Todorcević's fragments of $MA_{\aleph_1}$ and a recent work due to Bagaria-Shelah

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Todorcevic's fragments of $MA_{\aleph_1}$

and a recent result due to Bagaria-Shelah.

§ 1st part of Bagaria-Shelah: arXive:1502.0500v1.

Theorem (Harrington-Shelah)

If $\aleph_1^L = \aleph_1$ & MA, then

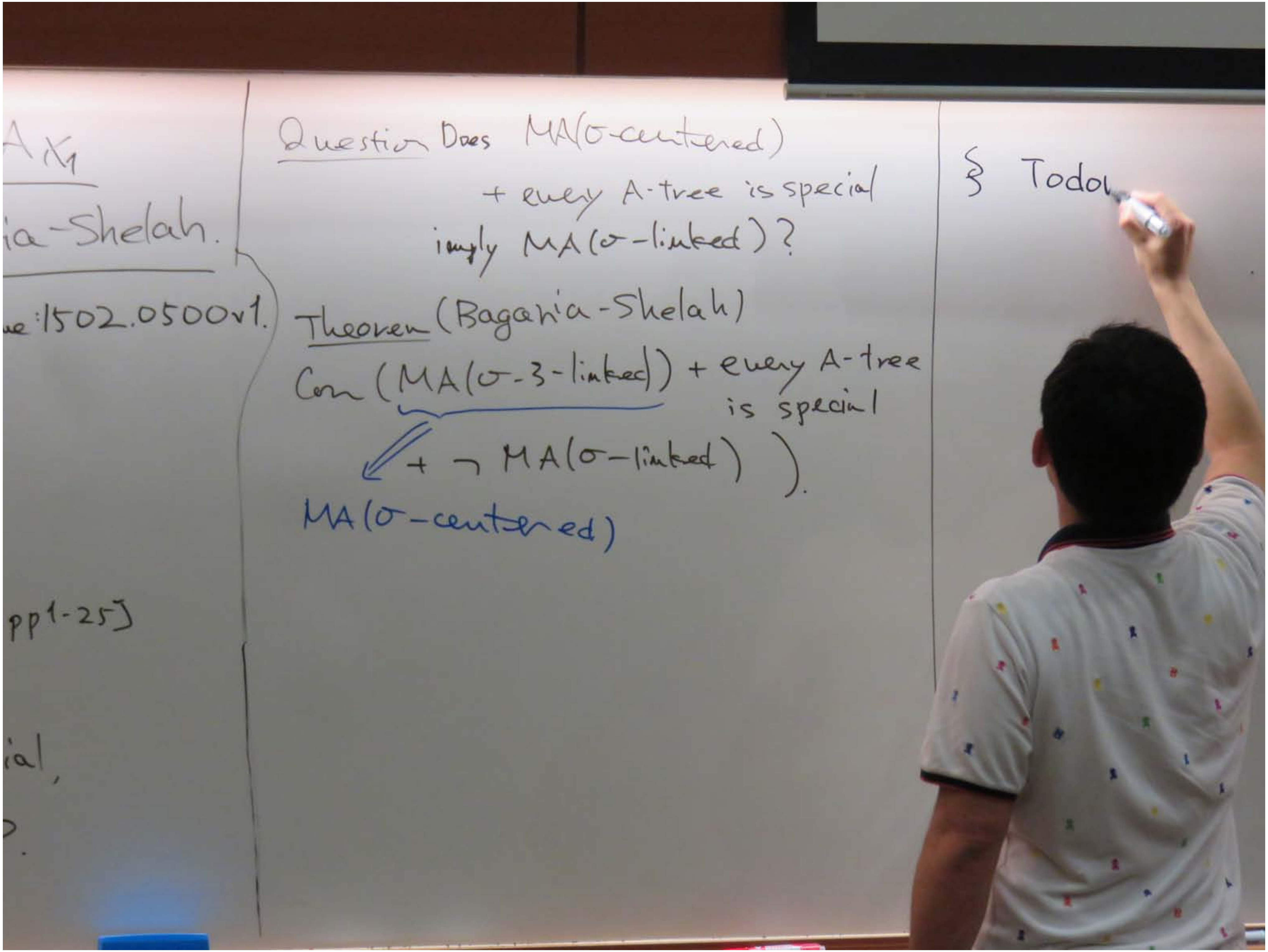
$\exists\ \Delta^1_3$-set of reals without BP.

