Fragments of Kripke-Platek set theory and the metamathematics of α-recursion theory

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Overview

Fragments of KP

- Main Question: How much foundation is needed to prove various theorems of recursion theory in set theoretic models?
- Language: $\mathcal{L}(\in)$.
- Fragments of KP: subtheories of KP including KP^-.
- KP^-: the theory obtained from the usual Kripke–Platek set theory KP by taking away the foundation scheme.

Recall: Axioms of KP

(i) Extensionality: $\forall x, y[\forall z(z \in x \leftrightarrow z \in y) \to x = y]$.

(ii) **Foundation**: If y is not a free variable in $\phi(x)$, then

$[\exists x \phi(x) \to \exists x(\phi(x) \wedge \forall y \in x \neg\phi(y))]$.

(iii) Pairing: $\forall x, y \exists z(x \in z \wedge y \in z)$.

(iv) Union: $\forall x \exists y \forall z \in x \forall u \in z(u \in y)$.

(v) Σ_0-Separation: $\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \phi(z)))$ for each Σ_0 formula ϕ.

(vi) Σ_0-Collection:

$\forall x[(\forall y \in x \exists z \phi(y, z)) \to \exists u \forall y \in x \exists z \in u \phi(y, z)]$ for each Σ_0 formula ϕ.

Γ-Foundation: Foundation restricted to formulas in Γ.

Compare this with weak system of arithmetic

	Fragments of KP	Fragments of PA
Language	$\mathcal{L}(\in)$	$\mathcal{L}(0, 1, +, \cdot)$
Main Question	Foundation	Induction
Base Theory	KP^{-}	PA^{-}: PA without Induction

First Questions

- What can be done without Foundation?
- Is the consideration of Fragments of KP meaningful?

What can be done without Foundation?

Proposition

KP^- proves the following:

(1) Strong Pairing: $\forall x, y \exists z\, (z = \{x, y\})$.

(2) Strong Union: $\forall x \exists y\, (y = \bigcup x)$.

(3) Δ_1-Separation and Σ_1-Collection.

(4) Strong Σ_1-Collection: Suppose f is a Σ_1 function. If $\mathrm{dom}(f)$ is a set, then $\mathrm{ran}(f)$ and $\mathrm{graph}(f)$ are sets.

(5) Ordered Pair: $\forall x, y \exists z\, (z = (x, y))$.

(6) Cartesian Product: $\forall x, y \exists z\, (z = x \times y)$.

Ordinals

Proposition ($\mathrm{KP}^- + \Sigma_0$-Foundation)

(1) $0 = \emptyset$ is an ordinal.

(2) If α is an ordinal, then $\beta \in \alpha$ is an ordinal and $\alpha + 1 = \alpha \cup \{\alpha\}$ is an ordinal.

(3) $<$ is a linear order on the ordinals.

(4) For every ordinal α, $\alpha = \{\beta : \beta < \alpha\}$.

(5) If C is a nonempty set of ordinals, then $\bigcap C$ and $\bigcup C$ are ordinals, $\bigcap C = \inf C = \mu\alpha(\alpha \in C)$ and $\bigcup C = \sup C = \mu\alpha(\forall\beta \in C(\beta \leq \alpha))$.

Transfinite Induction

Theorem (Transfinite Induction along the ordinals)

Suppose $M \models \Pi_1\text{-Foundation}$ *and* $I \colon M \to M$ *is a* Σ_1 *partial function. Then the partial function* $f \colon \mathsf{Ord}^M \to M, \delta \mapsto I(f \restriction \delta)$ *is well defined and* Σ_1*.*

Theorem (Transfinite $\in$-induction)

Let $M \models \mathrm{KP}^- + \Pi_1\text{-Foundation}$*, and* $I \colon M \to M$ *that is* Σ_1*-definable. Then there exists a* Σ_1*-definable* $f \colon M \to M$ *satisfying* $f(x) = I(f \restriction x)$ *for every* $x \in M$*.*

the Schröder–Bernstein Theorem

Theorem ($\mathrm{KP}^- + \Pi_1$-Foundation)

Let A, B be sets. If there are injections $A \to B$ and $B \to A$, then there is a bijection $A \to B$.

