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Limit-Computable Categoricity of Computable Structures

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Computably and Relatively Computably Categorical Structures

Let A be a *computable* structure.

- A is *computably categorical* if for all computable $B \cong A$, there is a computable isomorphism f from A onto B.

- A is *relatively computably categorical* if for all $B \cong A$, there is an isomorphism f from A onto B, which is computable relative to the atomic diagram of B.

- A is relatively computably categorical $\Rightarrow$ A is computably categorical

Examples

- $(\mathbb{Q}, <)$ is computably categorical (back-and-forth argument).

  Not every isomorphism from $(\mathbb{Q}, <)$ to a computable structure is computable (not *computably stable*).

  $(\omega, <)$ is not computably categorical.

  The field $\mathbb{Q}$ is computably categorical.

- (Miller-Shoutens 2013; Kudinov-Lvov?)
  There is a computable computably categorical field of *infinite* transcendence degree.

Computably Categorical but Not Relatively

- (Goncharov 1977)
  There is a computable structure (in fact, a rigid graph) that is computably categorical, but *not* relatively computably categorical.

- (Hirschfeldt-Khoussainov-Shore-Slinko 2002)
  There are computable computably categorical, but *not* relatively computably categorical: partial orders, lattices, 2-step nilpotent groups, commutative semigroups, integral domains.

- (Hirschfeldt-Kramer-Miller-Shlapentokh 2015)
  There is a computable computably categorical algebraic field, which is *not* relatively computably categorical.

Linear Orderings

- (Goncharov-Dzgoev 1980, Remmel 1981)
  A computable linear ordering A is *computably categorical iff*
  A has only finitely many successor pairs *iff*
  A is relatively computably categorical.

Boolean Algebras

- (LaRoche 1977, Goncharov-Dzgoev 1980, Remmel 1981)
  A computable Boolean algebra B is *computably categorical iff*
  B has finitely many atoms *iff*
  B is relatively computably categorical.

Algebraic Fields with Splitting Algorithm

- (Miller-Shlapentokh 2015)
  A computable algebraic field F with a splitting algorithm is *computably categorical iff* the *orbit relation*, $\{(a.b) \in F^2 : (\exists h \in Aut(F))[h(a) = b]\}$, is decidable *iff* F is relatively computably categorical.

- F has a *splitting algorithm* if it is decidable which polynomials in $F[x]$ are irreducible.

$\mathfrak{Abelian}$ p-$\mathfrak{Groups}$ (p is a prime number)

- $(g \in G - \{0\}) \Rightarrow (\exists m \geq 1)[o(g) = p^m]$
  $\mathbb{Z}(p^n)$ the cyclic group of order p^n
  $\mathbb{Z}(p^\infty)$ the quasicyclic (Prüfer) p-group
  the set of rationals in $[0, 1)$ of the form $\frac{i}{p^k}$ with addition modulo 1

- Descending chain of subgroups defined recursively
  $G = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_\alpha \supsetneq G_{\alpha+1} \supsetneq \cdots \supsetneq G_{\lambda(G)}$
  $G_{\alpha+1} = pG_\alpha$
  α limit: $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$

- *length* of $G \ = \ \lambda(G) \ = \ \mu\lambda[G_\lambda = G_{\lambda+1}]$
  $G_\alpha - G_{\alpha+1} \ = \ $ elements of *height* α

- G a countable Abelian p-group
  $\Rightarrow \lambda(G)$ is a countable ordinal

- G a computable Abelian p-group
  $\Rightarrow \lambda(G) \leq \omega_1^{CK}$

- G *reduced* $\Leftrightarrow$ $G_{\lambda(G)} = \{0\}$

- G a computable reduced Abelian p-group
  $\Rightarrow \lambda(G) < \omega_1^{CK}$

- $D(G) = G_{\lambda(G)} \neq \{0\}$ (unique) *divisible* part

- $G = D(G) \oplus G_{reduced}$

- (Goncharov 1980, Smith 1981)

  A computable Abelian p-group *G* is *computably categorical iff* *G* is isomorphic to:

  (1) $\bigoplus_{\alpha} \mathbb{Z}(p^\infty) \oplus F$, where $\alpha \leq \omega$ and F is finite, *or*

  (2) $\bigoplus_{n} \mathbb{Z}(p^\infty) \oplus F \oplus \bigoplus_{\omega} \mathbb{Z}(p^k)$, where $n, k \in \omega$ and F is finite.

