Axiomatizing some small classes of set functions

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A set-theoretic function $f(\vec{a})$ is Σ_1-definable in a fragment T if there exists a Σ_1-formula $\varphi(\vec{a}, b)$ such that $f(\vec{a}) = b \Leftrightarrow V \models \varphi(\vec{a}, b)$ for any $\vec{a}, b$, and $T \vdash \forall \vec{a} \exists ! b \, \varphi(\vec{a}, b)$.

A formal system axiomatizes a class of functions iff Σ_1-definable functions in it are exactly functions in the class.

1. rudimentary functions [Jensen], pp. 3-7.
2. primitive recursive functions [Jensen-Karp], pp. 8-9.
3. safe recursive set functions [Beckmann-Buss-Friedman], pp. 10-14.
4. predicatively computable set functions [A.] augmented with an ι-operator, pp. 15-30.

1 Rudimentary functions

Theorem 1 A set-theoretic function is rudimentary iff it is Σ_1-definable in the fragment KP *minus* Foundation schema.

$$\mathrm{KP}^- := \mathrm{KP\text{-}Foundation}$$

The set of rudimentary functions are generated from projections, pair, difference $a - b$ by operating composition and **(Bounded Union)**:

$$f(\vec{x}, z) = \bigcup \{g(\vec{x}, y) : y \in z\}.$$

(Σ_1-definability of rudimentary functions in KP^-)
For the bounded union

$$f(\vec{x}, z) = \bigcup \{g(\vec{x}, y) : y \in z\}$$

assume that $g(\vec{x}, y) = a$ is defined by a Σ_1-formula $\varphi_g(\vec{x}, y, a)$, $\forall y \in z \exists! a\, \varphi_g(\vec{x}, y, a)$. Pick a b such that $\forall y \in z \exists a \in b\, \varphi_g(\vec{x}, y, a)$ by (Δ_0-Collection). Then $f(\vec{x}, z) = \cup\{a \in b : \exists y \in z\, \varphi_g(\vec{x}, y, a)\}$.

The 'only-if' part is proved by a witnessing argument [Buss].
idea: given an implication $\exists a\, \varphi(x,a) \to \exists b\, \psi(x,b)$ of Σ_1-formulas, find a function f such that $\forall x, a[\varphi(x,a) \to \psi(x, f(x,a))]$.

$$\frac{a \in c(x) \wedge \varphi(x,a) \to \exists b\, \psi(x,b)}{\exists a \in c(x)\, \varphi(x,a) \to \exists b\, \psi(x,b)}$$

Problem. Given $a \in c(x) \wedge \varphi(x,a) \to \psi(x, f(x,a))$, find a $g(x)$ such that $\exists a \in c(x)\, \varphi(x,a) \to \psi(x, g(x))$.

Suppose $\exists a \in c(x)\, \varphi(x,a)$. Pick an $a \in c(x)$ such that $\varphi(x,a)$, and put $g(x) := f(x,a)$??

A choice function $a = a(x) \in \{a \in c(x) : \varphi(x,a)\}$!

Solution. Find a non-empty set of witnesses.

$$\frac{a \in c(x) \wedge \varphi(x,a) \to \emptyset \neq f(x,a) \subset \{b : \psi(x,b)\}}{\exists a \in c\, \varphi(x,a) \to \emptyset \neq g(x) \subset \{b : \psi(x,b)\}}$$

for $g(x) = \bigcup\{f(x,a) : a \in c(x), \varphi(x,a)\}$ by **(Bounded Union)**.

Assume that $\exists! a\, \varphi(x,a)$ is derivable in KP^-. We can find a rudimentary function $g(x)$ such that

$$\emptyset \neq g(x) \subset \{a : \varphi(x,a)\}$$

Since the set $\{a : \varphi(x,a)\}$ is a singleton for each x, $f(x) = \cup g(x)$ enjoys $\forall x\, \varphi(x, f(x))$ as desired.

Corollary 2 A predicate is rudimentary iff it is Δ_0 iff it is Δ_1-definable in KP^-.

Proof. Suppose $\forall\vec{x}[\neg\exists a\,\varphi_0(\vec{x},a) \leftrightarrow \exists a\,\varphi_1(\vec{x},a)]$ is derivable for some Δ_0-formulas φ_0, φ_1.

