The spectrum of κ-maximal cofinitary groups

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- We denote by $S(\kappa)$ the group of all permutations on κ. A subgroup $\mathscr{G}$ of $S(\kappa)$ is said to be κ-cofinitary if each of its non-identity elements has less than κ-many fixed points. A κ-cofinitary group is said to be a κ-maximal cofinitary group (abbreviated κ-mcg), if it is maximal among the κ-cofinitary groups under inclusion.

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Thus $\mathfrak{a}(\kappa) = \min C_\kappa(\mathrm{mad})$ and $\mathfrak{a}_g(\kappa) = \min C_\kappa(\mathrm{mcg})$.

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- Brendle, Spinas and Zhang obtained an analogous result regarding mcg's: if C is as above (and GCH holds), then there is a ccc generic extension in which $C_\omega(\mathrm{mcg}) = C$.

Theorem (V.F.)

(GCH) Let κ be a regular infinite cardinal and let C be a closed set of cardinals such that

1. $\kappa^+ \in C$, $\forall \nu \in C(\nu \geq \kappa^+)$,
2. $[\kappa^+, |C|] \subseteq C$,
3. $\forall \nu \in C(cof(\nu) \leq \kappa \rightarrow \nu^+ \in C)$.

Then there is a generic extension in which cofinalities have not been changed and such that $C = C_\kappa(mcg)$.

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- excluding certain cardinals as possible values for κ-mcg.

- A mapping $\rho : B \to S(\kappa)$ *induces* a κ-cofinitary representation of $\mathbb{F}_B$ if the canonical extension of ρ to a homomorphism $\hat{\rho} : \mathbb{F}_B \to S(\kappa)$ has the property that every non-identity element of $\text{im}(\hat{\rho})$ has $< \kappa$-many fixed points.

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- Given a κ-cofinitary representation ρ with domain B and a set A s.t. $A \cap B = \emptyset$, we will add generically a family of κ-permutations $\{g_a\}_{a \in A}$ such that the group $\mathscr{G}(\rho, A)$ generated by $\mathrm{im}(\hat{\rho})$ and $\{g_a\}_{a \in A}$ is κ-cofinitary.

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- We will work with approximations to the new generators of size $< \kappa$, i.e. with sets $s \in [A \times \kappa \times \kappa]^{<\kappa}$. For each $a \in A$ we interpret $s_a = \{(\gamma, \beta) : (a, \gamma, \beta)\}$ as a partial approximation to the generator g_a.

- To describe arbitrary members of $\mathscr{G}(\rho, A)$, we work with the set of words $\widehat{W}_{A\cup B}$ (referred to as good words) on the alphabet $A \cup B$ which start and end with a different letter, or a power of a single letter. Every word on $A \cup B$ is a conjugate of such a good word.

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- Every approximation s to the new generators gives an approximation to the permutations corresponding to arbitrary words $w \in \widehat{W}_{A\cup B}$. This approximation is denoted $e_w[s, \rho]$ and is obtained by substituting every appearance of a letter b from B with $\rho(b)$ and every appearance of a letter $a \in A$ with s_a. We refer to $e_w[s, \rho]$ as the evaluation of w given s and ρ.

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- $s \supseteq t$, $F \supseteq E$ and
- for all $\alpha \in \kappa$ and $w \in E$, if $e_w[s,\rho](\alpha) = \alpha$ then already $e_w[t,\rho](\alpha)$ is defined and $e_w[t,\rho](\alpha) = \alpha$.

In case $B = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$.

- The above poset is clearly $< \kappa$-closed. In analogy with the Knaster property, we will say that a poset $\mathbb{P}$ has the κ-Knaster property, if in every collection of κ-many conditions from $\mathbb{P}$ there are κ many which are pairwise compatible.
- The poset $\mathbb{Q}^{\kappa}_{A,\rho}$ is κ^+-Knaster.

Some Basic Properties

Let $(s,F) \in \mathbb{Q}^{\kappa}_{A,\rho}$, $a \in A$.

1. Domain Extension Let $\alpha \in \kappa \backslash \mathrm{dom}(s_a)$. Then there is an index set $I = I_{a,\alpha}$ such that $|\kappa \backslash I| < \kappa$ and for all $\beta \in I$ $(s \cup \{(a,\alpha,\beta)\}, F)$ extends (s,F).
2. Range Extension Let $\beta \in \kappa \backslash \mathrm{ran}(s_a)$. Then there is an index set $J = J_{a,\beta}$ such that $|\kappa \backslash J| < \kappa$ and for all $\alpha \in J$ $(s \cup \{(a,\alpha,\beta)\}, F)$ extends (s,F).