  A computable computably categorical Abelian p-group is relatively computably categorical.

Equivalence Structures

- (Calvert-Cenzer-Harizanov-Morozov 2006)

  A computable equivalence structure $A = (D, E)$ is *computably categorical iff* either
  (1) A finitely many finite equivalence classes, or
  (2) A has finitely many infinite classes, bounded character, and exactly one finite k with infinitely many classes of size k.

- The *character* of A:
  $\chi(A) = \{(k, n) : k, n > 0$ and A has $\geq n$ equivalence classes of size $k\}$
  Bounded character: k is bounded

- Every computable computably categorical equivalence structure is *relatively* computably categorical.

Trees and Nested Equivalences

- (Miller 2005)
  No computable tree $(T, \prec)$ of infinite height is computably categorical.

- (Lempp-McCoy-Miller-Solomon 2005)
  Every computable computably categorical tree of finite height is relatively computably categorical.

- A finitely nested equivalence structure $A = (D, E_1, \ldots, E_n)$:
  $(\forall a \in D)\,([a]_{E_1} \supseteq \cdots \supseteq [a]_{E_n})$ where $[a]_E$ is the E-equivalence class of a.

- (Leah Marshall, to appear)
  Every computable computably categorical finitely nested equivalence structure is relatively computably categorical.

Injection Structures

- $A = (D, f)$, where $f : D \to D$ is a $1-1$ function.
  For $a \in D$, the *orbit of* a (under f) is:

$$\mathcal{O}_f(a) = \{b \in D : (\exists n \in \omega)[f^n(a) = b \ \vee \ f^n(b) = a]\}$$

- Two types of *infinite orbits*:

  Z-orbits, which are isomorphic to $(\mathbb{Z}, Succ)$, in which every element is in $ran(f)$;

  ω-orbits, which are isomorphic to $(\omega, Succ)$ and have the form $\mathcal{O}_f(a) = \{f^n(a) : n \in \omega\}$ for some $a \notin ran(f)$.

- Hence, an injection structure is characterized by the number of orbits of size k for each finite k, and the number of orbits of types Z and ω.

- (Cenzer-Harizanov-Remmel 2014)

  A computable injection structure A is *computably categorical iff* A has finitely many infinite orbits *iff* A is relatively computably categorical.

Syntactic Approach to Ctaegorticity

- There is a syntactic condition that implies computable categoricity, and is equivalent to relative computable categoricity.
  The condition involves the existence of a certain Scott family.

- Let A be a *countable* structure.

  (*Scott Isomorphism Theorem*) There is an $L_{\omega_1\omega}$ sentence, called Scott sentence for A, the countable models of which are exactly the isomorphic copies of A.

  $L_{\omega_1\omega}$ language allows countable disjunctions and conjunctions.

  Scott sentence for A is derived from a family of $L_{\omega_1\omega}$ formulas defining the orbits (under automorphisms) of tuples in A.

Scott Families

Let A be a *countable* structure.

- A *Scott family for* A is a set Φ of $L_{\omega_1\omega}$ formulas, with a fixed finite tuple of parameters $\overline{c}$ in A, such that:

1. Each tuple $\overline{a}$ in A satisfies some $\psi(\overline{c}, \overline{x}) \in \Phi$, and

2. If $\overline{a}$, $\overline{b}$ are tuples in A (of the same length) satisfying the *same* formula $\psi(\overline{c}, \overline{x}) \in \Phi$, then there is an *automorphism* of A taking $\overline{a}$ to $\overline{b}$.

- (Goncharov 1975)

  A computable structure A is *relatively* computably categorical *iff* A has a c.e. Scott family of (finitary) existential formulas.

- (Cholak-Shore-Solomon 1999)

  There is a computably categorical (rigid) structure with *no* Scott family of *finitary* formulas.

