Towards the Effective Descriptive Set Theory

Victor Selivanov

A.P. Ershov Institute of Informatics Systems
Siberian Branch Russian Academy of Sciences

Sets and Computations, NUSingapore, 13.04.2015

Introduction

Classical DST deals with hierarchies in Polish spaces. It exists for more than a century already, is currently a highly developed part of mathematics with numerous applications. Theoretical Computer Science, in particular Computable Analysis motivated an extension of the classical DST to non-Hausdorff spaces, a noticeable progress was achieved for the ω-continuous domains and quasi-Polish spaces.
TCS especially needs an effective version of DST for effective versions of the mentioned spaces. A lot of work in this direction was done within Classical Computability Theory but only for the space ω and the Baire space $\mathcal{N}$.

Introduction

For a systematic work to develop the effective DST for effective Polish spaces see e.g. [Mo09]. There was also some work on the effective DST for effective domains and approximation spaces [Se06, Se08, BG12].
In this work we attempt to make a next step towards the "right" version of effective DST. The task seems non-trivial since even the recent search for the "right" effective versions of topological spaces for Computable Analysis resulted in proliferation of different notions of effective spaces of which it is quite hard to choose really useful ones.

Introduction

We prove effective versions of some classical results about measurable functions (in particular, any computable Polish space is an effectively open effectively continuous image of the Baire space, and any two perfect computable Polish spaces are effectively Borel isomorphic). We derive from this extensions of the Suslin-Kleene theorem, and of the effective Hausdorff theorem for the computable Polish spaces (this was recently established by Becher and Grigorieff with a different proof) and for the computable omega-continuous domains (this answers an open question from the paper by Becher and Grigorieff).

Wcb$_0$-spaces

By *weakly computable cb$_0$-space (wcb$_0$-space)* we mean a pair (X, τ) where X is a non-empty set of points and $\tau : \omega \to P(X)$ is a numbering of a base of a T_0-topology in X such that $\emptyset, X \in \tau[\omega]$, and for some computable functions f, g we have $\tau_x \cap \tau_y = \tau_{f(x,y)}$ and $\bigcup \tau[W_x] = \tau_{g(x)}$ (where $\{W_n\}$ is the standard numbering of c.e. sets. For a function $f : X \to Y$ and a set $A \subseteq X$, $h[A]$ denote the image $\{h(a) \mid a \in A\}$. Informally, $\tau[\omega]$ is a collection of open sets which is rich enough to have the usual closure properties of open sets effectively.

Wcb$_0$-spaces

If (X,τ) is a wcb$_0$-space then any non-empty subset Y of X has the induced structure τ_Y of wcb$_0$-space defined by $\tau_Y(n) = Y \cap \tau(n)$. For wcb$_0$-spaces (X,ξ) and (Y,η), by an *effective embedding of X into Y* we mean an injection $f : X \to Y$ such that $\lambda n.f[\xi] \equiv \eta_{f[X]}$. Obviously, f is an effective homeomorphism between (X,ξ) and $(f[X], \eta_{f[X]})$.
As morphisms between wcb$_0$-spaces (X,ξ) and (Y,η) we use *effectively continuous functions*, i.e. functions $f : X \to Y$ such that the numbering $\lambda n.f^{-1}(\eta_n)$ is reducible to ξ (in particular, $f^{-1}(A) \in \xi[\omega]$ whenever $A \in \eta[\omega]$).

Wcb_0-base structures

By *wcb_0-base structure* we mean a pair (X, β) where X is a non-empty set of points and $\beta : \omega \to P(X)$ is a numbering of a base of a T_0-topology in X such that there is a c.e. sequence $\{A_{ij}\}$ with $\beta_i \cap \beta_j = \bigcup \beta[A_{ij}]$ for all $i, j \geq 0$. Any wcb_0-space is a wcb_0-base structure, and any wcb_0-base structure (X, β) induces the wcb_0-space (X, β^*) where $\beta^*(n) = \bigcup \beta[W_n]$. We say that a wcb_0-base structure (X, β) induces a wcb_0-space (X, τ) if $\tau \equiv \beta^*$.