The generic cofinitary representation

If G is $\mathbb{Q}^{\kappa}_{A,\rho}$-generic, then the mapping $\rho_G : A \cup B \to S(\kappa)$, which is defined by

- $\rho_G \restriction B = \rho$ and
- $\rho_G(a) = \bigcup\{s_a : \exists F(s,F) \in G\}$ for every $a \in A$,

induces a κ-cofinitary representation of $A \cup B$ which extends ρ.

Complete Embeddings and Quotients

Let $A_0 \subseteq A$ and $A_1 = A \backslash A_0$. Then:

- $\mathbb{Q}^\kappa_{A_0,\rho} \lessdot \mathbb{Q}^\kappa_{A,\rho}$,
- $\mathbb{Q}_{A,\rho} = \mathbb{Q}_{A_0,\rho} * \mathbb{Q}_{A_1,\rho_{\dot{G}}}$, where $\dot{G}$ is the canonical name for the $\mathbb{Q}_{A_0,\rho}$-generic filter.

Generic Hitting

Let $\rho : B \to S(\kappa)$ induce a κ-cofinitary representation and let $\sigma \in S(\kappa) \setminus \mathrm{im}(\hat{\rho})$ be such that $\langle \mathrm{im}(\hat{\rho}) \cup \{\sigma\} \rangle$ is κ-cofinitary. Let $a \notin B$. Then for every $\Omega \in \kappa$ the set of all $(s, F) \in \mathbb{Q}^{\kappa}_{\{a\},\rho}$ such that for some $\alpha > \Omega$ we have $s_a(\alpha) = \sigma(\alpha)$ is dense in $\mathbb{Q}^{\kappa}_{\{a\},\rho}$.

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As an immediate corollary we obtain that if G is $\mathbb{Q}^{\kappa}_{\{a\},\rho}$-generic, then there are κ-many α such that $\rho_G(a)(\alpha) = \sigma(\alpha)$. Thus in particular, in $V^{\mathbb{Q}^{\kappa}_{\{a\},\rho}}$ the group $\langle \mathrm{im}(\hat{\rho}_G) \cup \{\sigma\}\rangle$ is not κ-cofinitary.

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If $|A| > \kappa$ then $\mathbb{Q}^{\kappa}_{A,\rho}$ adds a maximal κ-cofinitary group.

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Proof:

Let G be $\mathbb{Q}^{\kappa}_{A,\rho}$-generic. Suppose that $\operatorname{im}\hat{\rho}_G$ is not maximal. Then there is a $\sigma \notin \operatorname{im}(\hat{\rho}_G)$ such that $\langle \operatorname{im}(\hat{\rho}_G) \cup \{\sigma\}\rangle$ is cofinitary. By the κ^+-c.c., there is $A_0 \subset A$, $|A_0| = \kappa$ such that $\sigma \in V[H]$ where $H = G \cap \mathbb{Q}_{A_0,\rho}$. Take any $a \in A \backslash A_0$. Then by the Generic Hitting Lemma in $V[G]$ we have that $\rho_G(a)(\alpha) = \sigma(\alpha)$ for κ-many α, which is a contradiction. $\square$

Theorem (V.F.)

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2. *if* $|C| \geq \kappa^+$ *then* $[\kappa^+, |C|] \subseteq C$,
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Then there is a generic extension in which cofinalities have not been changed and such that $C = C_\kappa(mcg)$.

Proof:

For each $\xi \in C$, let $I_\xi := \{(\gamma, \xi) : \gamma < \xi\}$ and let $I = \bigcup_{\xi \in C} I_\xi$. Let $\mathbb{P} = \prod_{\xi \in C} \mathbb{Q}^\kappa_{I_\xi}$ with supports of size $< \kappa$.

Lemma A

$\mathbb{P}$ is $< \kappa$-closed and κ^+-Knaster.

Proof of Lemma A:

Let $\{p_\alpha\}_{\alpha<\kappa^+}$ be given. Without loss of generality $\{\mathrm{supt}(p_\alpha)\}_{\alpha<\kappa^+}$ form a Δ-system with root R_0, where $|R_0|<\kappa$.

- For $p\in\mathbb{P}$ and $\xi\in\mathrm{supt}(p)$ recall that $p(\xi)\in\mathbb{Q}^\kappa_{I_\xi}$. That is $p(\xi)=(s^\xi,F^\xi)$ where $s^\xi\in[I_\xi\times\kappa\times\kappa]^{<\kappa}$ and $F^\xi\in[\widehat{W_{I_\xi}}]^{<\kappa}$. By $\mathrm{oc}(p(\xi))$ we denote the set of all letters of I_ξ which occur in (s^ξ,F^ξ). Thus in particular $\mathrm{oc}(p(\xi))\in[I_\xi]^{<\kappa}$.
- Since $\{\prod_{\xi\in R_0}\mathrm{oc}_A(p_\alpha)(\xi)\}_{\alpha<\kappa^+}$ are κ^+-many sets each of size $<\kappa$, by the Δ-system lemma we can assume that they form a Δ-system with root Δ where $\Delta=\prod_{\xi\in R_0}\Delta_\xi$.