Pick rudimentary functions f_0, f_1 such that

$$(\emptyset \neq f_0(\vec{x}) \subset \{a : \varphi_0(\vec{x},a)\}) \text{ or } (\emptyset \neq f_1(\vec{x}) \subset \{a : \varphi_1(\vec{x},a)\})$$

Then $\exists a \in f_1(\vec{x})\,\varphi_1(\vec{x},a) \rightarrow \exists a\,\varphi_1(\vec{x},a) \rightarrow \neg\exists a\,\varphi_0(\vec{x},a) \rightarrow \neg(\emptyset \neq f_0(\vec{x}) \subset \{a : \varphi_0(\vec{x},a)\}) \rightarrow (\emptyset \neq f_1(\vec{x}) \subset \{a : \varphi_1(\vec{x},a)\}) \rightarrow \exists a \in f_1(\vec{x})\,\varphi_1(\vec{x},a)$. Hence $\exists a \in f_1(\vec{x})\,\varphi_1(\vec{x},a) \leftrightarrow \exists a\,\varphi_1(\vec{x},a)$.

$\square$

2 Primitive recursive functions

The set of primitive recursive functions is generated from projections, null, conditional, and $M(a,b) = a \cup \{b\}$, and operating composition and set recursion:

$$f(x, \vec{y}) = h(x, \vec{y}, \{f(z, \vec{y}) : z \in x\}).$$

Theorem 3 [Rathjen]
A set-theoretic function is primitive recursive in a Δ_0-function $\mathsf{g}(\vec{x})$ iff it is Σ_1-definable in $\mathrm{KP}^- + \Sigma_1$-Foundation$+\forall \vec{x} \exists ! y(\mathsf{g}(\vec{x}) = y)$.

Σ_1-Foundation$+\Delta_0$-Collection suffices for the existence of the transitive closure $\mathrm{TC}(x)$ of x, and Σ-recursion of functions.

Theorem 3 and Corollary 2 are extended to Π_1-functions g.

1. A function is primitive recursive in $\mathsf{g}(\vec{x})$ iff it is $\Sigma_1(\mathsf{g})$-definable in $\mathrm{KP}^-(\mathsf{g})+\Sigma_1(\mathsf{g})$-Foundation$+\forall\vec{x}\exists! y(\mathsf{g}(\vec{x})=y)$.
2. A predicate is primitive recursive in $\mathsf{g}(\vec{x})$ iff it is $\Delta_1(\mathsf{g})$-definable in $\mathrm{KP}^-(\mathsf{g})+\Sigma_1(\mathsf{g})$-Foundation$+\forall\vec{x}\exists! y(\mathsf{g}(\vec{x})=y)$.

As for rudimentary functions, the 'only-if' parts are shown by a witnessing argument with non-empty sets of witnesses.

3 Safe recursive functions

Arguments of each function $f(\vec{x}/\vec{a})$ in the class are divided to normal arguments $\vec{x}$ and safe arguments $\vec{a}$ a là [Bellantoni-Cook]. The class SRSF of safe recursive set functions, is obtained from rudimentary set functions on safe arguments by safe composition schema and predicative set (primitive) recursion schema.

(Bounded Union)

$$f(\vec{x}/\vec{a}, b) = \bigcup_{c \in b} g(\vec{x}/\vec{a}, c).$$

(Safe Composition Scheme)

$$f(\vec{x}/\vec{a}) = h(\vec{r}(\vec{x}/-)/\vec{t}(\vec{x}/\vec{a})).$$

(Safe Set Recursion Scheme)

$$f(x, \vec{y}/\vec{a}) = h(x, \vec{y}/\vec{a}, \{f(z, \vec{y}/\vec{a}) : z \in x\}).$$

Expand the language by augmenting a predicate $\mathcal{D}$, denoting a transitive class for normal arguments.

We say that a set-theoretic function $f(\vec{x}/\vec{a})$ is $\Sigma_1^{\mathcal{D}}$-definable in T if there exists a Σ_1-formula (in the language of set theory) $\varphi(\vec{x}, \vec{a}, b)$ such that $f(\vec{x}/\vec{a}) = b \Leftrightarrow V \models \varphi(\vec{x}, \vec{a}, b)$ for any $\vec{x}, \vec{a}, b$, and $T \vdash \forall \vec{x} \subset \mathcal{D} \forall \vec{a} \exists ! b\, \varphi(\vec{x}, \vec{a}, b)$.