Back-and-Forth Construction

- If a computable structure A has a c.e. Scott family Φ of existential formulas, then A is *relatively* computably categorical.

- *Proof sketch*
  $\overline{c}$ parameters; let $(A, \overline{c}) \cong (B, \overline{d})$.
  Will construct an isomorphism f computable in the atomic diagram of B.
  $f = \bigcup_s f_s, \ \ f_s \subseteq f_{s+1}$.
  Assume f_s maps $\overline{c}\hat{\ }\overline{a} \to \overline{d}\hat{\ }\overline{b}$,
  $q \in A$, where $q \notin \overline{c}\hat{\ }\overline{a}$.
  Find $\psi(\overline{c}, \overline{x}, y) \in \Phi$ and $r \in B$ such that

$$A \models \psi(\overline{c}, \overline{a}, q) \ \wedge \ B \models \psi(\overline{d}, \overline{b}, r).$$

  Let $f_{s+1}(q) = r$.

- A computable structure A is *relatively* computably categorical *iff* A has a c.e. Scott family of computable $\mathbf{\Sigma}_1$ formulas.

  Computable $\mathbf{\Sigma}_1$ formula: $\bigvee_{i \in I} \exists \overline{u_i}\, \theta_i(\overline{x}, \overline{u_i})$, I is c.e. and θ_i's quantifier-free

- $\mathfrak{Classification\ of\ Computable\ Formulas}$ (for computable ordinals α)
  A computable $\mathbf{\Sigma}_0$ or $\mathbf{\Pi}_0$ formula is a finitary quantifier-free formula.
  A computable $\mathbf{\Sigma}_\alpha$ formula, $\alpha > 0$, is a *c.e. disjunction*

  $$\bigvee_{i \in I} \exists \overline{u_i}\, \theta_i(\overline{x}, \overline{u_i}),$$

  where each θ_i is computable $\mathbf{\Pi}_\beta$ for some $\beta < \alpha$.
  A computable $\mathbf{\Pi}_\alpha$ formula, $\alpha > 0$, is a *c.e. conjunction*

  $$\bigwedge_{i \in I} \forall \overline{v_i}\, \psi_i(\overline{x}, \overline{v_i}),$$

  where each ψ_i is computable $\mathbf{\Sigma}_\beta$ for some $\beta < \alpha$.

- Computable $\mathbf{\Sigma}_2$ formula:

$$\bigvee_{j \in J} \exists \overline{u}_j \bigwedge_{i \in I} \forall \overline{v}_{ij} \theta_{ij}(\overline{x}, \overline{u}_j, \overline{v}_{ij}),$$

  where I and J are c.e. and θ_{ij}'s are quantifier-free.

- (Ash 1986)
  A relation defined in a computable structure A by a computable $\mathbf{\Sigma}_\alpha$ formula (computable $\mathbf{\Pi}_\alpha$ formula, respectively) is $\mathbf{\Sigma}^0_\alpha$ ($\mathbf{\Pi}^0_\alpha$, respectively).

Δ^0_α-Categoricity and Relative Δ^0_α-Categoricity

- Let α be a computable ordinal.

  (i) A is Δ^0_α-*categorical* if for all computable $B \cong A$, there is a Δ^0_α isomorphism f from A onto B.

  (ii) A is *relatively* Δ^0_α-*categorical* if for all $B \cong A$, there is an isomorphism f from A onto B, which is Δ^0_α relative to the atomic diagram of B.

- Δ^0_1-categorical: computably categorical.
  Δ^0_2-categorical: limit computably categorical or $\mathbf{0}'$-categorical, where $\mathbf{0}'$ is the Turing degree of the halting set.

- $\mathcal{A}$ is relatively Δ^0_n-categorical $\Rightarrow$ $\mathcal{A}$ is Δ^0_n-categorical

Equivalence of Semantic and Syntactic Conditions

- (Ash-Knight-Manasse-Slaman 1989, Chisholm 1990)

  A computable structure A is *relatively* Δ^0_α-categorical *iff*

  A has a c.e. Scott family of computable Σ_α formulas *iff*

  A has a Σ^0_α Scott family of computable Σ_α formulas
  (*A* has a *formally* Σ^0_α Scott family)

Examples

- $(\omega, <)$ is relatively Δ^0_2-categorical.
  Any computable equivalence structure is relatively Δ^0_3-categorical.
  Any computable injection structure is relatively Δ^0_3-categorical.