Proof cnt.'d:

- For every α let $p_\alpha(\xi) = (s^{\alpha,\xi}, F^{\alpha,\xi})$. Then the sets $\{\prod_{\xi\in R_0} s^{\alpha,\xi}\restriction\Delta_\xi \times \kappa \times \kappa\}_{\alpha<\kappa^+}$ must coincide on a set of size κ^+, since $|\prod_{\xi\in R_0}(\Delta_\xi \times \kappa \times \kappa)| = \kappa$.
- Thus there is some $t = \prod_{\xi\in R_0} t^\xi$ such that for all $\xi \in R_0$ and (wlg) all $\alpha < \kappa^+$ we have that $s^{\alpha,\xi}\restriction\Delta \times \kappa \times \kappa = t_\xi$.
- This implies that $\prod_{\xi\in R_0}(s^{\alpha,\xi} \cup s^{\beta,\xi}, F^{\alpha,\xi} \cup F^{\beta,\xi})$ is a common extension of $p_\alpha\restriction R_0$ and $p_\beta\restriction R_0$. Thus we can find a subset if size κ^+ of pairwise compatible conditions. $\square$

Lemma B

In $V^{\mathbb{P}}$ there is a κ-mcg of size ξ for all$\xi \in C$.

Proof

Let $\xi_0 \in C$ and let $\mathscr{G}_{\xi_0}$ be the mcg added by $\mathbb{Q}^{\kappa}_{I_{\xi_0}}$. We will show that $\mathscr{G}_{\xi_0}$ remains maximal in $V^{\mathbb{P}}$. If not then there are a $p \in \mathbb{P}$ and a $\mathbb{P}$-name for a κ-cofinitary permutation τ such that $p \Vdash_{\mathbb{P}} \langle$"$\text{im}(\hat{\rho}_{\xi_0}) \cup \{\dot{\tau}\}\rangle$ is a κ-cofin. group". Wlg τ is a nice name. Since $\mathbb{P}$ is κ^+-cc there are κ-many antichains $\{B_\alpha\}_{\alpha\in\kappa}$ each of size κ, such that $\forall b \in B_\alpha \exists \beta_b \in \kappa$ with $b \Vdash_{\mathbb{P}} \dot{\tau}(\alpha) = \check{\beta}_b$.

Proof of Lemma B, cnt.'d:

- For $b \in B_\alpha$ let $K_{\alpha,b}$ denote the support of b. Then the set $C' = [(\bigcup_{\alpha\in\kappa, b\in B_\alpha} K_{\alpha,b}) \cup \text{supt}(p)] \setminus \{\xi_0\}$ is of size at most κ.
- Let $A_{\xi_0} = [\bigcup_{\alpha\in\kappa, b\in B_\alpha} \text{oc}(b(\xi_0))] \cup \text{oc}(p(\xi_0))$. That is A_{ξ_0} is the collection of all letters from I_{ξ_0} occurring in τ and p.
- Let $\bar{\mathbb{P}} = \prod_{\xi\in C'} \mathbb{Q}^\kappa_\xi$ with supports of size $< \kappa$ and $\bar{\mathbb{Q}} = \mathbb{Q}^\kappa_{A_{\xi_0}}$. Then $\mathbb{Q}^\kappa_{A_{\xi_0}} \lessdot \mathbb{Q}^\kappa_{I_{\xi_0}}$. Also p is in $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$ and τ is a $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$-name for a κ-cofinitary permutation. Furthermore

$$p \Vdash_{\bar{\mathbb{P}}\times\bar{\mathbb{Q}}} \text{“}\langle \text{im}(\hat{\rho}_{\xi_0}) \cup \{\dot{\tau}\}\rangle \text{ is a } \kappa\text{-cofin. group”}.$$

Proof, cnt.'d:

- Since A_{ξ_0} is of size at most κ and I_{ξ_0} is of size $\xi_0 > \kappa$, there is some $a \in I_{\xi_0} \setminus A_{\xi_0}$. Let G be $\bar{\mathbb{P}} \times \bar{\mathbb{Q}}$ generic and $p \in G$. Then in $V[G]$ by the Generic Hitting Lemma we have that $\Vdash_{\mathbb{Q}^{\kappa}_{I_{\xi_0} \setminus A_{\xi_0}, \rho_{A_{\xi_0}}}}$ "$\forall \Omega < \kappa \exists \beta > \Omega(\rho_{I_{\xi_0} \setminus A_{\xi_0}}(a)(\beta) = \tau(\beta))$".
- However $(\bar{\mathbb{P}} \times \mathbb{Q}^{\kappa}_{A_{\xi_0}}) * \mathbb{Q}^{\kappa}_{I_{\xi_0} \setminus A_{\xi_0}, \rho_{A_{\xi_0}}} = \bar{\mathbb{P}} \times (\mathbb{Q}^{\kappa}_{A_{\xi_0}} * \mathbb{Q}^{\kappa}_{I_{\xi_0} \setminus A_{\xi_0}, \rho_{A_{\xi_0}}}) = \bar{\mathbb{P}} \times \mathbb{Q}^{\kappa}_{I_{\xi_0}}$. Therefore $p \Vdash_{\bar{\mathbb{P}} \times \mathbb{Q}^{\kappa}_{I_{\xi_0}}}$ "$\forall \Omega < \kappa \exists \beta > \Omega(\rho_{\xi_0}(a)(\beta) = \dot{\tau}(\beta))$". Since $\mathbb{P} \times \bar{\mathbb{Q}}^{\kappa}_{I_{\xi_0}} \lessdot \mathbb{P}$, we reach a contradiction.

$\square$

Lemma C

In $V^{\mathbb{P}}$ for every $\lambda \notin C$ there are no κ-maximal cofinitary groups of size λ.

The proof follows very closely to Blass's proof regarding the spectrum of maximal almost disjoint families on ω and relies on homogeneity properties shared by the two constructions.

Proof of Lemma C:

We will show that for every $\lambda \notin C$, λ is not $OD({}^{\kappa}\kappa)$ definable. That is we will show that in $V[G]$ if $\langle X_\alpha \rangle_{\alpha\in\lambda}$ is a sequence of $OD({}^{\kappa}\kappa)$ definable sets which covers ${}^{\kappa}\kappa$, then there is a proper subsequence which also covers ${}^{\kappa}\kappa$. Fix such a sequence and for each α an ordinal Θ_α and a function $u_\alpha \in {}^{\kappa}\kappa$ such that in $V[G]$, X_α is the Θ_α-th set definable from u_α.

Let μ be the largest element of C below λ. Then $cof(\mu) \geq \kappa^+$. By GCH (in V) we have $\mu^\kappa = \mu$. Recursively we will define a sequence $\langle M_\gamma \rangle_{\gamma\in\kappa^+}$, where $|M_\gamma| \leq \mu$ for all γ, such that the X_α's with indexes in $\bigcup M_\gamma$ cover ${}^{\kappa}\kappa$. Let $M_0 := \emptyset$ and for γ limit, let $M_\gamma := \bigcup_{\delta<\gamma} M_\delta$.

For each $\alpha \in \lambda$ choose $J_\alpha \subseteq I = \bigcup_{\xi \in C} I_\xi$ of size κ such that for every p which is involved either in $\dot{u}_\alpha$ or in $\dot{\Theta}_\alpha$ and each ξ in the support of p we have that $\mathrm{oc}(p(\xi)) \subseteq J_\alpha$. Let S be the union of $\{I_\gamma : \gamma \in \mu \cap C\}$ and $\{J_\alpha : \alpha \in \lambda\}$. Then $|S| = \lambda$.

K-support

Let $K \subseteq S$ be of size μ such that $\bigcup_{\gamma \in \mu \cap C} I_\gamma \subseteq K$. A subset J of I such that $|J| = \kappa$ is called a K-support for the name $\dot{x}$ of a function in ${}^\kappa\kappa$ if

- for every p involved in $\dot{x}$ and every ξ in the support of p we have that $\mathrm{oc}(p(\xi)) \subseteq J$ and
- if $J \cap I_\gamma \backslash K \neq \emptyset$ then $|J \cap I_\gamma \backslash K| = \kappa$.

Since every $\gamma \in C \backslash (\mu \cup \{\mu\})$, $\gamma > \lambda$, we have $|I_\gamma \backslash S| = |I_\gamma \backslash K| = \gamma$. Thus whenever we are given a K as above and a name for a function in ${}^\kappa\kappa$, we can assume that it has a K-support.

Proof of Lemma C

Let $\mathscr{G}$ be the group of those permutations of I that map each I_γ into itself and that fixes all members of K. Then $\mathscr{G}$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each p to a condition $g(p)$ naturally defined from g and p.

