Pure Patterns and Ordinal Numbers

Gunnar Wilken

Okinawa Institute of Science and Technology Graduate University
Japan

Sets and Computations
Institute of the Mathematical Sciences
National University of Singapore
17 April 2015

Patterns of Embeddings

Gödel's program of using large cardinals to solve mathematical incompleteness inspired Tim Carlson to initiate his program of "Patterns of Embeddings".

Heuristics: Axioms of infinity closely related to ordinal notations.

Goal: Find "ultra fine structure" for large cardinal axioms based on embeddings, complementary to inner model theory at stages missing inner model construction.

Elementary Patterns of Resemblance

Elementary Patterns of Resemblance: First steps into Patterns of Embeddings.

Binary relations code elementary substructurehood, no codings of embeddings involved.

Elementary Patterns of Resemblance (in short: patterns) are finite structures of nested forests, possibly with underlying arithmetic structure: Finite isomorphism types of structures of ordinals.

Applications

Patterns give rise to large ordinal notation systems.

Proof-theoretic analysis of theories of numbers and sets.

Rich combinatorial properties allow for strong independence results.

Definition

$$\mathcal{R}_1 := (\mathrm{Ord}; \leq, \leq_1)$$

where $\leq_1$ is defined by recursion on β as follows:

$$\alpha \leq_1 \beta \quad :\Leftrightarrow \quad (\alpha; \leq, \leq_1) \preceq_{\Sigma_1} (\beta; \leq, \leq_1)\,.$$

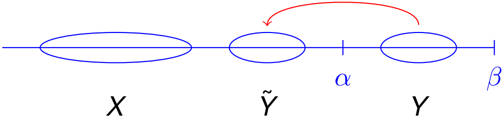
Remark

Carlson discovered this structure when he verified a conjecture by William Reinhardt, namely that Epistemic Arithmetic is consistent with a formalization of the statement I know that I am a Turing Machine, see [Carlson 2000].
The relation $\leq_1$ naturally surfaced in the guise of the following

Criterion for Σ_1-Elementary Substructures

Lemma

$\alpha \leq_1 \beta$ iff



For any finite X, Y there is a $\tilde{Y}$ s.t.

$$X < \tilde{Y} < \alpha, \quad X \cup \tilde{Y} \cong_{\leq, \leq_1} X \cup Y.$$

Example: $\alpha \leq_1 \alpha + 1 \iff \alpha \in \mathrm{Lim}.$

Lemma

$\{\beta \mid \alpha \leq_1 \beta\}$ is a closed interval.

Definition

$$\mathrm{lh}(\alpha) := \begin{cases} \max\{\beta \mid \alpha \leq_1 \beta\} & \text{if that exists} \\ \infty & \text{otherwise.} \end{cases}$$

Theorem [Carlson 1999]

- $\mathcal{R}_1 \cong \mathcal{R}_1 \cap [\alpha + 1, \infty)$ for all α.
- $\mathrm{lh}(\varepsilon_0 \cdot (1 + \eta)) = \infty$ for all η.
- For $\alpha =_{\text{CNF}} \omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$ ($n > 0$) with $\alpha_n =_{\text{ANF}} \rho_1 + \ldots + \rho_m < \alpha$ we have

$$\mathrm{lh}(\alpha) = \alpha + \mathrm{lh}(\rho_1) + \ldots + \mathrm{lh}(\rho_m).$$

$\mathcal{R}_n$ in general

Definition

$$\mathcal{R}_n := (\mathrm{Ord}; (\leq_i)_{0 \leq i \leq n})$$

where $\leq_0 := \leq$ is standard and the relations $\leq_1, \ldots, \leq_n$ are defined simultaneously by recursion on β:

$$\alpha \leq_j \beta \quad :\Leftrightarrow \quad (\alpha; (\leq_i)_{0 \leq i \leq n}) \preceq_{\Sigma_j} (\beta; (\leq_i)_{0 \leq i \leq n}) \,.$$

End-Extension Property of the R_n

Observation: For the least β such that there exists α such that

$$\mathcal{R}_{n+1} \models \alpha <_{n+1} \beta$$

we have

$$\mathcal{R}_{n+1} \restriction_{(\leq_i)_{0 \leq i \leq n}} \cap \beta = \mathcal{R}_n \cap \beta.$$

