Large Lindelöf spaces with points G_δ

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Arhangel'skii's inequality

All topological spaces are assumed to be T_1.

A space X is Lindelöf if every open cover has a countable subcover.

Fact 1 (Arhangel'skii (1969))

If X is Hausdorff, Lindelöf, and first countable, then the cardinality of X is $\leq 2^\omega$.

Fact 2 (Arhangel'skii)

If X is Hausdorff, then $|X| \leq 2^{L(X)+\chi(X)}$.

- $L(X)$, Lindelöf degree of X, is the least infinite cardinal κ such that every open cover of X has a subcover of size $\leq \kappa$.
- $\chi(X)$: the character of X.

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Arhangel'skii's question

Question 3 (Arhangel'skii (1969))

Can the first countability be weakened to be points G_δ?

A space X is points G_δ if for each $x \in X$, the set $\{x\}$ is a G_δ-set.

Why points G_δ?

Fact 4 (Arhangel'skii (19??))

If X is Lindelöf and with points G_δ, then the cardinality of X is strictly less than the least measurable cardinal.

Fact 5 (Shelah (19??))

If κ is weakly compact, then there is no Lindelöf space X with point G_δ such that $|X| = \kappa$.

Fact 6 (Juhasz (19??))

For each cardinal κ, if κ is strictly less than the least measurable cardinal, then there is a (non-Hausdorff) Lindelöf space of size $> \kappa$ and with points G_δ.

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Fact 7 (Arhangel'skii, Shapirovskii)

If X is Hausdorff, then $|X| \leq 2^{L(X)+t(X)+\psi(X)}$.

Definition 8

- For $x \in X$, $\psi(x, X) = \min\{|\mathcal{U}| : \mathcal{U}$ is a family of open sets, $\bigcap \mathcal{U} = \{x\}\} + \aleph_0$.
- The pseudocharacter of X, $\psi(X)$, is $\sup\{\psi(x, X) : x \in X\}$.

- $t(X)$, the titghtness number of X, is the least infinite cardinal κ such that for every $A \subseteq X$ and $x \in \overline{A}$, there is $B \subseteq A$ of size $\leq \kappa$ such that $|B| \leq \kappa$ and $x \in \overline{B}$.

Note that $\psi(X) + t(X) \leq \chi(x)$.

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Fact 9

For each cardinal κ, there are X and Y such that

1. X is normal, $\psi(X) = t(X) = \omega$, but $|X| = \kappa$.
2. Y is Hausdorff, compact (so $L(Y) = \omega$), $t(Y) = \omega$, but $|Y| = \kappa$.

Hence $L(X) + t(X)$ and $t(X) + \psi(X)$ cannot give an upper bound of the cardinality of the space X.

Question 10 (rephrased)

If X is Hausdorff, does $|X| \leq 2^{L(X)+\psi(X)}$?

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Consistency results

Fact 11 (Shelah (1978), Gorelic (1993))

It is consistent that ZFC+Continuum Hypothesis+"there exists a regular Lindelöf space with points G_δ and of size 2^{ω_1}".

Question 12 (still open)

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Fact 13 (Dow (2014?))

Suppose $\diamondsuit^*$ holds, that is, there exists $\langle \mathcal{A}_\alpha : \alpha < \omega_1 \rangle$ such that

1. $\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha)$, $|\mathcal{A}_\alpha| \leq \omega$.
2. For every $A \subseteq \omega_1$, the set $\{\alpha < \omega_1 : A \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club in ω_1.

Then there exists a zero-dimensional Hausdorff Lindelöf space with points G_δ and of size 2^{ω_1}.

Inspired by Dow's construction, we introduce a new construction of large regular Lindelöf spaces with points G_δ, and we show that the statement that no large regular Lindelöf space with points G_δ is a large cardinal property (if it is consistent).

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Main theorem

Theorem 14

Suppose that either:

1. There exists a regular Lindelöf P-space with pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$,
2. CH$+$there exists an ω_1-Kurepa tree, or
3. CH$+\square(\omega_2)$holds.

Then there exists a regular Lindelöf space with points G_δ and of size $> 2^\omega$.

Fact 15 (Jensen (197?))