3.1 A fragment T_2 for SRSF

$T_2 := \mathrm{KP}^- + (\Sigma_1^{\mathcal{D}}\text{-Foundation}) + (\Sigma_1\text{-Submodel Rule})$

($\Sigma_1^{\mathcal{D}}$-Foundation)

$$\forall y \in \mathcal{D}[\forall x \in y \exists a\, \varphi(x, a) \to \exists a\, \varphi(y, a)] \to \forall y \in \mathcal{D} \exists a\, \varphi(y, a)$$

(Σ_1-Submodel Rule)

$$\frac{\forall \vec{x} \subset \mathcal{D} \exists a\, \varphi(\vec{x}, a)}{\forall \vec{x} \subset \mathcal{D} \exists a \in \mathcal{D}\, \varphi(\vec{x}, a)}$$

and an axiom saying that $\mathcal{D}$ is transitive.

Theorem 4 A set-theoretic function is in SRSF iff it is $\Sigma_1^{\mathcal{D}}$-definable in T_2.

($\Sigma_1^{\mathcal{D}}$-definability of SRSF-functions in T_2)

($\Sigma_1^{\mathcal{D}}$-Foundation) suffices for **(Predicative Set Recursion)**
$f(x, \vec{y}/\vec{a}) = h(x, \vec{y}/\vec{a}, \{f(z, \vec{y}/\vec{a}) : z \in x\})$.
(Bounded Union) by (Δ_0-Coll).

(Σ_1-Submodel Rule) suffices for **(Safe Composition)**
$f(\vec{x}/\vec{a}) = h(\vec{r}(\vec{x}/-)/\vec{t}(\vec{x}/\vec{a}))$.

Corollary 5 A predicate is in SRSF iff it is $\Delta_1^{\mathcal{D}}$-definable in T_2.

4 Predicatively computable set functions with ι-operator

PCSF-functions are generated from projections, pair, null, union $\cup(-/a)$, conditional, **(Safe Separation)**

$$f(-/\vec{a}, c) = \{b \in c : h(-/\vec{a}, b) \neq 0\}$$

(Safe Composition) and **(Predicative Set Recursion)**.

A function on $\mathbb{HF}$ is poly time computable iff it is in PCSF.

Theorem 6 (Polysize)
For each $f(\vec{x}/\vec{a}) \in$ PCSF, the size of the transitive closure $\mathrm{TC}(f(\vec{X}/\vec{A}))$ of $f(\vec{X}/\vec{A})$ for $\vec{X}, \vec{A} \subset \mathbb{HF}$ is bounded by the sum of the sizes of $\mathrm{TC}(\vec{A})$ and a polynomial of the sizes of $\mathrm{TC}(\vec{X})$.

Difficulty in axiomatizing PCSF is due to lack of **(Bounded Union)**. Without it we can not collect witnesses to a set.

$$\frac{a \in c(x) \wedge \varphi(x,a) \rightarrow \emptyset \neq f(x,a) \subset \{b : \psi(x,b)\}}{\exists a \in c\, \varphi(x,a) \rightarrow \emptyset \neq \bigcup \{f(x,a) : a \in c(x), \varphi(x,a)\} \subset \{b : \psi(x,b)\}}$$

Let us restrict our attention to derivations in which existential quantifiers occur only as uniqueness quantifires $\exists ! b$.

$$\frac{a \in c(x) \wedge \varphi(x,a) \rightarrow \exists ! b\, \psi(x,b)}{\exists a \in c(x)\, \varphi(x,a) \rightarrow \exists ! b\, \psi(x,b)}$$

If $f(x,a)$ is the unique witness of b in $\psi(x,b)$ for any $a \in c(x)$ with $\varphi(x,a)$, then for $g(x) = \iota b[\exists a \in c(x)(\varphi(x,a) \wedge f(x,a) = b)]$, we obtain $\exists a \in c(x)\, \varphi(x,a) \rightarrow \psi(x, g(x))$.