- (Lempp-McCoy-Miller-Solomon 2005)
  For every $n \geq 1$, there is a computable tree of finite height, which is Δ^0_{n+1}-categorical but not Δ^0_n-categorical.

- (Calvert-Harizanov-Knight-Quinn 2006)

  Assume that G is a computable reduced Abelian p-group.
  If $\lambda(G) = \omega \cdot n$, then G is not Δ^0_{2n-1}-categorical.

Extra Decidability on Categorical Structures

- (Goncharov 1975)
  Assume that A is 2-*decidable*. If A is computably categorical, then A is relatively computably categorical.

  A is n-*decidable* if $\mathbf{\Sigma}^0_n$ elementary diagram of A is computable.

- (Ash 1987)
  Let $\alpha > 1$ be a computable ordinal.
  Under some additional decidability on A, if A is $\mathbf{\Delta}^0_\alpha$-categorical, then A is relatively $\mathbf{\Delta}^0_\alpha$-categorical.

- (Kudinov 1996)
  There is a 1-*decidable* structure that is computably categorical, but *not* relatively computably categorical.

- (Cholak-Goncharov-Khoussainov-Shore 1999)
  There is a computable computably categorical structure A such that for every $a \in A$, the expanded structure (A, a) is *not* computably categorical.

- (T. Millar 1986)
  If a computably categorical structure A is 1-decidable, then any expansion of A by finitely many constants remains computably categorical.

- (Downey-Kach-Lempp-Turetsky 2013)
  Any 1-decidable computably categorical structure is relatively Δ^0_2-categorical.

Non-Relatively Δ^0_α-Categorical Structures

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon 2005)
  For every computable *successor* ordinal $\alpha = \beta + 1$, there is a computable structure that is Δ^0_α-categorical, but *not* relatively Δ^0_α-categorical.

- (Chisholm-Fokina-Goncharov-Harizanov-Knight-Quinn 2009)
  For every computable *limit* ordinal α, there is a computable structure that is Δ^0_α-categorical, but *not* relatively Δ^0_α-categorical.

- (Downey-Kach-Lempp-Lewis-Montalbán-Turetsky 2015)
  For every computable ordinal α, there is a computably categorical structure that is *not* relatively Δ^0_α-categorical.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon 2005)

  Let $\alpha \geq 2$ be a computable successor ordinal.
  There is a computable structure that is $\mathbf{\Delta}^0_\alpha$-categorical but
  not relatively $\mathbf{\Delta}^0_\alpha$-categorical.

- *Proof sketch.* (1) Relativize the proof for $\mathbf{\Delta}^0_1$ to $\mathbf{\Delta}^0_\alpha$.

  There is a rigid $\mathbf{\Delta}^0_\alpha$ directed graph G such that:

  (i) G has exactly one $\mathbf{\Delta}^0_\alpha$-isomorphic copy, up to $\mathbf{\Delta}^0_\alpha$ isomorphism,
  (ii) G does not have $\mathbf{\Sigma}^0_\alpha$ Scott family of (finitary) existential formulas.

- *Proof sketch continued.* (2) Code Δ^0_α directed graph G in a *computable* structure G^*, using a pair of structures B_0, B_1 such that B_0 codes $G \models a \to b$ and B_1 codes $G \models \neg(a \to b)$.

- B_0 and B_1 are *computable* structures, for which the standard *back-and-forth relations* $\leq_\beta$ for $\beta < \alpha$ are uniformly c.e.

- B_0 and B_1 satisfy the *same* infinitary Π_β sentences for $\beta < \alpha$.

- B_0 satisfies *some* computable Π_α sentence that is not true in B_1, and *vice versa.*

- B_0 and B_1 are uniformly relatively Δ^0_α-categorical.

- (3) Show that:

  (i) G^* is Δ^0_α-categorical
  (G had exactly one Δ^0_α-isomorphic copy, up to Δ^0_α isomorphism);

  (ii) G^* does not have formally Σ^0_α Scott family
  (G did not have Σ^0_α Scott family of (finitary) existential formulas).