C.e. cb$_0$-spaces

By *c.e. cb$_0$-space* we mean a wcb$_0$-space (X, τ) such that the predicate $\tau n \neq \emptyset$ is c.e. The notion of a c.e. cb$_0$-base structure is obtained by a similar strengthening of the notion of wcb$_0$-base structure. Note that if (X, β) is a c.e. cb$_0$-base structure then (X, β^*) is a c.e. cb$_0$-space. Similar spaces were introduced and studied in [GSW07, KK08, Se08] under different names. For such spaces (and for wcb$_0$-spaces) one can naturally define the notions of computable function and show that the computable functions coincide with the effectively continuous ones.

Many popular spaces (e.g., the discrete space ω of naturals, the space of reals $\mathbb{R}$, the domain $P\omega$, the Baire space $\mathcal{N} = \omega^\omega$, the Cantor space $\mathcal{C} = 2^\omega$ and the Baire domain $\omega^{\leq\omega} = \omega^* \cup \omega^\omega$ of finite and infinite sequences of naturals) are c.e. cb$_0$-spaces.

Computable Polish spaces

Any computable metric space (X, d, ν) [We00] gives rise to a c.e. cb_0-base structure (X, β) were $\beta_{\langle m,n\rangle} = B(\nu_m, \varkappa_m)$ is the basic open ball with center ν_m and radius $\varkappa_m$ ($\varkappa$ is a computable numbering of the rationals).

By *computable Polish space* we mean a c.e. cb_0-space (X, τ) induced by a computable complete metric space (X, d, ν), i.e. $\tau \equiv \beta^*$. Most of the popular Polish spaces are computable.

Computable ω-continuous domains

By *computable ω-continuous domain* [AJ94] we mean a pair $(X, \{b_n\})$ where X is an ω-continuous domain and $\{b_n\}$ is a numbering of a (domain) base in X modulo which the approximation relation $\ll$ is c.e. Any computable ω-continuous domain $(X, \{b_n\})$ gives rise to a c.e. cb_0-base structure (X, β) where $\beta_n = \{x \mid b_n \ll x\}$. Most of the popular ω-continuous domains are computable.

As we will see below, both computable Polish spaces and computable ω-continuous domains have some attractive effective DST-properties. In contrast, arbitrary c.e. cb_0-spaces (though certainly of interest to Computable Analysis) seem too general to admit a reasonable effective DST.

Computable quasi-Polish spaces?

Thus, it makes sense to search for a subclass of c.e. cb$_0$-spaces with good effective DST that contains both computable Polish spaces and computable ω-continuous domains. A similar problem in classical DST was resolved by M. de Brecht who suggested the important notion of a quasi-Polish space, so it makes sense to search for a natural effective version of quasi-Polish spaces. A reasonable candidate was suggested in [BG12].

A *convergent approximation space* is a triple $(X, \mathcal{B}, \ll)$ consisting of a T_0-space X and a binary relation $\ll$ on a basis $\mathcal{B}$ such that for all $U, V, T \in \mathcal{B}$: $U \ll V$ implies $V \subseteq U$, $U \subseteq T$ and $U \ll V$ imply $T \ll V$, for any $x \in U$ there is $W \in \mathcal{B}$ with $x \in W \gg U$, any sequence $U_0 \ll U_1 \ll \cdots$ is a neighborhood basis of some point.

Effective convergent approximation spaces

By *effective convergent approximation space* we mean a triple $(X, \beta, \ll)$, $\beta : \omega \to P(X)$, where $(X, \beta[\omega], \ll)$ is a convergent approximation space such that the relation $\beta_m \ll \beta_n$ is c.e. and any β_n is non-empty. In particular, (X, β) is a c.e. cb_0-base structure. We immediately obtain the following effectivization of Proposition 3.5 in [BG12]:

Proposition

Computable Polish spaces and computable ω-continuous domains maybe naturally considered as effective convergent approximation spaces.