We are going to enlarge the class PCSF by introducing Russell's ι-operator (definite description), cf. PCSF^+ in [Beckmann-Buss-Friedman-Müller-Thapen].

The ι-operator describes an object $\iota x.A(x)$ for a predicate $A(x)$: $\iota x.A(x)$ denotes the unique element x enjoying $A(x)$ if there exists a unique such x. Otherwise put $\iota x.A(x) = \emptyset$.

There remains some room for the class PCSF to be extended, still holding Theorem 6(Polysize), and keeping the extensionality of functions under encoding: if the codes G and H encode the same set (in $\mathbb{HF}$), then the codes $F(G)$ and $F(H)$ should encode the same set.

The class PCSF^ι is closed under (ι): if $g \in \mathsf{PCSF}^\iota$, then so is

(ι) $f(\vec{x}/\vec{a}, c) = \iota d(\exists b \in c(g(\vec{x}/\vec{a}, b) = d))$.

This means that when the range $g"c = \{g(\vec{x}/\vec{a}, b) : b \in c\}$ is a singleton, $f(\vec{x}/\vec{a}, c)$ denotes the unique element, and $f(\vec{x}/\vec{a}, c) = \emptyset$ otherwise.

Obviously Theorem 6(Polysize) holds for the enlarged class PCSF^ι, and each function in this class enjoys the extensionality under encoding.

Let $\Delta_0(\mathsf{PCSF}^\iota)$ denote the set of bounded formulas in the language with function symbols in the class PCSF^ι. $\Sigma_1!(\mathsf{PCSF}^\iota)$ denotes the set of formulas $\exists! a\, \varphi$ with $\varphi \in \Delta_0(\mathsf{PCSF}^\iota)$.

<u>A formal system T_3^ι</u> for axiomatizing PCSF^ι: $\varphi \in \Delta_0(\mathsf{PCSF}^\iota)$.

1. $\forall x \in \mathcal{D} \exists a (a = \mathrm{TC}(x))$ and $(\Delta_0(\mathsf{PCSF}^\iota)$-Sep).
2. $(\Delta_0^{\mathcal{D}}(\mathsf{PCSF}^\iota)$-Replacement): $y \in \mathcal{D}$ is a 'domain' of a function. $\forall y \in \mathcal{D}[\forall x \in y \exists! a\, \varphi(x,a) \to \exists c \forall x \in y\, \varphi(x, c'x)]$.
3. $(\Sigma_1^{\mathcal{D}}!(\mathsf{PCSF}^\iota)$-Fund): $\mathcal{D}$ is weakly wellfounded. $\forall y \in \mathcal{D}[\forall x \in y \exists! a\, \varphi(x,a) \to \exists! a\, \varphi(y,a)] \to \forall y \in \mathcal{D} \exists! a\, \varphi(y,a)$.
4. $(\Sigma_1!(\mathsf{PCSF}^\iota)$-Submodel Rule)

$$\frac{\forall \vec{x} \subset \mathcal{D} \exists! a\, \varphi(\vec{x}, a)}{\forall \vec{x} \subset \mathcal{D} \exists y \in \mathcal{D}\, \varphi(\vec{x}, y)}$$

Problem. It is open for us how to axiomatize PCSF^ι-predicates.

A function $f(\vec{x}/\vec{a})$ is $\Sigma_1^{\mathcal{D}}$!-definable in T if there exists a Σ_1!-formula $\varphi(\vec{x}, \vec{a}, b)$ such that $f(\vec{x}/\vec{a}) = b \Leftrightarrow V \models \varphi(\vec{x}, \vec{a}, b)$ and $T \vdash \forall \vec{x} \subset \mathcal{D} \forall \vec{a} \exists ! b \, \varphi(\vec{x}, \vec{a}, b)$.

1. $T_3 := \mathrm{TC} + (\Delta_0\text{-Sep}) + (\Delta_0^{\mathcal{D}}\text{-Rpl}) + (\Sigma_1^{\mathcal{D}}!\text{-Fund}) + (\Sigma_1!\text{-SmR})$ $\Sigma_1^{\mathcal{D}}$!-defines PCSF-functions.
2. Each $\Sigma_1^{\mathcal{D}}$!-definable function in T_3 is in PCSF^{ι}, but not shown in PCSF.