Non-Relatively Δ^0_2-Categorical Tree

- (Fokina-Friedman-Harizanov-Turetsky, to appear)
  There is a Δ^0_2-categorical tree of finite height,
  which is not relatively Δ^0_2-categorical.

- *Proof sketch.* Build a computable tree $\mathcal{T}$.
  Diagonalize against all potential Scott families: consider all pairs $(\mathcal{X}, \overline{c})$, where $\mathcal{X}$ is a c.e. family of computable Σ_2 formulas and $\overline{c}$ is a finite tuple of elements from the domain of $\mathcal{T}$.
  Assure that every isomorphic computable tree is $\mathbf{0}'$-isomorphic to $\mathcal{T}$.

- *Open Problems:* Characterize computable relatively Δ^0_2-categorical trees of finite height.
  Characterize computable Δ^0_2-categorical trees of finite height.

Relatively Δ^0_2-Categorical Equivalence Structures

- (Calvert-Cenzer-Harizanov-Morozov 2006)
  A computable equivalence structure A is *relatively* Δ^0_2*-categorical iff*:
  (1) A has finitely many infinite equivalence classes, or
  (2) A has bounded character.

- *Open Problem:* Characterize computable Δ^0_2*-categorical* equivalence structures.

Khisamiev Functions

- A function $s : \omega^2 \to \omega$ is a Khisamiev *s-function* if for every i and t:
  (i) $s(i,t) \leq s(i,t+1)$, and the limit $m_i = \lim_{t\to\infty} s(i,t)$ exists.
  s is called Khisamiev *s_1-function* if, in addition:
  (ii) $m_0 < m_1 < \cdots < m_i < m_{i+1} < \cdots$

- Let A be a computable equivalence structure with *finitely* many infinite equivalence classes and infinite character $\chi(A)$.

  There exists a computable Khisamiev s-function with limits m_i such that:

  $$(k,n) \in \chi(A) \Leftrightarrow card(\{i : k = m_i\}) \geq n$$

  If $\chi(A)$ is unbounded, then there is a computable Khisamiev s_1-function such that A contains an equivalence class of size m_i for every i.

- A set K of pairs of nonzero natural numbers is simply called a *character* if for all $n > 0$ and k:

$$(k, n+1) \in K \Rightarrow (k, n) \in K$$

- Fix a $\mathbf{\Sigma}_2^0$ character K, and $r \in \omega$.

  If there is a computable Khisamiev s_1-function such that $(m_i, 1) \in K$ for all i, then there is a computable equivalence structure A with $\chi(A) = K$ and exactly r infinite equivalence classes.

- (Calvert-Cenzer-Harizanov-Morozov 2006)
  Let A be a computable equivalence structure with infinitely many infinite equivalence classes and with unbounded character, which has a computable Khisamiev s_1-function. Then A is *not* $\mathbf{\Delta}_2^0$-categorical.

Non-Relatively Δ^0_2-Categorical Equivalence Structure

- (Kach-Turetsky 2009)
  There is a computable Δ^0_2-categorical equivalence structure M, which is *not* relatively Δ^0_2-categorical.

  M has infinitely many infinite equivalence classes
  and unbounded character,
  but has no computable Khisamiev s_1-function,
  and has only finitely many equivalence classes of size k for any finite k.

Relatively Δ^0_2-Categorical Abelian p-Groups

- The *period* of a group *H* is $max\{o(h) : h \in H\}$ if finite, and ∞ otherwise.

- (Calvert-Cenzer-Harizanov-Morozov 2009)
  A computable Abelian p-group *G* is *relatively* Δ^0_2-*categorical iff*

  (1) *G* is isomorphic to $\bigoplus_{\alpha} \mathbb{Z}(p^{\infty}) \oplus H$, where $\alpha \leq \omega$ and *H* has finite period; *or*

  (2) all elements in *G* are of finite height
  (equivalently, reduced with $\lambda(G) \leq \omega$).

- A computable equivalence structure A is relatively $\mathbf{\Delta}^0_2$-categorical *iff*:
  (1) A has finitely many infinite equivalence classes, or
  (2) A has bounded character.