Effective hierarchies

In any wcb$_0$-space (X, ξ) one can naturally define [Se08] effective versions of the classical hierarchies of DST [Mo09, Ke95]. Following a tradition of DST, we denote levels of the effective hierarchies in the same manner as levels of the corresponding classical hierarchies, using the lightface letters Σ, Π, Δ instead of the boldface $\mathbf{\Sigma}, \mathbf{\Pi}, \mathbf{\Delta}$ used for the classical hierarchies.

Effective Borel hierarchy

First let us sketch the definition of the effective Borel hierarchy. We start with the numbering $\xi_W(n) = \bigcup \xi(W_n)$ of the effective open sets in X. Let $\beta : \omega \to P(X)$ be the numbering of finite Boolean combinations of effective open sets induced by ξ_W and the Gödel numbering of Boolean terms. *Finite effective Borel hierarchy* in (X, ξ) is the sequence $\{\Sigma^0_n(X, \xi)\}_{n<\omega}$ defined as follows: $\Sigma^0_0(X, \xi) = \{\emptyset\}$; $\Sigma^0_1(X, \xi)$ is the class of effective open sets equipped with the numbering ξ_W; $\Sigma^0_2(X, \xi)$ is the class of sets $\bigcup \beta(W_x)$, $x \geq 0$, equipped with the numbering induced by W; $\Sigma^0_n(X, \xi)$ $(n \geq 3)$ is the class of sets $\bigcup \gamma(W_x)$, $x \geq 0$, equipped with the numbering induced by W, where γ is the numbering of $\Pi^0_{n-1}(X, \xi)$ induced by the numbering of $\Sigma^0_{n-1}(X, \xi)$ (which exists by induction).

Effective Borel hierarchy

The transfinite extension of $\{\Sigma^0_n(X,\xi)\}_{n<\omega}$ is also defined in a natural way. In place of ω_1 in classical DST one has to take the first non-computable ordinal ω_1^{CK}. In fact, to obtain reasonable effectivity properties one should enumerate levels $\Sigma^0_{(a)}$ of the transfinite hierarchy not by computable ordinals $\alpha < \omega_1^{CK}$ but rather by their names $|a|_O = \alpha$ in the well-known Kleene notation system $(O;<_O)$ ($a \mapsto |a|_O$ is a surjection from $O \subseteq \omega$ onto ω_1^{CK}). Levels of the transfinite version are defined in the same way as for the finite levels, using the effective induction along the well-founded set $(O;<_O)$. In this way we obtain the *effective Borel hierarchy* $\{\Sigma^0_{(a)}(X,\xi)\}_{a\in O}$.
In fact, the effective Borel hierarchy is extensional.

Effective Hausdorff hierarchy

For every ordinal α, define the operation D_α sending sequences of sets $\{A_\beta\}_{\beta<\alpha}$ to sets by

$$D_\alpha(\{A_\beta\}_{\beta<\alpha}) = \bigcup\{A_\beta \setminus \cup_{\gamma<\beta} A_\gamma | \beta < \alpha,\ r(\beta) \neq r(\alpha)\}.$$

For all ordinals α and classes of sets $\mathcal{C}$, let $D_\alpha(\mathcal{C})$ be the class of all sets $D_\alpha(\{A_\beta\}_{\beta<\alpha})$, where $A_\beta \in \mathcal{C}$ for all $\beta < \alpha$.
The *effective Hausdorff hierarchy* $\{\Sigma^{-1,\alpha}_{(a)}(X,\xi)\}_{a\in O}$, over $\Sigma^0_\alpha(X,\xi)$ is defined as follows: $\Sigma^{-1,\alpha}_{(a)}$ is the class of sets of the form $D_{|a|}(\{A_b\}_{b<_O a})$, where $\{A_b\}_{b<_O a}$ ranges over the uniform sequences of Σ^0_α-sets (naturally identified with sequences $\{A_\beta\}_{\beta<|a|}$). WARNING: the effective Hausdorff hierarchy is not extensional.