Actually T_3^ι in a language $\mathcal{L}^{(\omega)} = \bigcup_n \mathcal{L}^{(n)}$ is a union of increasing formal systems $T_3^{(n)}$ in $\mathcal{L}^{(n)}$.

$$T_3^{(n)} = \mathrm{TC} + (\Delta_0(\mathcal{L}^{(n)})\text{-Sep}) + (\Delta_0^{\mathcal{D}}(\mathcal{L}^{(n)})\text{-Rpl}) + (\Sigma_1^{\mathcal{D}}!(\mathcal{L}^{(n)})\text{-Fund}) + (\Sigma_1!(\mathcal{L}^{(n)})\text{-SmR})$$

Enlarge the language $\mathcal{L}^{(n)}$ to get $\mathcal{L}^{(n+1)}$ by adding function symbols $\mathsf{f}(\vec{x}/\vec{a})$ when $T_3^{(n)} \vdash \forall \vec{x} \subset \mathcal{D} \forall \vec{a} \exists! b\, \theta_{\mathsf{f}}(\vec{x}, \vec{a}, b)$ for $\theta_{\mathsf{f}} \in \Sigma_1!(\mathcal{L}^{(n)})$, and add an axiom $\forall \vec{x} \subset \mathcal{D} \forall \vec{a}\, \theta_{\mathsf{f}}(\vec{x}, \vec{a}, \mathsf{f}(\vec{x}, \vec{a}))$.

The introduced function symbol f for $\Sigma_1!(\mathcal{L}^{(n)})$-definable functions in $T_3^{(n)}$ may occur in bounded formulas of Separation, Replacement, Foundation and Submodel rule of $T_3^{(n+1)}$.

A function $f(\vec{x}/\vec{a})$ is $\Sigma_1^{\mathcal{D}}!(\mathcal{L}^{(\omega)})$-definable in T_3^ι if there exists a $\Sigma_1!(\mathcal{L}^{(\omega)})$-formula $\varphi(\vec{x},\vec{a},b)$ such that $f(\vec{x}/\vec{a}) = b \Leftrightarrow V \models \varphi(\vec{x},\vec{a},b)$ and $T_3^\iota \vdash \forall\vec{x} \subset \mathcal{D}\forall\vec{a}\exists! b\, \varphi(\vec{x},\vec{a},b)$.

Theorem 7 A set-theoretic function is in PCSF^ι iff it is $\Sigma_1^{\mathcal{D}}!(\mathcal{L}^{(\omega)})$-definable in T_3^ι.

($\Sigma_1!(\mathcal{L}^{(\omega)})$-definability of PCSF^ι-functions)

$$f(\vec{x}/\vec{a},c) = \iota d(\exists b \in c(g(\vec{x}/\vec{a},b) = d))$$

$\varphi_f(\vec{x},\vec{a},c,d)$ iff $\exists! e[\exists b \in c(\mathsf{g}(\vec{x},\vec{a},b) = e)] \wedge (\exists b \in c(\mathsf{g}(\vec{x},\vec{a},b) = d)$ or $d = \emptyset \wedge (c \neq \emptyset \longrightarrow \exists b_0, b_1 \in c(\mathsf{g}(\vec{x},\vec{a},b_0) \neq \mathsf{g}(\vec{x},\vec{a},b_1))$.
φ_f is a $\Sigma_1!(\mathcal{L}^{(\omega)})$-formula with a function symbol g.

The converse of Theorem 7:

> if a set-theoretic function is $\Sigma_1^{\mathcal{D}}!(\mathcal{L}^{(n)})$-definable in $T_3^{(n)}$, then it is in PCSF^{ι}.

is proved by induction on n. Let us assume that each function in $\mathcal{L}^{(n)}$ as well as each $\Delta_0(\mathcal{L}^{(n)})$-formula is in PCSF^{ι}.

Uniqueness conditions involve unbounded universal quantifiers $\mathrm{Unique}_a(\varphi) :\Leftrightarrow \forall a, b[\varphi(a) \land \varphi(b) \to a = b]$.

To control the unbounded universal quantifiers, we introduce classes X, i.e., $\forall a, b$ is restricted to $\forall a, b \in X$. Classes are generated recursively.