If $\square(\kappa)$ fails for some regular uncountable κ, then κ is weakly compact in L.

Corollary 16

If

- ZFC+CH+ "every regular Lindelöf space with points G_δ has cardinality $\leq 2^\omega$"

is consistent, then so is

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Key lemma

Lemma 17

Let Y be a regular Lindelöf space such that:

1. $\psi(Y) \leq \omega_1$.
2. For $y \in Y$, if $\psi(y, Y) = \omega_1$ then there exists $\langle O^y_\alpha : \alpha < \omega_1 \rangle$ such that
   - 2.1 O^y_α is clopen.
   - 2.2 $O^y_\alpha \supseteq O^y_{\alpha+1}$.
   - 2.3 $O^y_\alpha = \bigcap_{\beta<\alpha} O^y_\beta$ if α is a limit ordinal.
   - 2.4 $\bigcap_{\alpha<\omega_1} O^y_\alpha = \{y\}$.

Then there exists a regular Lindelöf space with points G_δ and of size $\max\{2^\omega, |Y|\}$.

Remark 18

If Y is a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$, then Y satisfies the assumptions of Lemma 17.

Theorem 19

Suppose that there exists a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$, Then there exists a regular Lindelöf space with points G_δ and of size $> 2^\omega$.

Remark 20

The statement that "there exists a regular Lindelöf P-space of pseudocharacter $\leq \omega_1$ and of size $> 2^\omega$" is independent from ZFC.

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Theorem 21

Suppose that either:

1. CH + there exists an ω_1-Kurepa tree, or
2. CH + $\square(\omega_2)$ holds.

Then there exists a regular Lindelöf space of size $> 2^\omega$ which satisfies the assumption of key Lemma, In particular there exists a regular Lindelöf space with points G_δ and of size $> 2^\omega$.

Fact 22

1. The statement that "there is an ω_1-Kurepa tree" is independent from ZFC (Silver).
2. The statement that "$\square(\omega_2)$ holds" is independendet from ZFC (Jensen).

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Outline of a proof of the lemma

Fix a space Y and sequences $\langle O^y_\alpha : \alpha < \omega_1 \rangle$ for $y \in Y$ with $\psi(y, Y) = \omega_1$. Let

- $Y_0 = \{y \in Y : \psi(y, Y) \leq \omega\}$.
- $Y_1 = \{y \in Y : \psi(y, Y) = \omega_1\}$.

Let $Z = {}^\omega 2$ (Cantor space)

The underlying set of our space X is $Y_0 \cup (Y_1 \times Z)$.
(Clearly $|X| = \max\{|Y|, 2^\omega\}$.)

cont.

For $A \subseteq Y$, let

- $[\![A]\!] = (A \cap Y_0) \cup ((A \cap Y_1) \times Z)$.

Note that:

- For $\gamma < \omega_1$, $\bigcup_{\gamma \leq \alpha < \omega_1} [\![O^y_\alpha]\!] \setminus [\![O^y_{\alpha+1}]\!]$ is a (clopen) partition of $[\![O^y_\gamma]\!] \setminus (\{y\} \times Z)$.
- $x \in [\![O^y_0]\!] \setminus (\{y\} \times Z)$

  $\Rightarrow$ there is a unique $\alpha < \omega_1$ with $x \in [\![O^y_\alpha]\!] \setminus [\![O^y_{\alpha+1}]\!]$.

Fix an injection $\pi : \omega_1 \to Z$, and let $z_\alpha = \pi(\alpha)$ for $\alpha < \omega_1$.
For $y \in Y_1$, $\gamma < \omega_1$, and an open $W \subseteq Z$, let

- $U(y, \gamma, W) = (\{y\} \times W) \cup \{[\![O^y_\alpha]\!] \setminus [\![O^y_{\alpha+1}]\!] : \alpha \geq \gamma, z_\alpha \in W\}$.

cont.

Then the topology on X is generated by the family:

$$\{[\![V]\!] : V \subseteq Y \text{ is open}\}$$
$$\cup \{U(y,\gamma,W) : y \in Y_1,\ \gamma < \omega_1,\ W \subseteq Z \text{ is open}\}.$$

- For $y \in Y_0$, the family $\{[\![V]\!] : V \subseteq Y$ is an open neighborhood of $y\}$ is a local base at y.
- For $y \in Y_1$ and $z \in Z$, the family $\{U(y,\gamma,W) : \gamma < \omega_1, W \subseteq Z$ is an open neighborhood of $z\}$ is a local base at $\langle y,z\rangle$.