Let $\mathscr{G}$ be the group of those permutations of I that map each I_γ into itself and that fixes all members of K. Then $\mathscr{G}$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each p to a condition $g(p)$ naturally defined from g and p.

More precisely: let $p \in \mathbb{P}$, $\xi \in \mathrm{supt}(p)$ and $p(\xi) = (s^\xi, F^\xi)$ where $s^\xi \in [I_\xi \times \kappa \times \kappa]^{<\kappa}$, $F^\xi \in [W_{I_\xi}]^{<\kappa}$. Then let $\mathrm{supt}(g(p)) := \mathrm{supt}(p)$. For $\xi \in \mathrm{supt}(p)$, let $g(p(\xi)) := (g(s^\xi), g(F^\xi))$ where $\mathrm{oc}(g(s^\xi)) = g(\mathrm{oc}(s^\xi))$ and for every $(\alpha, \xi) \in \mathrm{oc}(g(s^\xi)) = g(\mathrm{oc}(s^\xi))$ if $(\alpha_0, \xi) \mapsto (\alpha, \xi)$ then $[g(s^\xi)]_{(\alpha,\xi]} := s^\xi_{(\alpha_0,\xi)}$. Furthermore for a word $w \in F^\xi$ define $g(w)$ to be the word obtained by substituting every appearance of a letter $a = (\alpha, \xi)$ in w with $g(\alpha, \xi)$. Then let $g(F^\xi)$ be the set of all $g(w)$ for $w \in F^\xi$.

Let $\mathscr{G}$ be the group of those permutations of I that map each I_γ into itself and that fixes all members of K. Then $\mathscr{G}$ acts as a group of automorphisms on the notion of forcing $\mathbb{P}$ by sending each p to a condition $g(p)$ naturally defined from g and p.

- Note that each such automorphism g preserves not only maximal antichains, but also the forcing relation. In particular, if J is a support of a name $\dot{x}$,then $g(J)$ is a support of the name $g(\dot{x})$. If in addition g fixes all members of J, then it also fixes the name $\dot{x}$.
- If J is a support then its $\mathscr{G}$-orbit is determined by $J \cap K$ and $\bar{J} = \{\gamma \in C : J \cap I_\gamma - K \neq \emptyset\}$. That is, if J' is another support with $J' \cap K = J \cap K$ and $\bar{J}' = \bar{J}$, then there is $g \in \mathscr{G}$ with $g(J) = J'$.

This implies that there are only μ many orbits of supports. Indeed:

- Since $J \cap K$ is of size $\leq \kappa$ and $|K| = \mu = \mu^\kappa$, there are only μ possibilities for $J \cap K$.
- If $[\kappa^+, |C|] \neq \emptyset$, then $[\kappa^+, |C|] \subseteq C$. Thus in this case $|C| \leq \mu$.
- If $[\kappa^+, |C|] = \emptyset$, i.e. $|C| \leq \kappa$, then since $\mu \geq \kappa^+$ we have again $|C| \leq \mu$. Thus there are no more than $\mu^\kappa = \mu$ many possibilities for $\bar{J} \in [C]^{\leq\kappa}$.

Proof of Lemma C

For each $\mathscr{G}$-orbit of supports, fix a member J such that $J \cap S = J \cap K$. Such orbits are referred to as standard supports. For each fixed support J there are only $\kappa^\kappa = \kappa^+$ (by GCH in V) many names. Since $\mu \geq \kappa^+$, there are only μ-many names that have standard supports.

For each name $\dot{x}$ with a standard support, fix a set $A = A(\dot{x}) \in [\lambda]^{\leq\kappa} \cap V$ such that $\mathbb{P}$ forces "$(\exists \alpha \in \check{A})\dot{x} \in \dot{X}_\alpha$". Let $B = \bigcup\{A(\dot{x}) : \dot{x}$ has a standard support$\}$. Then $|B| \leq \mu$.

We will proceed with the successor step in the inductive definition of $\langle M_\sigma \rangle_{\sigma<\kappa^+}$. Let

$$K_\sigma = \bigcup_{\alpha \in M_\sigma} J_\alpha \cup \bigcup_{\gamma \leq \mu \cap C} I_\gamma.$$

Then $|K_\sigma| = \mu$. Let $M_{\sigma+1}$ be obtained from K_σ in the same way that B was obtained from K above. Then $|M_{\sigma+1}| \leq \mu$. Define $M = \bigcup_{\sigma \in \kappa^+} M_\sigma$ and $K = \bigcup_{\sigma \in \kappa^+} K_\sigma$.

Let $\dot{x}$ be a $\mathbb{P}$-name for a function in ${}^\kappa\kappa$. We will show that $\mathbb{P}$ forces that "$(\exists \alpha \in M) \dot{x} \in \dot{X}_\alpha$".