1. Each singleton $\{f(\vec{x}/\vec{a})\}$ for $f \in \mathsf{PCSF}^{\iota}$ is a class.
2. For classes X, Y, $X \cup Y$ is a class.
3. If $X(a)$ is a class and $f \in \mathsf{PCSF}^{\iota}$, then $\bigcup\{X(a) : a \in f(\vec{x}/\vec{a})\}$ is a class.

If φ is a bounded formula in $\mathcal{L}(\mathsf{PCSF}^{\iota})$, then so is the formula $\forall d \in X\, \varphi(d)$ for each class X.

A witness b of a Σ_1!-formula $\exists! a\, \varphi$ wrt X is a unique witness in X, i.e., $b \in X \wedge \varphi(b) \wedge \forall a \in X(\varphi(a) \rightarrow a = b)$.

1. $w!^X_\varphi(b) :\Leftrightarrow \varphi$ if φ is a bounded formula.
2. $w!^X_{\exists! c\, \psi(c)}(b) :\Leftrightarrow b \in X \wedge \psi(b) \wedge \mathrm{Unique}^X_c(\psi(b))$
   where $\mathrm{Unique}^X_c(\psi(b)) :\Leftrightarrow \forall c \in X(\psi(c) \rightarrow b = c)$.
3. $w!^X_{\forall x \in y \exists! c\, \psi(x,c)}(b)$ iff b is a function on y s.t. $\forall x \in y[w!^X_{\exists! c\, \psi(x,c)}(b'x)]$.

$w!^X_\varphi(b)$ is a bounded formula in the language $\mathcal{L}(\mathsf{PCSF}^\iota)$ for each class X.

A witnessing function $f_X(x/a, b)$ for derivable implications of $\Sigma_1!$-formulas $\varphi(x, a) \to \psi(x, a)$ may depend uniformly on classes X, $w!^X_\varphi(b) \to w!^X_\psi(f_X(x/a, b))$.

When f is defined from $j, k, g, h \in \mathsf{PCSF}^\iota_X$ and $\varphi(\vec{x}, \vec{a}) \in \Sigma_1!$ by cases

$$f(\vec{x}/\vec{a}) = \begin{cases} j(\vec{x}/\vec{a}) & \text{if } \forall x \in g(\vec{x}/\vec{a})[w!^X_\varphi(h(\vec{x}, x/\vec{a}))] \\ k(\vec{x}/\vec{a}) & \text{otherwise} \end{cases}$$

then $f \in \mathsf{PCSF}^\iota_X$.

Each $f \in \mathsf{PCSF}^\iota_X$ denotes a function in PCSF^ι depending uniformly on classes X.

The following Lemma 8 yields the converse of Theorem 7.

Lemma 8 Assume that an implication

$$\mathcal{D}(\vec{x}) \wedge \sigma \rightarrow \neg\mathrm{Unique}_a(\theta) \vee \varphi$$

is derivable in $T_3^{(n)}$ for $\Sigma_1!(\mathcal{L}^{(n)})$-formulas σ, φ and bounded θ.

Then there exist a class $X = X(\vec{x}/\vec{a}, b)$ and a function $f_X(\vec{x}/\vec{a}, b) \in \mathsf{PCSF}^{\iota}_X$ such that

$$w!^X_\sigma(b) \rightarrow \neg\mathrm{Unique}^X_a(\theta) \vee w!^X_\varphi(f_X(\vec{x}/\vec{a}, b)) \quad (1)$$

where $\neg\mathrm{Unique}^X_a(\theta) :\Leftrightarrow \exists a, b \in X(\theta(a) \wedge \theta(b) \wedge a \neq b)$.

Proof.

Case 0.

The case when two occurrences of a formula φ is contracted. Let e be defined by cases from c, d and a bounded formula $w!^X_\varphi(c)$.

$$e = \begin{cases} c & \text{if } w!^X_\varphi(c) \\ d & \text{otherwise} \end{cases}$$

Then $w!^X_\varphi(c) \vee w!^X_\varphi(d) \to w!^X_\varphi(e)$.
Note that $w!^X_\varphi(c)$ is in PCSF^ι for each X.