“Respecting” Forests

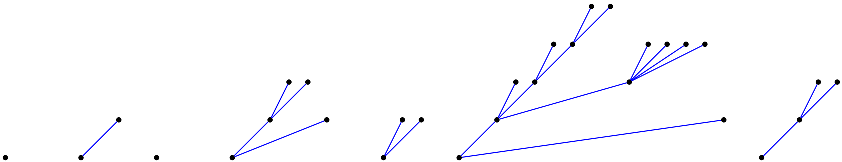
We summarize further immediate consequences of Σ_i-elementary substructurehood:

Lemma

For any $\mathcal{R}$-structure we have

- $\leq_i$ is a forest for $1 \leq i \leq n$
- $\leq_{i+1}$ respects $\leq_i$:
  - $\leq_{i+1} \subseteq \leq_i$ and
  - $\alpha \leq_i \beta \leq_i \gamma \;\&\; \alpha \leq_{i+1} \gamma \;\; \Rightarrow \;\; \alpha \leq_{i+1} \beta$

Example: A Respecting Forest for $\mathcal{R}_1$



The elements are $<$-increasing from left to right. Edges indicate $<_1$-connections.

In presence of an underlying arithmetic structure, decomposable elements occur only at leaves!

Elementary Patterns of Resemblance

Definition

An pattern for $\mathcal{R}_n$ is a finite structure with relations $(\leq_i)_{0 \leq i \leq n}$ that is isomorphic to a substructure of $\mathcal{R}_n$.

Visualization of patterns: Finite directed graphs (forests) whose edges are colored according to the strongest $\leq_i$-relation between any two elements.

It turns out that the class of patterns comprises the finite respecting forests (conditions in presence of arithmetic).

Isominimality and the Core

Isominimality

$P \subseteq \mathcal{R}_n$ is isominimal if

- $P \subseteq_{\text{fin}} \mathcal{R}_n$ and
- for any $Q \subseteq \mathcal{R}_n$ such that $P \cong Q$:

$$Q \leq_{\text{pw}} P \;\Rightarrow\; Q = P$$

Core of $\mathcal{R}_n$

$$\text{Core}(\mathcal{R}_n) \quad := \quad \bigcup \{P \subseteq \mathcal{R}_n \mid P \text{ isominimal}\}$$

Pattern Notations

Theorem

For any pattern P for $\mathcal{R}_n$ ($n = 1, 2$) there exists a unique isominimal $P^\star \subseteq \mathcal{R}_n$ such that $P \cong P^\star$.

Theorem

1. $\mathrm{Core}(\mathcal{R}_1) = \varepsilon_0$ [Carlson 1999, 2001]
2. $\mathrm{Core}(\mathcal{R}_1^+) = |\Pi^1_1 - \mathrm{CA}_0|$ [W 2007c]
3. $\mathrm{Core}(\mathcal{R}_2) = |\Pi^1_1 - \mathrm{CA}_0|$ [W]

The fact that $\mathrm{Core}(\mathcal{R}_1^+)$ and variants of $\mathrm{Core}(\mathcal{R}_2)$ and $\mathrm{Core}(\mathcal{R}_2^+)$ are isomorphic to some recursive ordinal was first shown in [Carlson 2001 and 2009].

It is conjectured that $\mathrm{Core}(\mathcal{R}_2^+)$ reaches beyond $|\mathrm{KPI}|$ and that $\mathrm{Core}(\mathcal{R}_3)$ has the same order type (future work).

Proof-theoretic Ordinals

Examples: We give characterizations of well-known proof-theoretic ordinals. These results are shown in Carlson 1999, W 2006, 2007b, Carlson/W 2012b.
The first equality in each example refers to $(\mathcal{R}_1^+, \leq_1^+)$, the second to $(\mathcal{R}_2, \leq_1, \leq_2)$.

1. $|\,\mathrm{PA}\,| = \min \alpha\ \alpha \leq_1^+ \alpha + \alpha = \min \alpha\ \exists \beta, \gamma\ \ \alpha <_1 \beta <_2 \gamma,$
2. $|\,\mathrm{ID}_1\,| = \min \alpha\ \exists \beta\ \alpha <_1^+ \beta <_1^+ \beta + \beta$
   $= \min \alpha\ \exists \beta, \gamma, \delta\ \ \alpha <_1 \beta <_2 \gamma <_2 \delta,$
3. $|\,\mathrm{ID}_n\,| = \min \alpha\ \exists \beta_1, \ldots, \beta_n\ \ \alpha <_1^+ \beta_1 <_1^+ \ldots <_1^+ \beta_n <_1^+ \beta_n + \beta_n$
   $= \min \alpha\ \exists \beta_1, \ldots, \beta_{n+2}\ \ \alpha <_1 \beta_1 <_2 \ldots <_2 \beta_{n+2},$
4. $|\,\mathrm{ID}_{<\omega}\,| = \min \alpha\ \ \alpha <_1^+ \infty = \min \alpha\ \ \alpha <_1 \infty = |\Pi_1^1 - \mathrm{CA}_0|.$