Let $J \subset I$ of size κ such that for every p involved in $\dot{x}$ and every ξ in the support of p we have $\mathrm{oc}(p(\xi)) \subseteq J$. Fix $\sigma < \kappa^+$ such that $J \cap K \subseteq K_\sigma$. For each $\gamma \in C$ such that $J \cap I_\gamma - K_\sigma \neq \emptyset$, we have that $\gamma > \lambda (> \mu)$. Then $|I_\gamma - K| = \lambda$. Thus enlarging J is necessary we can assume that it is a K_σ-support and $J \cap K \subseteq K_\sigma$.

Consider the group of all permutations of I which fix K_σ and map each I_γ to itself. There is $g \in \mathscr{G}$ such that $g(J)$ is a K_σ-standard support. Then neither J nor $g(J)$ meets $K_{\sigma+1} - K_\sigma$ and so there is a permutation h which agrees with g on J and with the identity map on $K_{\sigma+1} - K_\sigma$. In particular $h(J) = g(J)$ is standard and h leaves $K_{\sigma+1}$ pointwise fixed.

Since $h(\dot{x})$ has standard support $h(J)$, it is one of the μ names for which we chose a set $A = A(h(\dot{x}))$ to include in $M_{\sigma+1}$. Thus

$$\Vdash_{\mathbb{P}} \text{“}(\exists\alpha \in \check{A})h(\dot{x}) \in \dot{X}_\alpha\text{”},$$

which implies that

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{A}[h(\dot{x}) \text{ is in the } \dot{\Theta}_\alpha\text{th set ordinal-definable from } \dot{u}_\alpha]\text{”}.$$

However $A \subseteq M_{\sigma+1}$ and $\forall\alpha \in A(J_\alpha \subseteq K_{\sigma+1})$. Thus h fixes J_α pointwise, and so h fixes $\dot{\Theta}_\alpha$ and $\dot{u}_\alpha$. Therefore

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{A}[h(\dot{x}) \text{ is in the } h(\dot{\Theta}_\alpha)\text{th set ordinal-definable from } h(\dot{u}_\alpha)]\text{”}.$$

Now since h preserves the forcing relation, we have

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{A}[\dot{x} \text{ is in the } \dot{\Theta}_\alpha \text{th set ordinal-definable from } \dot{u}_\alpha]\text{”}.$$

Now since $M_{\sigma+1} \subseteq M$ we obtain that

$$\Vdash_{\mathbb{P}} \text{“}\exists\alpha \in \check{M}(\dot{x} \in \dot{X}_\alpha)\text{”},$$

which completes the proof that λ is not $OD({}^\kappa\kappa)$-definable.

Following standard notation, $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$ denotes the κ-Cohen poset, e.g. the poset of all partial functions from κ to κ of cardinality $< \kappa$ with extension relation superset.

Theorem (V.F.)

(GCH) There is a κ-Cohen indestructible κ-maximal cofinitary group.

Proof:
Let $\{\langle p_\xi, \dot{\tau}_\xi\rangle : \kappa \leq \xi < \kappa^+, \xi \in \mathrm{Succ}(\kappa^+)\}$ enumerate all pairs $\langle p, \tau\rangle$ where $p \in \mathrm{Fn}_{<\kappa}(\kappa,\kappa)$ and τ is a name for a κ-cofinitary permutation. Recursively we will construct a family $\{\rho_\xi\}_{\kappa\leq\xi<\kappa^+}$ of κ-cofinitary representations such that

1. for all ξ, $\rho_\xi : \xi \to S(\kappa)$,
2. for all $\eta < \xi$ $\rho_\eta = \rho_\xi \restriction \eta$, and
3. $\bigcup_{\kappa\leq\xi<\kappa^+} \rho_\xi : \kappa^+ \to S(\kappa)$ induces a cofinitary representation $\hat{\rho}$ such that $\mathrm{im}(\hat{\rho})$ is a κ-maximal cofinitary group, which is $\mathrm{Fn}_{<\kappa}(\kappa,\kappa)$-indestructible.

