^2$. Let $\mathcal{P}$ be the poset of all finite $X \subseteq \gamma$ such that $[X]^2 \subseteq K_0$. Then $\mathcal{P}$ is a ccc poset which does not have γ as its pre-caliber. Recall the standard fact that every $\aleph_0$-inacessible regular cardinal is pre-caliber of every ccc poset.

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Theorem (Raghavan-T., 2014)

$\gamma \to (\gamma, \omega+2)^2$ implies γ-SH when γ is $\aleph_0$-accessible and regular.

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Define

$$[T]^2 = K_0 \cup K_1$$

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Then $[T]^2 = K_0 \cup K_1$ witnesses $\gamma \not\to (\gamma, \omega + 2)^2$.

Remark

$[T]^2 = K_0 \cup K_1$ is **not** a witnesses to $\gamma \not\to (\gamma, \omega : 2)^2$.

Corollary

$\aleph_{\omega+1} \not\to (\aleph_{\omega+1}, \omega + 2)^2$ *is consistent with GCH.*

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Recall that we have previously established this with the continuum large and in particular with $2^{\aleph_0} = \aleph_{\omega+1}$.

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Problem

Is $\aleph_{\omega+1} \to (\aleph_{\omega+1}, \omega+2)^2$ *is consistent with GCH? With* $2^{\aleph_0} = \aleph_{\omega+1}$*? How about* $\aleph_{\omega+1} \to (\aleph_{\omega+1}, \alpha)^2$ *for all* $\alpha < \omega_1$*?*

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Problem

Is $\aleph_{\omega+1} \to (\aleph_{\omega+1}, \omega : 2)^2$ *is consistent with GCH? With* $2^{\aleph_0} = \aleph_{\omega+1}$*? Same question for the negative relation* $\aleph_{\omega+1} \not\to (\aleph_{\omega+1},\ \omega : 2)^2$.

$\gamma \to (\gamma, \ \omega : 2)^2$ and the Souslin Hypothesis

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Definition

Fix a coherent Souslin tree S. *By* $\mathrm{MA}_{\aleph_1}(S)$ *(respectively,* $\mathrm{PFA}(S)$*) we denote the forcing axiom for* $\aleph_1$ *dense sets and ccc posets that preserve the tree* S *(respectively, proper posets that preserve* S*).*

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$\omega_1 \to (\omega_1,\ \omega : 2)^2$ *does not imply SH.*

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Proposition

$\omega_1 \to (\omega_1,\ \omega : 2)^2$ *does not imply SH.*

Proof.

Observe that the proof of Laver's theorem above shows that $\mathrm{MA}_{\aleph_1}(S)$ implies $\omega_1 \to (\omega_1,\ \omega : 2)^2$. $\square$

Theorem (Raghavan - T., 2014)

$\mathrm{PFA}(S)$ *implies that S forces* $\omega_1 \to (\omega_1, \alpha)^2$ *for all* $\alpha < \omega_1$.

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The **P-ideal dichotomy** *is the consequence of PFA stating that for every P-ideal* I *of countable subsets of some index-set* X, *either there is uncountable* $Y \subseteq X$ *such that all countable subsets of* Y *belong to* I *or the set* X *can be decomposed into countably many subsets that contain no infinite subsets that are in* I.

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Problem (T., 2011)

Assume PID. Is $\mathfrak{b} = \omega_2$ *equivalent to the statement that* $\omega_1 \to (\omega_1, \alpha)^2$ *for all* $\alpha < \omega_1$*?*

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Recall the following that has been mentioned above

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If γ is a real-valued measurable cardinal then $\gamma \to (\gamma, \alpha)^2$ for all $\alpha < \omega_1$

Remark

Recall that we have established above that γ simply carrying an $\aleph_1$-saturated normal ideal does not imply $\gamma \to (\gamma, \omega + 2)^2$.

Definition

Given infinite cardinals α, β, γ, recall that a poset $\mathcal{P}$ has **caliber** (γ, β, α) *if for every $\mathcal{X} \subseteq \mathcal{P}$ of cardinality γ there exists $\mathcal{Y} \subseteq \mathcal{X}$ of cardinality β such that all subsets of $\mathcal{Y}$ of cardinality α have extensions in $\mathcal{P}$.*

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Theorem (Laver, 1978)

If γ is a measurable cardinal then there is a forcing notion (that preserves cardinals if GCH holds) that adds at least γ reals and forces that γ supports a normal $(\gamma, \gamma, \aleph_0)$-saturated ideal. Moreover, $2^{\aleph_0} = \gamma$ can be arranged.

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Unlike Kunen's assumption that $\mathcal{P}(\gamma)/I$ supports a measure, Laver's assumption generalizes to higher cardinals. So we have, for example, the following.

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Theorem (Laver, 1978)

$2^{\aleph_1} \to (2^{\aleph_1}, \alpha)^2$ for all $\alpha < \omega_2$ is consistent relative the consistency of a measurable cardinal.

The σ-finite chain condition and $\mathfrak{b} \to (\mathfrak{b}, \alpha)^2$

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If γ supports a normal ideal I so that the quotient algebra $\mathcal{P}(\gamma)/I$ satisfies the σ-finite chain condition then $\gamma \to (\gamma, \alpha)^2$ for all $\alpha < \omega_1$.

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Question

What is the consistency strength of the statement that $\mathfrak{b} \to (\mathfrak{b}, \alpha)^2$ for all $\alpha < \omega_1$?

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The same result hold for any α-support product of posets of cardinalities at most β for any pair of cardinals $\alpha, \beta < \gamma$.

Weakly compact cardinals and Sacks reals

Let $\mathcal{S}_\delta$ be the standard poset for adding side-by-side δ Sacks reals.

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Question

Assume CH. What does $\mathcal{S}_{\omega_2}$ force about $\gamma = \aleph_2 = 2^{\aleph_0}$?
For example, does $\mathcal{S}_{\omega_2}$ force

$$2^{\aleph_0} \to (2^{\aleph_0}, \alpha)^2 \text{ for all } \alpha < \omega_1?$$

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$2^{\aleph_1} \to (2^{\aleph_1}, \alpha)^2$ for all $\alpha < \omega_2$ is consistent relative the consistency of a weakly compact cardinal.

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Similarly, working with Sacks subsets of ω_2, we get

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(III) While $\mathfrak{b} \not\to (\mathfrak{b},\ \omega : 2)^2$ when $\mathfrak{b} = \omega_1$ it is consistent that $\mathfrak{b} \to (\mathfrak{b}, \alpha)^2$ for all $\alpha < \omega_1$.

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(IV) There are many natural open problems whose solution might require blends of techniques from different areas of set theory. For example, the consistency of $\aleph_{\omega+1} \to (\aleph_{\omega+1}, \omega + 2)^2$ might be one such problem.