Lemma 23

X is regular.

Lemma 24

X is points G_δ

1. If $y \in Y_0$, fix open sets $V_n \subseteq Y$ $(n < \omega)$ with $\{y\} = \bigcap_{n<\omega} V_n$. Then $\{y\} = \bigcap_{n<\omega} [\![V_n]\!]$.
2. Suppose $\langle y, z \rangle \in X$.

   Point: If $x \in [\![O^y_\alpha]\!] \setminus [\![O^y_{\alpha+1}]\!]$ and $z_\alpha \notin W \subseteq Z$, then $x \notin U(y, \gamma, W)$.

   Fix open sets $W_n \subseteq Z$ with $\{z\} = \bigcap_{n<\omega} W_n$, and a large $\gamma < \omega_1$. Then we have $\{\langle y, z \rangle\} = \bigcap_{n<\omega} U(y, \gamma, W_n)$.

Lemma 25

X is Lindelöf.

Let $\mathcal{U}$ be an open cover of X.

- For $y \in Y_0$, we can take an open $V_y \subseteq Y$ with $y \in V_y$ and $[\![V_y]\!] \subseteq U$ for some $U \in \mathcal{U}$.
- For $y \in Y_1$, since $\{y\} \times Z$ is homeomorphic to Z, we can find countably many $U_n \in \mathcal{U}$ with $\{y\} \times Z \subseteq \bigcup_{n<\omega} U_n$. Then we can find an open $V_y \subseteq Y$ with $\{y\} \times Z \subseteq [\![V_y]\!] \subseteq \bigcup_{n<\omega} U_n$.
- Y is Lindelöf and $\{V_y : y \in Y\}$ is an open cover of Y, hence there are countably many $y_n \in Y$ with $Y \subseteq \bigcup_{n<\omega} V_n$.
- The family $[\![V_n]\!]$ $(n < \omega)$ induces a countable subcover of $\mathcal{U}$.

How to construct a space satisfying the assumptions of Key lemma

$T \subseteq {}^{<\omega_2}2$: tree

A branch of T is a maximal chain of T.

Lemma 26

Suppose that there exists a tree $T \subseteq {}^{<\omega_2}2$ such that:

1. T has no branch of size ω_2.
2. T does not contain an isomorphic copy of Cantor tree ${}^{\leq\omega}2$.

Suppose T has κ cofinal branches. Then there exists a zero-dimensional Hausdorff Lindelöf space Y of size $\max\{|T|, \kappa\}$ such that Y satisfies the assumptions in Lemma 17.

Remark 27

Suppose CH. If T is an ω_1-Kurepa tree, then

1. T has more than 2^ω many branches,
2. T does not have a branch of size ω_2, and
3. T does not contain an isomorphic copy of Cantor tree.

Fact 28 (Todorcevic)

Suppose $\square(\omega_2)$ holds. Then there exists a tree $T \subseteq {}^{<\omega_2}$ such that

1. T is an ω_2-Aronszajn tree.
2. T does not contain an isomorphic copy of Cantor tree.

Corollary 29

Suppose CH"$+\omega_1$-Kurepa tree exists", or CH$+\square(\omega_2)$ holds, then there is a regular Lindelöf space with points G_δ and of size $> 2^\omega$.

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Constructing a space from tree

Fix a tree $T \subseteq {}^{\omega_2}2$.

For $\alpha < \omega_2$, let $T_\alpha = T \cap {}^{\alpha}2$, the α-th level of T.

Let B be the set of all branches of T.

Our space Y is $B \cup T$.

For $t \in T \cup B$, let $[t] = \{s \in T \cup B : t \subseteq s\}$.