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Example: $\varepsilon_0 \cdot \omega <_2 \varepsilon_0 \cdot (\omega + 1)$

Lemma

If $\alpha <_2 \beta$ then α is the $\sup$ of an infinite $<_1$-chain.

Proof. For any $\rho < \alpha$ we have $\beta \models \exists x \, \forall y > x \, (\rho < x <_1 y)$. Hence the same holds true in α. We obtain $\rho_1 <_1 \rho_2 <_1 \rho_3 <_1 \ldots <_1 \alpha$. □

The least candidate for such α is $\epsilon_0 \cdot \omega$!

Lemma

Suppose $\alpha <_2 \beta$ and $X, Z \subseteq_{\text{fin}} \alpha$ with $X < Z$. If

$$\alpha \models \forall r \, \exists \tilde{Z} \, (r < \tilde{Z} \wedge \text{“}X \cup \tilde{Z} \cong X \cup Z\text{”})$$

then the same holds in β.

The least candidate for such β is $\epsilon_0 \cdot (\omega + 1)$!

Example: ... continued

Lemma

Suppose $\alpha <_2 \beta$, $X \subseteq_{\text{fin}} \alpha$, and $Y \subseteq_{\text{fin}} [\alpha, \beta)$. Then there exist cofinally many $\tilde{Y} \subseteq \alpha$ such that $X < \tilde{Y}$, $X \cup \tilde{Y} \cong X \cup Y$, and for any $y \in Y$ such that $y <_1 \beta$ the corresponding $\tilde{y} \in \tilde{Y}$ satisfies $\tilde{y} <_1 \alpha$.

Proof. Let $\{y \in Y \mid y <_1 \beta\} = \{y_1, \ldots, y_k\}$. Note that for any parameter $\xi < \alpha$

$$\beta \models \exists \tilde{Y} > \xi \, \forall r > \tilde{Y} \left(\text{“}X \cup \tilde{Y} \cong X \cup Y\text{”} \wedge \bigwedge_{i=1}^{k} \tilde{y}_i <_1 r \right)$$

which then also holds in α. □

How to show Σ_n-Elementarity

Proposition

Suppose $\alpha < \beta$. If for all $X \subseteq_{\text{fin}} \alpha$ and all $Y \subseteq_{\text{fin}} [\alpha, \beta)$ there exists $\tilde{Y}$ such that:

1. $X < \tilde{Y} < \alpha$ and
2. $\exists h : X \cup \tilde{Y} \xrightarrow{\cong} X \cup Y$ such that for all $\tilde{Y}^+$ with $\tilde{Y} \subseteq \tilde{Y}^+ \subseteq_{\text{fin}} \alpha$

$$\exists h^+ \supseteq h, Y^+ \supseteq Y \quad \text{s.t.} \quad h^+ : X \cup \tilde{Y}^+ \xrightarrow{\cong} X \cup Y^+$$

then $\alpha <_2 \beta$.

In $\mathcal{R}_2$ the converse holds as well! See [Carlson/W 2012b].

This type of criterion generalizes to $\leq_n$!

The WQO-Result for Pure Patterns of Order 2

Definition

For patterns P, Q an injection $h : P \hookrightarrow Q$ is a covering of P into Q iff

$$x \leq_i^P y \quad \Rightarrow \quad h(x) \leq_i^Q h(y).$$

Theorem [Carlson]

The collection of pure patterns of order two $\mathcal{P}_2$ is well-quasi ordered with respect to coverings.

Independence

Theorem [W]

There is a mapping $p_2 : \mathrm{Core}(\mathcal{R}_2) \to \mathcal{P}_2$ such that

$$\alpha > \beta \quad \Rightarrow \quad p_2(\alpha) \not\hookrightarrow p_2(\beta)$$

hence

$$\mathcal{P}_2 \text{ is wqo wrt coverings} \quad \Rightarrow \quad (|\Pi^1_1 - \mathrm{CA}_0|, <) \text{ is well-ordered.}$$

Therefore the above wqo-result is independent of $\Pi^1_1 - \mathrm{CA}_0$, or equivalently $\mathrm{KP}\ell_0$.