L^M

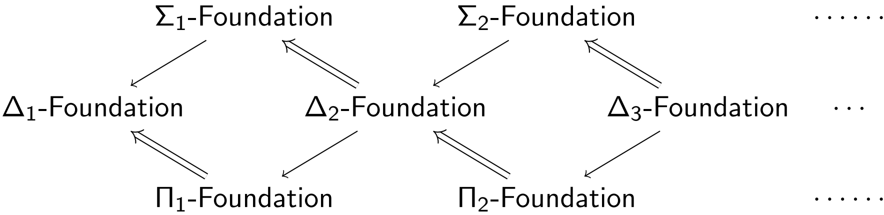
Let $M \models \mathrm{KP}^- + \Pi_1$-Foundation.

By a transfinite induction, we may define L^M along Ord^M:

$$L_0^M = \emptyset,$$
$$L_{\alpha+1}^M = L_\alpha^M \cup \mathrm{Def}^M(L_\alpha^M),$$
$$L_\lambda^M = \bigcup_{\alpha<\lambda} L_\alpha^M \quad \text{where } \lambda \text{ is limit.}$$

Here, $\mathrm{Def}^M(x)$ denotes the collection of all definable subsets of x in the sense of M. Let $L^M = \bigcup_{\alpha \in \mathrm{Ord}^M} L_\alpha^M$.

Is the consideration of Fragments of KP meaningful?



Lemma (Ramón Pino [1])

Let $n \in \mathbb{N}$*. Then*

$\mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}\text{-Collection} + \Pi_{n+1}\text{-Foundation} + \mathrm{V} = \mathrm{L}$

proves the following statement.

> *For every* $\delta \in \mathsf{Ord}$*, there exists a sequence* $(\alpha_i)_{i \leq \delta}$ *in which* $\alpha_0 = 0$ *and* $\alpha_{i+1} = \min\{\alpha > \alpha_i : \mathrm{L}_\alpha \preceq_n \mathrm{L}\}$ *for each* $i < \delta$*.*

Theorem (Ressayre [1])

$\mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}\text{-Collection} + \Sigma_{n+1}\text{-Foundation}(+\mathrm{V} = \mathrm{L}) \nvdash \Pi_{n+1}\text{-Foundation}$ *for all* $n \in \mathbb{N}$.

Sketch of the Proof (Essentially Ressayre)

- Start with a countable $M \models \mathrm{KP}^- + \mathrm{Infinity} + \Sigma_{n+1}\text{-Collection} + \Pi_{n+1}\text{-Foundation} + \mathrm{V} = \mathrm{L}$ in which $\omega^M = \omega$ but Ord^M is not well-ordered.
- Take a nonstandard $\delta \in \mathsf{Ord}^M$. Let $(\alpha_i)_{i \leq \delta+\delta}$ be a sequence of ordinals given by Lemma.

- As δ is nonstandard, there are continuum-many initial segments of Ord^M between δ and $\delta + \delta$.
- So there must be one that is not definable in M.
- Take any initial segment $I \subseteq \mathsf{Ord}^M$ with this property.
- $K = \bigcup_{i \in I} \mathrm{L}_{\alpha_i}^M$ is the model we want.

Claim

1. $K \preceq_n M$.
2. $K \models \Sigma_{n+1}\text{-Collection}$.
3. $K \models \Sigma_{n+1}\text{-Foundation}$.
4. $K \not\models \Pi_{n+1}\text{-Foundation}$.

Theorem

$\mathrm{KP}^- + \Sigma_{n+1}$-Collection $+ \Pi_{n+1}$-Foundation $+ \mathrm{V} = \mathrm{L} \vdash$

Σ_{n+1}-Foundation *for all* $n \in \mathbb{N}$.