Effective Luzin hierarchy

The *effective Luzin hierarchy* is the family of pointclasses $\{\Sigma^1_n\}_{n<\omega}$ defined by induction as follows:
$\Sigma^1_0(X,\xi) = \Sigma^0_2(X,\xi)$,
$\Sigma^1_{n+1}(X,\xi) = \{pr_X(A) \mid A \in \Pi^1_n(\mathcal{N} \times X)\}$
where $pr_X(A)$ is the projection of A along the $\mathcal{N}$-axis.
In this way we obtain the sequence $\{\Sigma^1_n(X,\xi)\}$ of pointclasses in any wcb$_0$-space (X,ξ).

Properties of effective hierarchies

The introduced hierarchies have many properties well known in particular cases from effective DST [Ro67, Mo09]: the natural inclusions of levels of any given hierarchy, the mutual inclusions between levels of different hierarchies, the closure of any level under certain set-theoretic operations and under preimages of effectively continuous functions. For a future reference, we only give an example of such a property related to subspaces.

Properties of effective hierarchies

Proposition

Let (X,τ) be a wcb$_0$-space, (Y,τ_Y) its subspace, and Γ a level of an introduced hierarchy. Then $\Gamma(Y)=\{Y\cap A\mid A\in\Gamma(X)\}$.

We also give the following effective version of a result in [Br13].

Proposition

For any wcb$_0$-space (X,τ), the equality relation $=_X$ on X is in $\Pi^0_2(X\times X)$.

Main notions

For levels Γ, E of the effective Borel hierarchy and for any wcb_0-spaces X, Y, let $\Gamma E(X, Y)$ (resp. $\Gamma E[X, Y]$) denote the class of functions $f : X \to Y$ such that $f^{-1}(B) \in E(X)$ for each $B \in E(Y)$ effectively in B, (resp. $f[A] \in E(Y)$ for each $A \in \Gamma(X)$ effectively in A). In the case $\Gamma = E$ we abbreviate $\Gamma E(X, Y)$ to $\Gamma(X, Y)$ and $\Gamma E[X, Y]$ to $\Gamma[X, Y]$.
The introduced notions are effective versions of the corresponding notions from [MSS12] and include some notions already considered in Computable Analysis (see e.g. [We00, Bra05]). In particular, $\Sigma^0_1(X, Y)$ is the class of effectively continuous functions, $\Sigma^0_1[X, Y]$ is the class of effectively open functions, $\Sigma^0_2\Sigma^0_1(X, Y)$ is the class of effectively Σ^0_2-measurable (or effective Baire class 1) functions.

First result

Our first result is an effective version of the classical fact that any Polish space is a continuous open image of the Baire space [SG09, Theorem 1.3.7]:

Theorem

Let X be a computable Polish space or a computable ω-continuous domain. Then there exist functions $f : \mathcal{N} \to X$ and $s : X \to \mathcal{N}$ such that $f \circ s = id_X$, $f \in \Sigma^0_1(\mathcal{N}, X) \cap \Sigma^0_1[\mathcal{N}, X]$, and $s \in \Sigma^0_2\Sigma^0_1(X, \mathcal{N}) \cap \Pi^0_2[X, \mathcal{N}]$.

First result

Proof sketch. Let $(X, \beta, \ll)$ be the effective convergent approximation space for X from the proof of Proposition 1. Since the relation "$\beta_m \ll \beta_n$" is c.e., there is a computable function $g : \omega^+ \to \omega$ such that $g(n) = n$ for each $n < \omega$ and $\{g(\sigma n) \mid n < \omega\} = \{m \mid \beta_{g(\sigma)} \ll \beta_m\}$ for each $\sigma \in \omega^+$. In particular, for the sets $U_\sigma := \beta_{g(\sigma)}$ we then have $X = \bigcup_n U_n$, $U_\sigma = \bigcup_n U_{\sigma n}$, and $U_\sigma \ll U_{\sigma n}$.

First result

For any $p \in \mathcal{N}$, let $f(p) \in X$ be the unique element with the neighborhood base $\{U_{p[n+1]}\}$ [BG12]. Note that if X is a computable Polish space then $f(p) = lim_n x_n$ (where x_n is the center of the ball $U_{p[n+1]}$), and if X is a computable ω-continuous domain then $f(p) = sup\{b_{g(p)[n+1]} \mid n < \omega\}$ (obviously, $b_{g(p[1])} \ll b_{g(p[2])} \ll \cdots$). Therefore, $f : \mathcal{N} \to X$ is computable, hence $f \in \Sigma^0_1(\mathcal{N}, X)$.