Proof cnt.'d:

Let ρ_κ be a cofinitary representation of κ given by $\mathbb{Q}_\kappa^\kappa$. Suppose for all $\xi : \kappa \leq \xi < \eta$, ρ_ξ has been defined and $\eta = \xi + 1$ for some ξ. Consider the pair $\langle p_\xi, \dot{\tau}_\xi \rangle$. If

- $p_\xi \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)}$ "$\tau_\xi \notin \mathrm{im}(\hat{\rho}_\xi)$", and
- $p_\xi \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)}$ "$\langle \mathrm{im}(\hat{\rho}_\xi) \cup \{\dot{\tau}_\xi\} \rangle$ is a κ-cofin. group",

then proceed as follows:

Let $q \leq p_\xi$. Then

$$q \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)} \text{“}\langle \mathrm{im}(\hat{\rho}_\xi) \cup \{\dot{\tau}_\xi\}\rangle \text{ is a cofin. group”}.$$

The Generic Hitting implies that if G is $\mathrm{Fn}_{<\kappa}(\kappa,\kappa)$-generic and $q \in G$, then in $V[G]$ for all $\Omega \in \kappa$ the set

$$D_{\dot{\tau}_\xi[G],\Omega} = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : \exists \alpha > \Omega(s(\alpha) = \tau_\xi[G](\alpha))\}$$

is dense. Thus for every $\Omega \in \kappa$ and every $(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi}$ there are $q' \leq_{\mathrm{Fn}_{<\kappa(\kappa,\kappa)}} q$, $\alpha > \Omega$ and $(s',F') \leq (s,F)$ such that

$$q' \Vdash_{\mathrm{Fn}_{<\kappa(\kappa,\kappa)}} \check{s}'(\alpha) = \dot{\tau}_\xi(\alpha).$$

Proof cnt.'d:

Therefore the set

$$D^q_\Omega = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : \exists \alpha > \Omega \exists q' \leq q (q' \Vdash s(\alpha) = \dot{\tau}_\xi(\alpha))\}$$

is dense in $\mathbb{Q}_{\{\xi\},\rho_\xi}$. Now let $G \subseteq \mathbb{Q}^\kappa_{\{\xi\},\rho_\xi}$ be a filter meeting the dense sets

- $D^{\text{domain}}_\alpha = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : \alpha \in \text{dom}(s)\}$,
- $D^{\text{range}}_\alpha = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : \alpha \in \text{range}(s)\}$,
- $D_w = \{(s,F) \in \mathbb{Q}_{\{\xi\},\rho_\xi} : w \in F\}$ and D^q_Ω,

where $\alpha, \Omega \in \kappa$, $q \leq_{\text{Fn}_{<\kappa}(\kappa,\kappa)} p_\xi$ and $w \in \widehat{W}_{\{\xi\}\cup\xi}$.

Since these are only κ many dense sets and the forcing notion $\mathbb{Q}^{\kappa}_{\{\xi\},\rho_\xi}$ is $<\kappa$-closed such a filter G exists. Then the mapping $\rho_{\xi+1}:\xi+1\to S(\kappa)$ where

- $\rho_{\xi+1}\restriction\xi=\rho_\xi$,
- $\rho_{\xi+1}(\xi)=\bigcup\{s:\exists F(s,F)\in G\}$

induces a κ-cofinitary representation extending ρ_ξ.

Claim

$p_\xi \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)}$ "$\forall\Omega \in \kappa \exists \alpha > \Omega(\tau_\xi(\alpha) = \rho_{\xi+1}(\xi)(\alpha))$".

Proof:

Suppose not. Then there are $q \leq p_\xi$ and $\Omega \in \kappa$ such that

$$q \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)} \text{“}\{\alpha : \dot{\tau}_\xi(\alpha) = \rho_{\xi+1}(\xi)(\alpha)\} \subseteq \check{\Omega}\text{”}.$$

Then let $(s, F) \in G \cap D^\Omega_q$. Then there are $\alpha > \Omega$ and $q' \leq_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)} q$ such that $q' \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)} \dot{\tau}_\xi(\alpha) = s(\alpha)$. It remains to observe that $\rho_{\xi+1}(\xi)(\alpha) = s(\alpha)$ and so we have reached a contradiction.

If ξ is a limit, then define $\rho_\xi := \bigcup_{\eta<\xi} \rho_\eta$ and note that $\rho_\xi : \xi \to S(\kappa)$ induces a cofinitary representation.

Indeed, let $w \in \mathbb{F}_\xi$. Then there is a good word $w' \in \widehat{W_\xi}$ such that for some $u \in W_\xi$ we have $w = u^{-1} w' u$. However in each of those words there are only finitely many letters involved and so there is $\eta < \kappa^+$ such that w, u, w' are in fact elements in W_η. Then $e_{w'}[\rho_\xi] = e_{w'}[\rho_\eta]$ and since by Inductive Hypothesis ρ_η induces a κ-cofinitary representation we have that the set of all fixed points of $e_{w'}[\rho_\xi]$ is of cardinality smaller than κ. However $|\mathrm{fix}(e_w[\rho_\xi])| = |\mathrm{fix}(e_{w'}[\rho_\xi])|$, which completes our argument.