Theorem (Bagaria)

If $\aleph_1^L = \aleph_1$ & MA(σ-centered)

& every Aronszajn tree is special,

then $\exists\ \Delta^1_3$-set of reals without BP.



Question Does MA(σ-centered)
+ every A-tree is special
imply MA(σ-linked)?
Theorem (Bagaria-Shelah)
Con (MA(σ-3-linked) + every A-tree is special
+ ¬ MA(σ-linked)).
MA(σ-centered)
§ Todo
ia-Shelah.
1502.0500v1.
pp1-25]

§ Todorcevic's fragments of $MA_{\aleph_1}$

Definition (Todorcevic) $n \geq 2$.

A partition $[\omega_1]^n = K_0 \dot\cup K_1$ is called ccc

$[\omega_1]^{<\aleph_0}$

if $\mathbb{P}_{K_0} := \{ v \in [\omega_1]^{<\aleph_0} \mid [v]^n \subseteq K_0 \}$,

$[v]^{<\aleph_0}$

$\leq_{\mathbb{P}_{K_0}} := \supseteq$

is ccc. & $[\omega_1]^1 \subseteq K_0$.

$\mathcal{K}_n' \equiv \forall$ partition

$MA_{\aleph_1}$

is called ccc

$]^n \subseteq k_0 \{$

$]^{<\aleph_0}$

$\subseteq k_0$.

$\mathcal{K}_n' \equiv \forall$ partition $[\omega_1]^{n \; (<\omega)} = k_0 \cup k_1$ ccc

$\exists I \in [\omega_1]^{\aleph_1}$ s.t. $[I]^n \subseteq k_0$.

$\mathcal{K}_{<\omega}'$ $\qquad [I]^{<\aleph_0}$

<u>Definition</u> (Todorcevic) For $n \geq 2$,

$\mathcal{K}_n \equiv \forall$ ccc $\mathbb{P}$ $\forall I \in [\mathbb{P}]^{\aleph_1}$

$\mathcal{K}_{<\omega}$ $\quad \exists I' \in [I]^{\aleph_1}$ s.t. I' is n-linked

has the finite-compatibility property.

Velickovic-Todorcevic

$$
\begin{tikzcd}
 & \mathcal{K}_{<\omega} \arrow[r] \arrow[dd, leftrightarrow] & \cdots \arrow[r] & \mathcal{K}_{n+1} \arrow[r] \arrow[dd] & \mathcal{K}_n \arrow[r] \arrow[dd] & \cdots \arrow[r] & \mathcal{K}_3 \arrow[r] \arrow[dd] & \mathcal{K}_2 \arrow[dd] \\
MA_{\aleph_1} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & & & & & & & \\
 & \mathcal{K}_{<\omega}' \arrow[r] & \cdots \arrow[r] & \mathcal{K}_{n+1}' \arrow[r] & \mathcal{K}_n' \arrow[r] & \cdots \arrow[r] & \mathcal{K}_3' \arrow[r] & \mathcal{K}_2'
\end{tikzcd}
$$

patibility

$K_3 \to K_2$

$\downarrow \qquad \downarrow$

$K_3' \to K_2'$

Note: $K_2' \to$ SH,

every A-tree is special,

every (ω_1,ω_1)-gap is ind.,

$\mathfrak{b} > \aleph_1$.

$K_3' \to$ add$(\mathcal{N}) > \aleph_1$,

$\forall$ nonprincipal uf U on ω,

$(2^{\omega_1}, <_{lex}) \hookrightarrow {}^{\omega}\omega/U$ $\Big\} \rightsquigarrow$ Connection to Automatic Continuity.

$K_4' \to$ every ladder system coloring can be uniformized. $\Big\} \to \exists$ a non-free Whitehead group

$\exists$ almost disjoint coding of subsets of ω_1.

Question Are there other implications in the diagram.

Connection to

* Automatic Continuity,

→ $\exists$ a non-free Whitehead group

...sets of ω_1.

...in the diagram.

Theorem (Larson–Todorcevic)

Let S be a coherent Suslin tree.

- $\Vdash_S$ "$\exists$ ladder system coloring which cannot be uniformized.

  $(\neg K_4')$.