Then the topology of Y is generated by the following family:

$$\{\{t\} : t \in T, \mathrm{cf}(\mathrm{dom}(t)) \leq \omega\}$$
$$\cup\{[\rho \restriction \xi] : \rho \in B, \xi < \mathrm{dom}(\rho), \mathrm{cf}(\xi) \leq \omega\}$$
$$\cup\{[\rho \restriction \xi] \setminus ([\rho^\frown 0] \cup [\rho^\frown 1]) : \rho \in \bigcup\{T_\alpha : \mathrm{cf}(\alpha) = \omega_1\}, \xi < \mathrm{dom}(\rho), \mathrm{cf}(\xi) \leq \omega\}.$$

It is easy to check that Y is T_1 and zero-dimensional.

- If $t \in T_\alpha$ with $\mathrm{cf}(\alpha) \le \omega$, then t is an isolated points of Y.
- If $t \in B$ with $\mathrm{cf}(\mathrm{dom}(t)) = \omega$, then fix an increasing cofinal sequence $\langle \xi_i : i < \omega \rangle$ with limit $\mathrm{dom}(t)$. Then $\{t\} = \bigcap_{i<\omega}[t \restriction \xi_i + 1]$, so $\psi(t, Y) = \omega$.
- If $t \in B \cup T$ with $\mathrm{cf}(\mathrm{dom}(t)) = \omega_1$, then $\psi(t, Y) = \omega_1$. Fix an increasing continuous sequence $\langle \xi_i : i < \omega_1 \rangle$ with limit $\mathrm{dom}(t)$. Then $[t \restriction \xi_i] \setminus ([t^\frown 0] \cup [t^\frown 1])$ $(i < \omega_1)$ are clopen, continuously decreasing, and $\{t\} = \bigcap_{i<\omega_1}([t \restriction \xi_i] \setminus ([t^\frown 0] \cup [t^\frown 1]))$.

Lemma 30

Y is Lindelöf.

Fix an open cover $\mathcal{U}$ of Y. Let T' be the set of all $t \in T \cup B$ such that $[t]$ cannot be covered by countable subfamily of $\mathcal{U}$. We see that T' is empty, then $\mathcal{U}$ has a countable subfamily which covers $[\emptyset] = Y$.

If T' is non-empty, we can check that T' is downward closed and branching. Hence we can define $f : {}^{<\omega}2 \to T'$ such that

- $\sigma \subseteq \tau \Rightarrow f(\sigma) \subseteq f(\tau)$.
- $f(\sigma^\frown 0) \neq f(\sigma^\frown 1)$.

Since T does not have an isomorphic copy of the Cantor tree, there is $\rho \in {}^{\omega}2$ such that $t^* = \bigcup_{n<\omega} f(\rho \restriction n) \notin T$. Then $t^* \in B$, and there is $O \in \mathcal{U}$ and $\xi < \mathrm{dom}(t^*)$ with $[t^* \restriction \xi] \subseteq O$. Then there is σ with $t^* \restriction \xi \subseteq \sigma$ and $[f(\sigma)] \subseteq O$, this contradicts to $f(\sigma) \in T'$.

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Forcing which adds a large Lindelöf space

Proposotion 31

Cohen forcing $\mathbb{C}$ (actually any c.c.c. forcing adding new reals) forces the statement that "there exists a regular Lindelöf space with points G_δ and of size $(2^{\omega_1})^V$.

Proof.

In $V^{\mathbb{C}}$, the tree $({}^{<\omega_1}2)^V$ does not contain an isomorphic copy of Cantor tree. □

When $L(X) > \omega$

Proposotion 32

Suppose $V = L$. For each regular cardinal κ, there is a regular Lindelöf space X with points G_δ, $L(X) = \kappa$, and $|X| = \kappa^{++}(> 2^\kappa)$.

Hence, under $V = L$, the inequality $|X| \leq 2^{L(X)+\psi(X)}$ does not hold.

Question 33

1. Is it consistent that that ZFC+"every regular Lindelöf space with points G_δ has cardinality $\leq 2^\omega$"?
2. Is it consistent that that ZFC+"every regular Lindelöf c.c.c. space with points G_δ has cardinality $\leq 2^\omega$"?

   (Gorelic's space satisfies the c.c.c.)
3. Is it consistent that that ZFC+"there is a regular Lindelöf space with points G_δ and of size $> 2^{\omega_1}$?

Thank you for your attention!

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