With this the inductive construction of the sequence $\langle \rho_\xi \rangle_{\kappa \leq \xi < \kappa^+}$ is complete. Let $\rho := \bigcup_{\kappa \leq \xi < \kappa^+} \rho_\xi$.

Claim

im$(\hat{\rho})$ is a κ-mcg which is κ-Cohen indestructible.

Proof:
Let G be $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$-generic filter. Suppose

$$V[G] \vDash (\mathrm{im}(\hat{\rho}) \text{ is not a } \kappa \text{ maximal cof. group}).$$

Then

$$V[G] \vDash \exists \tau (\tau \notin \mathrm{im}(\hat{\rho}) \wedge \langle \mathrm{im}(\hat{\rho}) \cup \{\tau\} \rangle \text{ is a } \kappa \text{ cofin. group}).$$

Proof cnt'd.:

Therefore there is $p \in G$ and a $\mathrm{Fn}_{<\kappa}(\kappa, \kappa)$-name for a cofinitary permutation $\dot{\tau}$ such that

$$p \Vdash_{\mathrm{Fn}_{<\kappa}(\kappa,\kappa)} (\tau \notin \mathrm{im}(\hat{\rho}) \wedge \langle \mathrm{im}(\hat{\rho}) \cup \{\dot{\tau}\}\rangle \text{ is a } \kappa\text{-cofin. group}).$$

There is $\xi : \kappa \leq \xi < \kappa^+$, successor such that $\langle p, \tau\rangle = \langle p_\xi, \tau_\xi\rangle$. Then by construction

$$p \Vdash \forall \Omega \exists \alpha > \Omega(\rho(\xi + 1)(\alpha) = \dot{\tau}(\alpha)),$$

which is a contradiction. □

Theorem (V.F.)

(GCH) Let $\kappa^{++} \leq \lambda$ be regular uncountable cardinals and let $\mathbb{P} = Fn_{<\kappa}(\lambda \times \kappa, \kappa)$. Then in $V^{\mathbb{P}}$ every κ-maximal cofinitary group is either of size κ^+ or of size $2^\kappa = \lambda$.

An isomorphism of names argument shows that in the generic extension there are no κ-maximal cofinitary groups of size μ, where $\kappa^+ < \mu < 2^\kappa$.

Let $\mathscr{C}$ denote either of the following sets: set of all κ-maximal cofinitary groups, the set of κ-maximal almost disjoint families, the set of κ-almost disjoint permutations on ${}^\kappa\kappa$, the set on κ-almost disjoint functions on ${}^\kappa\kappa$. Then:

Theorem (V.F.)

(GCH) Let κ be a regular uncountable cardinal and let C be a closed set of cardinals such that

1. $\kappa^+ \in C, \forall \nu \in C(\nu \geq \kappa^+)$,
2. $[\kappa^+, |C|] \subseteq C$ *and*
3. $\forall \nu \in C(cof(\nu) \leq \kappa \to \nu^+ \in C)$.

Then there is a generic extension in which cofinalities have not been changed and such that $C = \{|\mathscr{G}| : \mathscr{G} \in \mathscr{C}\}$.

Theorem (V.F., S.D. Friedman)

Assume GCH. Let E be an Easton index function and let $\mathbb{P} = \mathbb{P}(E)$ be the Easton product. Then in $V^{\mathbb{P}}$ for every $\kappa \in \mathrm{dom}(E)$ we have that $\mathfrak{a}(\kappa) = \mathfrak{a}_g(\kappa) < \mathfrak{d}(\kappa)$.

Theorem (V.F., S.D. Friedman)

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Can we control the spectra of κ-mad families (resp. κ-m.c.g.) globally?

- The question of obtaining an optimal set of conditions on the potential spectrum of m.c.g. and also m.a.d. families, even on ω is still open.

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- The question of obtaining an optimal set of conditions on the potential spectrum of m.c.g. and also m.a.d. families, even on ω is still open. In a recent paper S. Shelah and O. Spinas that the requirements $\aleph_1 \in C$ and $\forall\lambda \in C(\mathrm{cof}(\lambda) = \omega \to \lambda^+ \in C)$ in Blass's theorem are not necessary.
- An analogous weakening on the requirements which we impose on the spectrum of κ-maximal cofinitary groups is of interest.
- There are still many open questions regarding the possible sizes of κ-mad families and κ-maximal cofinitary groups. For example, it is not known if consistently $\mathrm{cof}(\mathfrak{a}(\kappa)) = \kappa$, neither if consistently $\mathrm{cof}(\mathfrak{a}_g(\kappa)) = \kappa$.

Thank you!