First result

For any $x \in X$, define $p = s(x) \in \mathcal{N}$ as follows. If X is a computable Polish space then (by induction on i) $p(i) := \mu n(x \in U_{p[i]n})$, and if X is a computable ω-continuous domain then $p(i) := \mu n(x \in U_{p[i]n} \wedge \forall j < i(b_j \ll x \to b_j \in U_{p[i]n}))$. Then we clearly have $f \circ s = id_X$, in particular f is surjective (in the "Polish case" this is obvious while in the "ω-continuous case" the second conjunction summond guarantees that $x = sup\{b_{g(s(x))[n+1]} \mid n < \omega\} = f(s(x)))$. The same argument shows that $f[\sigma \cdot \mathcal{N}] = U_\sigma$ for each $\sigma \in \omega^+$, hence $f \in \Sigma^0_1[\mathcal{N}, X]$. □

Effective retracts

For wcb_0-spaces X and Y, we say that X is an *effective retract* of Y iff there exist effectively continuous functions $s : X \to Y$ (called a section) and $r : Y \to X$ (called a retraction) such that $r \circ s = id_X$. We will use the following

Effective retracts

Proposition

Let (X, ξ) and (Y, η) be wcb_0-spaces and Γ a non-zero level of the effective Borel hierarchy.

1. *If $f : X \to Y$ is an effective embedding with $f[X] \in \Gamma(Y)$ then $f \in \Gamma[X, Y]$.*
2. *If X is an effective retract of Y via a section-retraction pair (s, r) then $s \in \Pi^0_2[X, Y]$.*

Second result

Our second result is an effective version of the classical fact that any perfect Polish space contains a homeomorphic copy of the Cantor (or Baire) space (see e.g. [Ke95]):

Theorem

Let X be a computable Polish space with a perfect basic open ball, or a computable reflective ω-algebraic domain, or a computable 2-reflective ω-algebraic domain. Then there exists an effective embedding $g : \mathcal{C} \to X$ such that $g \in \Sigma^0_1(\mathcal{C}, X) \cap \Pi^0_2[\mathcal{C}, X]$. The same holds with $\mathcal{N}$ in place of $\mathcal{C}$.

Second result

Proof sketch. In the "Polish case", there is a basic open ball B which is perfect in the subspace topology. Clearly, there is a computable sequence $\{B_\sigma\}_{\sigma\in 2^*}$ of basic open balls such that $B_\emptyset = B$, $B_{\sigma 0}\cap B_{\sigma 1}=\emptyset$, the closure $\bar{B}_{\sigma i}$ of $B_{\sigma i}$ is contained in B_σ, and $diam(B_{\sigma i}) \leq diam(B_\sigma)/2$.
For any $p\in\mathcal{C}$, let $g(p)$ be the unique element of $\bigcap_n B_{p[n]}$. Then $g:\mathcal{C}\to X$ is a computable topological embedding, hence $g\in\Sigma^0_1(\mathcal{C},X)$. Since

$$g[\mathcal{C}] = \bar{B}_\emptyset \cap (\bar{B}_0\cup\bar{B}_1)\cap(\bar{B}_{00}\cup\bar{B}_{01}\cup\bar{B}_{10}\cup\bar{B}_{11})\cap\cdots$$

and $\bar{B}_\sigma\in\Pi^0_1(X)$ uniformly in σ, $g[\mathcal{C}]\in\Pi^0_1(X)$. By Proposition 4, $g\in\Pi^0_1[\mathcal{C},X]$, and also $g\in\Pi^0_2[\mathcal{C},X]$.