  $(2^{\omega_1}, <_{lex}) \not\hookrightarrow {}^{\omega}\omega/U$ for a nonp. uf U on ω in the ground model"

  $(\neg K_3')$

- $MA_{\aleph_1}(S) \Rightarrow \Vdash_S$ "every A-tree is special".

Definition (Larson–Todorcevic)

$[\omega_1]^2 = K_0$ is K_0 has the rectangle refining property (rec)

if $\forall I, J \in [\omega_1]^{\aleph_1}$ $\exists I' \in [I]^{\aleph_1}$ $\exists J' \in [J]^{\aleph_1}$

such that $I' \cap J' = \phi$ &

$\forall \alpha \in I'$ $\forall \beta \in J'$, $\{\alpha\, \beta\} \in K_0$.

Example: If T is an A-tree,

then $K_0 := \{\{s,t\} \in [T]^2 \mid s \perp_T t\}$

has the rec.

Theorem (Larson–Todorcevic)

$MA_{\aleph_1}(S) \Rightarrow \Vdash_S$ "K_2'(rec)".

$\neg\exists$ almost d.c. of $\mathcal{P}(\omega_1)$.

a positive answer $\rightarrow$ of Katetov's Problem.

<u>Note</u> : $K_2'(\mathrm{REC}) \Rightarrow \mathrm{SH}$,

every (ω_1, ω_1)-gap is ind.,

$\underset{\sim}{b} > \aleph_1$.

Maybe, $K_2'(\mathrm{REC})$ doesn't imply that every Aronszajn tree is special.

(See [APAL 161 (2010), pp 469-487]
[APAL 162 (2011), pp 752-754])

...fining property

$...' \in [J]^{\aleph_1}$

$... K_0.$

$..._T t \}$

$\mathcal{P}(\omega_1).$

a positive answer → of Katetov's Problem.

In [Proc. AMS 138 (2001), pp 2961-2971]:

Definition

$$\mathrm{FSCO}_1 := \{ \mathbb{P} \mid \mathbb{P} \subseteq [\omega_1]^{<\aleph_0}, \ \leq_{\mathbb{P}} = \supseteq, \ \ldots \}$$

Define $\mathrm{FSCO}_2 \subseteq \mathrm{FSCO}_1$.

For $\mathbb{P} \in \mathrm{FSCO}_1$,
$\mathbb{P}$ has the rec. if
$\forall I, J \in [\mathbb{P}]^{\aleph_1}$,
if $I \cup J$ forms a Δ-system,
then $\exists I' \in [I]^{\aleph_1} \ \exists J' \in [J]^{\aleph_1}$ s.t.
$\forall p \in I' \ \forall q \in J'. \ p \not\perp_{\mathbb{P}} q.$

Note $\{ \mathbb{P}_{K_0} \mid [\omega_1]^2 = K_0 \cup K_1 \}$

). pp 2961-2971]:

$[\omega_1]^{<\aleph_0}$,

$\supseteq$,

}

FSCO$_1$.

-system,

$\in [J]^{\aleph_1}$ s.t.

Δ $\mathbb{P}$ {.

<u>Note</u> $\{\mathbb{P}_{K_0} \mid [\omega_1]^2 = K_0 \cup K_1\} \subseteq$ REC in FSCO$_2$

$\subseteq$ REC in FSCO$_1$.

· MA$_{\aleph_1}$(REC in FSCO$_2$)

$\Rightarrow$ every ladder system coloring can be uniformized.

· MA$_{\aleph_1}(S) \Rightarrow \Vdash_S$ "$\mathcal{K}_{<\omega}$(REC in FSCO$_2$)"

So Con($\mathcal{K}_{<\omega}$(REC in FSCO$_2$)

+ $\neg$MA$_{\aleph_1}$(REC in FSCO$_2$)).

$\mathcal{K}'_n \equiv \forall$ partiti

$\exists I \in [\omega$

$\mathcal{K}'_{<\omega}$

<u>Definition</u> (Todorce

$\mathcal{K}_n \equiv \forall$ ccc $\mathbb{P}$

$\mathcal{K}_{<\omega}$ $\exists I' \in [$

Velickovic-Todorcevi

$$\begin{tikzcd}
 & \mathcal{K}_{<\omega} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \\
\mathrm{MA}_{\aleph_1} & \\
 & \mathcal{K}'_{<\omega} \arrow[ul, leftrightarrow]
\end{tikzcd}$$

... in $FSCO_2$,

... in $FSCO_1$.