Case 1.

$$\frac{\sigma \rightarrow (\theta(s_0) \land \theta(s_1) \land s_0 \neq s_1) \lor \varphi}{\sigma \rightarrow \neg\mathrm{Unique}_a(\theta) \lor \varphi}\ (\exists)$$

For $\neg\mathrm{Unique}_a^X(\theta)$ it suffices to have $\{s_0, s_1\} \subset X$.
This means that we need to augment two elements s_0, s_1 to a class X_0 of the upper sequent, $X = X_0 \cup \{s_0, s_1\}$.
Although the function $f_X(x/a, b)$ may differ $f_{X_0}(x/a, b)$, these depend classes X, X_0 uniformly in the sense that the 'definitions' of these functions coincide.
Furthermore as we shall see it, requirements on classes are monotonic, i.e., if X and f_X enjoys (1), then so does a larger class $Y \supset X$ (and f_Y).

Case 2. For an eigenvariable d (suppressing $\neg\mathrm{Unique}_a(\theta)$)

$$\frac{d \in t \land \gamma(d) \to \varphi}{\exists c \in t\, \gamma(c) \to \varphi}$$

Let $X = \bigcup\{X_0(d) : d \in t\}$ for a class $X_0(d)$ of the upper sequent.

Assume that if $\gamma(d)$ and $d \in t$, then $w!^X_\varphi(h_X(x/a, d))$. We have $\forall d_0, d_1 \in t(\bigwedge_i \gamma(d_i) \to h_X(x/a, d_0) = h_X(x/a, d_1))$. Let

$$f_X(x/a) = \iota e[\exists d \in t(\gamma(d) \land h_X(x/a, d) = e)]$$

We obtain $\exists c \in t\, \gamma(c) \to w!^X_\varphi(f_X(x/a))$.

$\square$

Thank you for your attention!

Case 3. For $t' = \mathrm{TC}(t \cup \{t\})$

$$\frac{\forall y \in t'(\forall x \in y \exists! a\, \gamma(x,a) \rightarrow \exists! a\, \gamma(y,a)) \vee \varphi}{\mathcal{D}(t) \rightarrow \exists! a\, \gamma(t,a) \vee \varphi}\ (\Sigma_1^{\mathcal{D}}!\text{-Fund})$$

For $X = X(y,b)$ assume that for any $b : y \rightarrow V$ and $y \in t'$

$$\forall x \in y\, w!^{X}_{\exists! a\, \gamma(x)}(b'x) \rightarrow w!^{X}_{\exists! a\, \gamma(y)}(h_X(y/b)) \vee w!^{X}_{\varphi}(k_X(y/b))$$

Let $g(y/) = h_X(y/b_y)$ for $b_y = g \restriction y = \{\langle x, g(x/)\rangle : x \in y\}$.

Let $Y = \bigcup\{X(y,b_y) : y \in t'\}$. For any $y \in t'$

$$\forall x \in y\, w!^{Y}_{\exists! a\, \gamma(x)}(b'_y x) \rightarrow w!^{Y}_{\exists! a\, \gamma(y)}(h_Y(y/b_y)) \vee w!^{Y}_{\varphi}(k_Y(y/b_y))$$

Equivalently

$$\forall x \in y\, w!^{Y}_{\exists! a\, \gamma(x)}(g(x/)) \longrightarrow w!^{Y}_{\exists! a\, \gamma(y)}(g(y/)) \vee w!^{Y}_{\varphi}(k_Y(y/b_y))$$

If $\neg\forall x \in t'[w!^{Y}_{\exists! a\, \gamma(x)}(g(x/))]$, then $\exists x \in t'\, w!^{Y}_{\varphi}(k_Y(x/b_x)))$. Otherwise we obtain $w!^{Y}_{\exists! a\, \gamma(t)}(g(t/))$.

Therefore $w!^{Y}_{\exists! a\, \gamma(t)}(g(t/)) \vee w!^{Y}_{\varphi}(K)$, where $K = \bigcup\{k_Y(x/b_x) : w!^{Y}_{\varphi}(k_Y(x/b_x)), x \in t'\}$ with a singleton $\{k_Y(x/b_x) : w!^{Y}_{\varphi}(k_Y(x/b_x)), x \in t'\}$.

$\square$