Second result

Since there is an effective embedding $h : \mathcal{N} \to \mathcal{C}$ with $h[\mathcal{N}] \in \Pi^0_2(\mathcal{C}])$, $h \in \Pi^0_2[\mathcal{N}, \mathcal{C}]$ by Proposition 4. Thus, $g \circ h \in \Pi^0_2[\mathcal{N}, X]$.
In the "reflective case" we use the result in [Se05, Se06] that the domain $\omega^{\leq\omega}$ is an effective retract of X, let $s : \omega^{\leq\omega} \to X$ be the corresponding effectively continuous section. By Proposition 4, $s \in \Pi^0_2[\omega^{\leq\omega}, X]$. The inclusion $i : \mathcal{N} \to \omega^{\leq\omega}$ is an effective embedding such that $i[\mathcal{N}] \in \Pi^0_2(\omega^{\leq\omega})$, hence $s \circ i \in \Pi^0_2[\mathcal{N}, X]$. Since $\mathcal{C}$ is an effective retract of $\mathcal{N}$, the assertion also holds for the Cantor space. $\square$

Third result

Our third result is an effective version of the classical fact that any two uncountable Polish spaces are Borel isomorphic (see e.g. [Ke95]):

Theorem

Let X be a computable Polish space with a perfect basic open ball, or a computable reflective ω-algebraic domain, or a computable 2-reflective ω-algebraic domain. Then X is $\Delta^0_{<\omega}$-isomorphic to $\mathcal{N}$.

Third result

Proof. By Theorem 1, there is an injection $s : X \to \mathcal{N}$ such that $s \in \Sigma^0_2\Sigma^0_1(X, \mathcal{N}) \cap \Pi^0_2[X, \mathcal{N}]$. By Theorem 2, there is an injection $g : \mathcal{N} \to X$ such that $g \in \Sigma^0_1(\mathcal{N}, X) \cap \Pi^0_2[\mathcal{N}, X]$. Let h be the bijection between $\mathcal{N}$ and X obtained from g, s by the standard Schröder-Bernstein back-and-fourth argument. One easily checks that h is a desired $\Delta^0_{<\omega}$-isomorphism. $\square$

Statement of the problem

We say that a wcb$_0$-space X *satisfies the Suslin-Kleene theorem* iff $\bigcup\{\Sigma^0_\alpha(X) \mid \alpha < \omega_1^{CK}\} = \Delta^1_1(X)$ (since the inclusion from left to right holds for any X, the condition is equivalent to $\bigcup\{\Sigma^0_\alpha(X) \mid \alpha < \omega_1^{CK}\} \supseteq \Delta^1_1(X)$). Which wcb$_0$-spaces satisfy the Suslin-Kleene theorem? According to classical results of Kleene [Ro67], $\omega, \mathcal{N}$ are among these spaces.

A technical fact

The next theorem extends this to many natural spaces (note that for the perfect computable Polish spaces theorem follows from results in [Mo09]) but first we establish the following:

Proposition

Let (X,ξ) and (Y,η) be wcb_0-spaces, $f:X\to Y$ a function in $\Delta^0_{<\omega}(X,Y)$ and Γ an infinite level of the effective Borel hierarchy or a non-zero level of the effective Luzin hierarchy. Then $A\in\Gamma(Y)$ implies $f^{-1}(A)\in\Gamma(X)$ effectively w.r.t. the canonical numberings.

Suslin-Kleene theorem

Theorem

Let X be a computable Polish space with a perfect basic open ball, or a computable reflective ω-algebraic domain, or a computable 2-reflective ω-algebraic domain. Then X satisfies the Suslin-Kleene theorem.

Proof. By Theorem 3, there is a $\Delta^0_{<\omega}$-isomorphism $h : \mathcal{N} \to X$. Let $A \in \Delta^1_1(X)$. By Proposition 5, $h^{-1}(A) \in \Delta^1_1(\mathcal{N})$, hence $h^{-1}(A) \in \Sigma^0_\alpha(\mathcal{N})$ for some infinite computable ordinal α. By Proposition 5, $A \in \Sigma^0_\alpha(X)$. $\square$

Statement of the problem

We say that a wcb_0-space X *satisfies the effective Hausdorff theorem* iff $\Delta^0_2(X) = \bigcup\{\Sigma^{-1}_{(a)}(X) \mid a \in O\}$. Since the inclusion from right to left holds for any X, the equality is equivalent to the converse inclusion. Here we investigate which wcb_0-spaces satisfy the effective Hausdorff theorem. We need the following easy fact.