... can be

... in $FSCO_2$)"

... CO_2))).

§ Bagaria–Shelah's approach and conclusion.

<u>Definition</u> (Bagaria–Shelah) For $n \geq 2$,

$\mathbb{P}$ satisfies the property Pr_n if

$\forall \{p_\alpha \mid \alpha \in \omega_1\} \subseteq \mathbb{P}$

$\exists \{u_\xi \mid \xi \in \omega_1\} \subseteq [\omega_1]^{<\aleph_0}$ pairwise disjoint

s.t. $\forall \{\xi_i \mid i < n\} \in [\omega_1]^n$

$\exists \langle \alpha_i \mid i < n \rangle \in \prod_{i<n} u_{\xi_i}$ s.t. $\{p_{\alpha_i} \mid i < n\}$ has a common extension in $\mathbb{P}$.

<u>Example</u>: A specialization of A-tree by finite approx. has Pr_n.

<u>Note</u>: $K_2' \to$ SH,

every A-...

every $(\omega_1$, ...

$\mathfrak{b} > \aleph_1$.

$K_3' \to \mathrm{add}(\mathcal{N}) > \aleph_1$,

$\forall$ nonprincipal u...

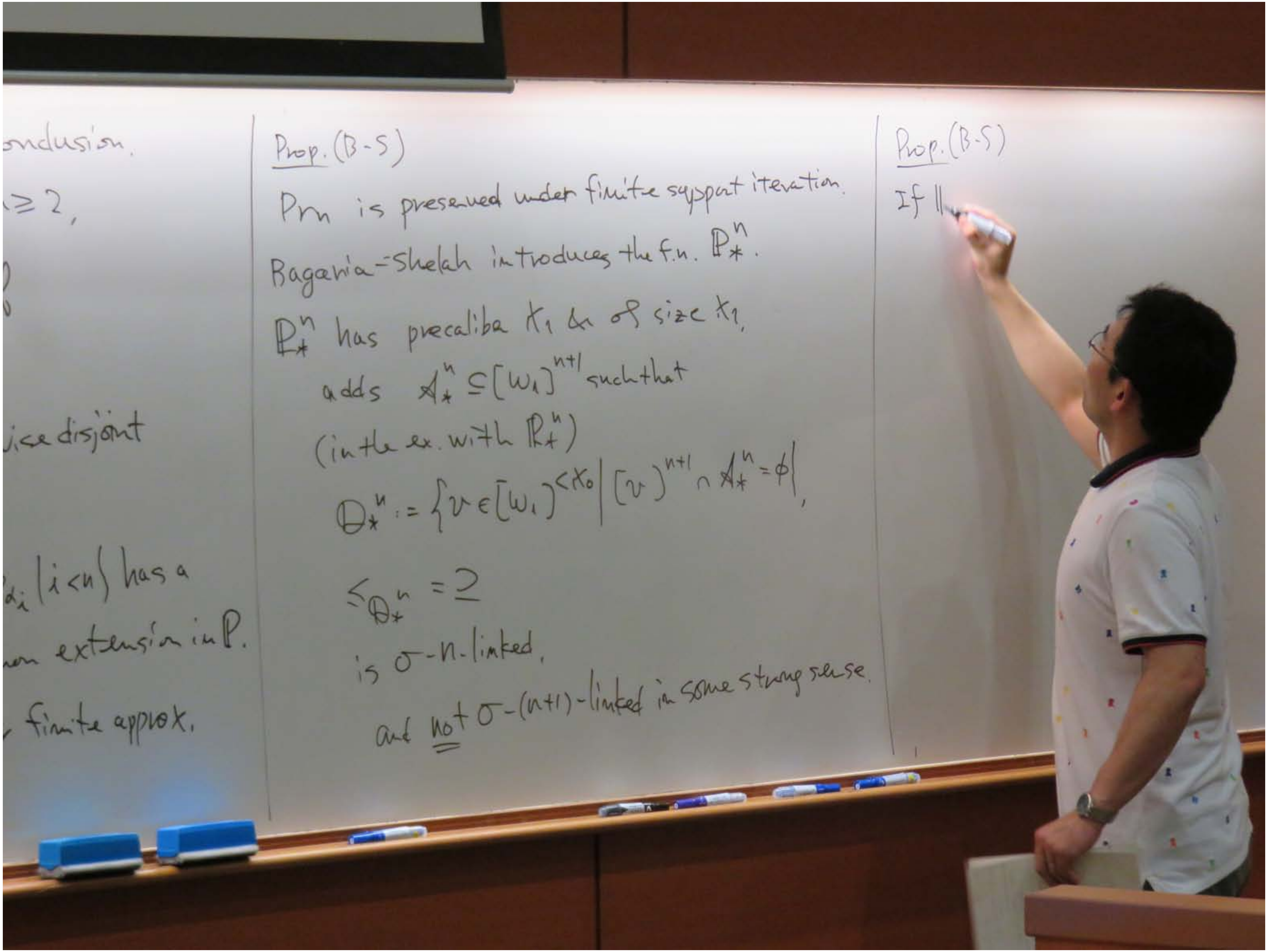
$(2^{\omega_1}, <_{lex})$ ...