Question

Does KP^-, Σ_{n+1}*-Collection, plus* Π_{n+1}*-Foundation (without* $V = L$*) prove* Σ_{n+1}*-Foundation?*

Question

Is Σ_{n+1}*-Foundation stronger than* Π_n*-Foundation?*

α-RecursionFragments of KP

	Fragments of PA	α-RecursionFragments of KP
Language	$\mathcal{L}(0, 1, +, \cdot)$	$\mathcal{L}(\in)$
Axioms	P^-, $I\Sigma_n$, etc	KPKP^-, Π_n-Foundation, etc
Models	Nonstandard models of arithmetic with restricted induction	L_α, where α isnonstandard Σ_1 admissible
Difficulty	lack of induction	lack of collection and foundation

Definition

Level 1-KPL denotes $\mathrm{KP}^- + \Pi_1$-Foundation $+ \mathrm{V} = \mathrm{L}$.

Notice Level 1-KPL $\vdash \Sigma_1$-Foundation.

Lemma

Suppose $M \models \mathrm{KP}^- + \Pi_1$-Foundation $+ \mathrm{V} = \mathrm{L}$. *Then there exists a* Δ_1 *bijection* $M \to \mathsf{Ord}^M$ *that preserves the relation* $\in$.

the Friedberg–Muchnik Theorem

- Now we will show the Friedberg–Muchnik Theorem in Level 1-KPL.
- M is a model of Level 1-KPL.
- The Sack–Simpson construction [2] in α-recursion theory uses the Σ_2-*cofinality* (of the ordinals), i.e., the least ordinal that can be mapped to a cofinal set of ordinals by a Σ_2 function.
- The existence of Σ_2 cofinality apparently needs much more foundation than Level 1-KPL can afford.

Σ_1 Projectum

Lemma (Level 1-KPL)

If there is a Σ_1 injection from the universe into an ordinal, then there is the least such an ordinal (Σ_1 Projectum).

Proof.

Suppose $\alpha \in M$ is an ordinal such that there is a Σ_1 injection from the universe into α. We claim $|\alpha| = \sigma 1 p$. Clearly, there is a Σ_1 injection from the universe into $|\alpha|$. Conversely, if we have a Σ_1 injection p from M into $\beta \leq |\alpha|$, then $p \restriction |\alpha|$ is in the model and is an injection into β. As $|\alpha|$ is a cardinal in M, $\beta = \alpha$. $\square$

Bounding Injury within Σ_1 Projectum

Lemma

Σ_1 *Projectum is a cardinal and also the largest one.*

Lemma (Sacks and Simpson)

Suppose $\alpha < \delta$ *and* δ *is a regular cardinal in* M. *If* $\{X_i : i < \alpha\}$ *is a uniform r.e. sequence in the model sets of ordinals with cardinality less than* δ. *Then* $\bigcup\{X_i : i < \alpha\}$ *is in the model and of cardinality less than* δ.

Similar for the case that cofinally many cardinals exist in the model.

Largest Cardinal $\aleph$ not Σ_1 Projectum

Definition

Suppose δ is an ordinal. We say δ is *(Σ_1) stable* if L_δ is a Σ_1 elementary substructure of the whole model.

Lemma (Level 1-KPL)

For every γ such that $\omega^M \leq \gamma$, there is a stable ordinal $\delta \geq \gamma$ with the same cardinality as γ.

Shore's Splitting Theorem

- Aim: Split a nonrecursive set into two incomparable nonrecursive sets.
- For a single requirement, we apply the classical method of preserving computation.
- To settle all requirements, we adopt the blocking method as in α-recursion theory.
- The problem is that, within Level 1-KPL, we may not have the Σ_2 cofinality of the Ord^M.
- Thus, here we use a modified version that came from arithmetic [3]. It is a modified version of that in α-recursion

Lemma

For any nondecreasing recursive sequence $\{\xi_s\}_s$*, either it is cofinal in* Ord^M *(we denote this by* $\lim_s \xi_s = \infty$*) or there is a stage s such that for all* $t > s$*,* $\xi_t = \xi_s$*.*

Questions not answered so far

1. Is there a model of Level 1-KPL without Σ_n projectum/cofinality for some $n \geq 2$?
2. Does Level 1-KPL imply the density theorem of r.e. degrees?
3. Does Level 1-$\mathrm{KPL} + \neg\Pi_2$-Foundation consistent with the existence of a minimal pair?

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Thank you!