Proposition

Let X, Y be wcb_0-spaces, $f : X \to Y$ an effectively continuous, effectively open surjection, and X satisfy the effective Hausdorff theorem. Then Y satisfies the effective Hausdorff theorem.

Effective Hausdorff theorem

Since $\mathcal{N}$ satisfies the effective Hausdorff theorem by [Se03] and any computable Polish space (as well as any computable ω-continuous domain) is an effectively continuous and effectively open image of $\mathcal{N}$ by Theorem 1, the next result is an immediate corollary of Proposition 6.

Theorem

Let X be a computable Polish space or a computable ω-continuous domain. Then X satisfies the effective Hausdorff theorem.

Effective Hausdorff theorem

The following particular cases of this result were known so far: In [Er68] the result was proved for the space ω, in [Se03] the fact was established for the Baire space, in [He06] it was obtained for the finite-dimensional Euclidean spaces, and for the computable Polish spaces the result was established in [BG12]. Our proof here is different from and shorter than the proof in [BG12]. The case of ω-continuous domain was left open in [BG12].

Conclusion

The effective DST is still in its early stage, in particular there are many open questions related to this paper. E.g., the “right” computable version of quasi-Polish space is still not clear to me (the proof of Theorem 1 does not seem to apply to arbitrary effective convergent approximation spaces). Also, the status of the effective Hausdorff-Kuratowski theorem seems to be widely open (to my knowledge, even for the Baire space).

References

S. Abramsky and A. Jung. Domain theory. In: *Handbook of Logic in Computer Science*, v. 3, Oxford, 1994, 1–168.

V. Becher, S. Grigorieff. Borel and Hausdorff hierarchies in topological spaces of Choquet games and their effectivization. To appear in *Mathematical structures in computer science*.

V. Brattka, Effective Borel measurability and reducibility of functions, *Mathematical Logic Quarterly*, 51:1 (2005), 19–44.

M. de Brecht. Quasi-Polish spaces. *Annals of pure and applied logic*, 164 (2013), 356–381.

References

- Yu.L. Ershov. On a hierarchy of sets 1,2,3. *Algebra i Logika*, 7, No 4 (1968), 15–47.
- T. Grubba, M. Schröder, K. Weihrauch. Computable metrization. Mathematical Logic Quarterly, 53 (2007), 381–395.
- A. Hemmerling. The Hausdorff-Ershov hierarchy in Euclidean spaces. *Archive for Mathematical Logic*, 45 (2006), 323–350.
- A. S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, 156, Springer, New York, 1995.

References

M.V. Korovina and O.V. Kudinov. Basic principles of Σ-definability and abstract computability. Schriften zur Theoretischen Informatik, Bericht Nr 08-01, Univeversität Siegen, 2008.

Y.N. Moschovakis. *Descriptive Set Theory*, North Holland, Amsterdam, 2009.

L. Motto Ros, P. Schlicht and V. Selivanov. Wadge-like reducibilities on arbitrary quasi-Polish spaces. To appear in *Mathematical structures in computer science*. arXiv:1304.1239 [cs.LO]

H. Rogers, jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, New York, 1967.

References

- V.L. Selivanov. Wadge degrees of ω-languages of deterministic Turing machines. Theoretical Informatics and Applications, 37 (2003), 67–83.
- V.L. Selivanov. Variations on the Wadge reducibility. *Siberian Advances in Math.*, **15** (2005), 44–80.
- V.L. Selivanov. Towards a descriptive set theory for domain-like structures. Theoret. Comput. Sci. **365** (2006), no. 3, 258–282.
- V.L. Selivanov. On the difference hierarchy in countably based T_0-spaces. *Electronic Notes in Theoretical Computer Science*, V. 221 (2008), 257-269.

References

Su Gao. *Invariant Descriptive Set Theory*. CRC Press, New York, 2009.

K. Weihrauch. *Computable Analysis*. Berlin, Springer, 2000.