$K_4' \to$ every ladder s...

can be unif...

$\exists$ almost disj...

<u>Question</u> ...


onclusion.
≥ 2,
ise disjoint
| i<n} has a
n extension in P.
finite approx.
Prop. (B-S)
Prm is preserved under finite support iteration.
Bagaria-Shelah introduces the f.n. $\mathbb{P}^n_*$.
$\mathbb{P}^n_*$ has precaliber $\aleph_1$ & of size $\aleph_1$.
adds $\mathcal{A}^n_* \subseteq [\omega_1]^{n+1}$ such that
(in the ex. with $\mathbb{P}^n_*$)
$\Theta^n_* := \{v \in [\omega_1]^{<\aleph_0} \mid [v]^{n+1} \cap \mathcal{A}^n_* = \emptyset\}$,
$\leq_{\Theta^n_*} = \supseteq$
is σ-n-linked,
and not σ-(n+1)-linked in some strong sense.
Prop. (B-S)
If ⊩

...der finite support iteration.

...oduces the f.n. $\mathbb{P}^n_*$.

... & of size $\aleph_1$,

...$]^{n+1}$ such that

...$^n_*)$

...$< \aleph_0 \mid [v]^{n+1} \cap A^n_* = \phi \}$,

...1)-linked in some strong sense.

Prop. (B-S)

If $\mathbb{P}$ satisfies Pr_n,

then $\Vdash_{\mathbb{P}}$ "$\neg MA_{\aleph_1}(\mathbb{Q}^n_*)$".

Theorem (B-S)

$Con(MA_{\aleph_1}(Pr_{n+1}) + \neg MA_{\aleph_1}(\sigma\text{-}n\text{-linked}))$.

Actually $\neg \mathcal{K}'_n$

Theorem

$Con(MA_{\aleph_1}(\text{REC in } FSCO_1) + \neg MA_{\aleph_1}(\sigma\text{-}3\text{-linked}) + \neg \mathcal{K}'_3)$

Theorem (B-5)

$$\mathrm{Con}\big(\underbrace{MA_{\aleph_1}(\mathrm{Pr}_{n+1})}_{\Downarrow\ MA_{\aleph_1}(\sigma\text{-centered})} + \neg MA_{\aleph_1}(\sigma\text{-}n\text{-linked})\big)$$

Actually $\neg K_n'$

Lemma (1) The REC in $FSCO_1$ is preserved under finite support iteration (i some sense)

(2) If $\mathbb{P} \in FSCO_1$ satisfies REC, then (in the extension with $\mathbb{P}^n_*$),

$$\Vdash \text{''} \neg MA_{\aleph_1}(\mathbb{Q}^n_*)$$

Lemma (1) The REC in $FSCO_1$ is preserved under finite support iteration (i some sense).

(2) If $\mathbb{P} \in FSCO_1$ satisfies REC,

then (in the extension with $\mathbb{P}^n_*$),

$$\Vdash_{\mathbb{P}} \text{"} \neg MA_{\aleph_1}(\mathbb{Q}^n_*),$$

$$\neg \exists I \in [\omega_1]^{\aleph_1} \text{ such that } [I]^{n+1} \cap \mathcal{A}^n_* = \phi\text{"}.$$

Conclusion

$$Con(MA_{\aleph_1}(\text{REC in } FSCO_1) + \neg MA_{\aleph_1}(\sigma\text{-3-linked}) + \neg \mathcal{K}'_3).$$