- Even for a group G of infinite period with
  only finitely many $\mathbb{Z}(p^\infty)$ components,
  G is not relatively $\mathbf{\Delta}^0_2$-categorical.

  This differs from equivalence structures,
  where each equivalence class is necessarily computable,
  but $D(G)$ need not be computable
  even when there is just one copy of $\mathbb{Z}(p^\infty)$.

Non-Relatively Δ^0_2-Categorical Abelian p-Group

- (Fokina-Friedman-Harizanov-Turetsky, to appear)
  There is a computable Δ^0_2-categorical Abelian p-group G, which is *not* relatively Δ^0_2-categorical.

- *Proof sketch.* $G \cong \bigoplus_\omega Z(p^\infty) \oplus \bigoplus_{k \in A} Z(p^k)$ for a suitable Δ^0_2 set A such that $D(G)$ is c.e.

- (Calvert-Cenzer-Harizanov-Morozov 2009)
  Let A be a computable group isomorphic to $\bigoplus\limits_{\alpha} \mathbb{Z}(p^\infty) \oplus H$, where all elements of H are of finite height. Then A is relatively Δ^0_3-categorical.

- *Open Problem*: Characterize computable Δ^0_2-categorical Abelian p-groups.

Δ^0_2-Categorical Injection Structures

- (Cenzer-Harizanov-Remmel 2014)
  A computable injection structure A is Δ^0_2-categorical *iff*
  A has finitely many orbits of type ω or finitely many orbits of type Z

  Every computable Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical.

Non-Relatively Δ^0_2-Categorical Torsion-Free Abelian Groups

- A *homogenous, completely decomposable,* abelian group G is a group of the form $\bigoplus_{i \in I} H$, where H is a subgroup of $(\mathbb{Q}, +)$.

- G is *computably categorical iff* G is of finite rank.

- (Fokina-Friedman-Harizanov-Turetsky, to appear)
  There is a computable, homogenous, completely decomposable, abelian group, which is Δ^0_2-categorical, but not relatively Δ^0_2-categorical.

- For P, a set of primes, $Q^{(P)}$ is the subgroup $(\mathbb{Q}, +)$ generated by $\{\frac{1}{p^k} \ : \ p \in P \wedge k \in \omega\}$.

- (Downey-Melnikov, to appear)
  A computable, homogenous, completely decomposable, abelian group G of infinite rank is Δ^0_2-*categorical iff* G is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is c.e. and the set (Primes $- P$) is semi-low.

- A set $S \subseteq \omega$ is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable from $\emptyset'$.

- (Fokina-Friedman-Harizanov-Turetsky, to appear)
  A computable, homogenous completely decomposable abelian group G of infinite rank is *relatively* Δ^0_2-*categorical iff* G is isomorphic to $\bigoplus_\omega Q^{(P)}$ where P is a *computable* set of primes.

Relatively Δ^0_2-Categorical Linear Orders

- (McCoy 2003)
  A computable linear order A is relatively Δ^0_2-categorical *iff*
  A is a sum of finitely many intervals, each of type

  $$m, \omega, \omega^*, \mathbb{Z}, \text{ or } n \cdot \eta,$$

  so that each interval of type $n \cdot \eta$ has a supremum and an infimum.

- *Open Problems:* Characterize computable Δ^0_2-*categorical* linear orders.

  Is there is a computable Δ^0_2-categorical linear order,
  which is *not* relatively Δ^0_2-categorical?

Relatively Δ^0_2-Categorical Boolean Algebras

- (McCoy 2003)
  A computable Boolean algebra $\mathcal{B}$ is *relatively* Δ^0_2-*categorical iff* $\mathcal{B}$ can be expressed as a finite direct sum of subalgebras

$$\mathcal{C}_0 \oplus \cdots \oplus \mathcal{C}_k$$

  where each $\mathcal{C}_k$ is either atomless, an atom, or a 1-atom.

- (Bazhenov 2014; Harris, to appear)
  Every computable Δ^0_2-categorical Boolean algebra is relatively Δ^0_2-categorical.

THANK YOU!