Proof of results

The results regarding Core($\mathcal{R}_2$) and independence build upon the work in [Carlson/W 2012b], where $\leq_1$ and $\leq_2$ are characterized in terms of a system of ordinal arithmetic developed in [W 2007a]. This characterization can be seen to be elementary recursive.

The notion of tracking chain (tc) determines successively greatest $\leq_i$-predecessors providing context information regarding nested $\leq_i$-connectivity components.

The maximal extension (me) of a tracking chain provides information about critical/largest immediate $\leq_i$-successors.

This is expressed by means of (local) elementary recursive enumeration functions κ and ν of relativized $\leq_1$- and $\leq_2$-connectivity components, respectively.

Closedness of Ordinal Patterns

All ordinals and notations range over the initial segment $|\Pi^1_1 - \mathrm{CA}_0|$. We may identify ordinals and their tracking chains.

Definition

A set M of tracking chains is closed iff it is closed under

1. initial tc's (omitting redundancies)
2. me
3. additive decomposition of κ-indices (minor technical adjustments)
4. additive decomposition of ν-indices
5. largest $\leq_2$-components (largest ν-indices)
6. parameters determining the spacing of $\leq_2$-successors ($\bar{\cdot}$ -operator) within largest $\leq_2$-components.

Describing Patterns in Normal Form

6*. Normal form condition for closures: parameters inserted below least $\leq_2$-successor of largest $\leq_2$-component.

Remark

Regarding normal forms for $\mathcal{R}_1^+$-patterns, see [Carlson/W 2012a].

Definition

For any ordinal α let $P(\alpha)$ be the closure of $\{\alpha\}$ under 1. - 5. and 6*.

Lemma

$P(\alpha)$ is finite.

Proof. Any path in $P(\alpha)$ can be extended only finitely many times. □

$P(\alpha)$ is called the normal form pattern representation of α when interpreted as structure over $(\leq, \leq_1, \leq_2)$.

Main Theorem

Theorem

Let $n \geq 0$ and $\{t_1, \ldots, t_n\} \cup P$ be a pure pattern of order 2 where $t_1 <_2 \ldots <_2 t_n <_1 P$ and $\exists p \in P\ t_n <_2 p$ if $n > 0$.
Set $m := \mathrm{Card}(P)$ and assume $m > 0$. Set $\tau_1 := |\,\mathrm{ID}_{n+m}\,|$ and $\tau_{i+1} := \mu_{\tau_i}$, $1 \leq i \leq n$, $\tau_0 := 1$.

1. Fixing $o(t_i) := o((\tau_1, \ldots, \tau_{i+1}))$, $1 \leq i \leq n$, there exists a unique, above $o(t_n)$ $\leq$-pointwise minimal ordinal covering $o : \{t_1, \ldots, t_n\} \cup P \ \rightarrow\ |\Pi^1_1 - \mathrm{CA}_0|$.
2. $o[P]$ is closed above $o(t_n)$.
3. o maps $\leq_i$-minimal elements of P to $o(t_n)$-$\leq_i$-minimal ordinals and $\mathrm{lh}_i/\mathrm{lh}_i^P$ and o commute: $\mathrm{lh}_i(o(p)) = o(\mathrm{lh}_i^P(p))$ for all $p \in P$, hence o is the above $o(t_n)$ isominimal realization of P.
4. If in particular P is a substructure of $\mathcal{R}_2$ which is closed above $o(t_n)$, then the isominimal realization of P above $o(t_n)$ is the identity.

Conclusion

Corollary

For any α the normal form pattern representation $P(\alpha)$ is isominimally realized by the identity.

It follows that $\mathrm{Core}(\mathcal{R}_2) = |\Pi^1_1 - \mathrm{CA}_0|$.

Corollary

$\alpha > \beta \quad \Rightarrow \quad P(\kappa_{\omega^\alpha}) \not\leftrightarrow P(\kappa_{\omega^\beta})$.

Hence the well-quasi-orderedness of pure patterns of order 2 wrt coverings is independent of $\Pi^1_1 - \mathrm{CA}_0$.