The Arithmetized Completeness Theorem

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This talk

Why would one formalize Gödel's Completeness Theorem in arithmetic?

Plan

1. Full induction
2. Restricted induction
3. Non-classical applications
4. Conclusion

First-order arithmetic

- $\mathscr{L}_{\mathrm{I}} = \{0, 1, +, \times, <\}$.
- PA consists of some basic algebraic axioms (PA^-) and an *induction axiom*

$$\theta(0) \wedge \forall x \ \big(\theta(x) \rightarrow \theta(x+1)\big) \rightarrow \forall x \ \theta(x)$$

  for every $\theta \in \mathscr{L}_{\mathrm{I}}$.
- ω is the *standard model* (of arithmetic).
- An $\mathscr{L}_{\mathrm{I}}$-structure M is *nonstandard* if $M \ncong \omega$.
- Let $M, K \models \mathsf{PA}^-$. Write $K \supseteq_{\mathrm{e}} M$ to mean K is an *end extension* of M, i.e., $K \supseteq M$ and

$$\forall k \in K \setminus M \ \ \forall m \in M \ \ \ k \geqslant m.$$

  Alternatively, we say M is a *cut* of K.
- No proper cut of a model of PA is definable.

The Arithmetized Completeness Theorem (ACT)

Example

ω is a cut of all models of PA^-, called the *standard cut*.

ACT (PA version)

Every consistent definable theory T in $M \models \mathrm{PA}$ has a definable model K in M. If, moreover, $T \supseteq \mathrm{PA}^-$, then we can view $K \supseteq_e M$.

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$$\forall k \in K \setminus M \ \ \forall m \in M \ \ k \geqslant m.$$

  Alternatively, we say M is a *cut* of K.
- No proper cut of a model of PA is definable.

Consistency implies satisfiability in an end extension

Theorem (Mostowski 1952, Kreisel–Lévy 1968)

PA is equivalent over $\mathrm{I}\Delta_0 + \exp$ to the *uniform reflection scheme*

$$\forall x \ \big(\theta(x) \to \mathrm{Con}(\theta(\check{x}))\big),$$

where $\theta \in \mathscr{L}_\mathrm{I}$.

Π_1/universal

Theorem (Mc Aloon 1978)

Let $a \in M \models \mathrm{PA}$ and $\theta(x)$ be an $\mathscr{L}_\mathrm{I}$-formula.
The following are equivalent.

(a) There is an extension $K \supseteq M$ satisfying $\mathrm{PA} + \theta(a)$.

(b) There is an end extension $K' \supseteq_\mathrm{e} M$ satisfying $\mathrm{PA} + \theta(a)$.

Proof of (a) $\Rightarrow$ (b)

$\mathrm{I}\Sigma_n \approx$ first n axioms of PA

- $\mathrm{Con}(\mathrm{I}\Sigma_n + \theta(a))$ is true in K and so in M for every $n \in \omega$.
- Thus $M \models \mathrm{Con}(\mathrm{I}\Sigma_\nu + \theta(a))$ for some nonstandard $\nu \in M$. □

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Π_1/universal

Theorem (Mc Aloon 1978)

Let $a \in M \models \mathrm{PA}$ and $\Theta(x)$ be a recursive set of $\mathscr{L}_{\mathrm{I}}$-formulas.
The following are equivalent.

(a) There is an extension $K \supseteq M$ satisfying $\mathrm{PA} + \Theta(a)$.

(b) There is an end extension $K' \supseteq_{\mathrm{e}} M$ satisfying $\mathrm{PA} + \Theta(a)$.

Proof of (a) ⇒ (b)

Θ_n = first n elements of Θ

- $\mathrm{Con}(\mathrm{I}\Sigma_n + \Theta_n(a))$ is true in K and so in M for every $n \in \omega$.
- Thus $M \models \mathrm{Con}(\mathrm{I}\Sigma_\nu + \Theta_\nu(a))$ for some nonstandard $\nu \in M$. □

Strong fragments of PA

- Δ_0 is the smallest set of $\mathscr{L}_{\mathrm{I}}$-formulas that
  - contains all atomic $\mathscr{L}_{\mathrm{I}}$-formulas; and
  - is closed under $\neg$, $\wedge$, $\vee$, and *bounded quantification*, i.e., $\forall v < t \ \cdots$ and $\exists v < t \ \cdots$.
- $\Sigma_n = \{\exists \bar{v}_1 \ \forall \bar{v}_2 \ \cdots \ \mathrm{Q}\bar{v}_n \ \theta(\bar{v}, \bar{x}) : \mathrm{Q} \in \{\forall, \exists\} \text{ and } \theta \in \Delta_0\}$.
- The dual is called Π_n.
- Formulas equivalent to both a Σ_n- and a Π_n-formula are Δ_n.
- $\mathrm{I}\Sigma_n$ consists of PA^- and the induction scheme for Σ_n-formulas.
- $\mathrm{B}\Sigma_n$ consists of $\mathrm{I}\Delta_0$ and the *collection scheme* for Σ_n-formulas, i.e., for all $\varphi \in \Sigma_n$,

$$\forall a \ \big(\forall x{<}a \ \ \exists y \ \varphi(x,y) \rightarrow \exists b \ \ \forall x{<}a \ \ \exists y{<}b \ \ \varphi(x,y)\big).$$

- $\exp$ is a sentence asserting the totality of $x \mapsto 2^x$ over $\mathrm{I}\Delta_0$.

Theorem (Parsons 1970, Parikh 1971, Paris–Kirby 1978)

$\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ for all $n \in \omega$; and $\mathrm{I}\Sigma_1 \vdash \exp$ but $\mathrm{B}\Sigma_1 \nvdash \exp$.

ACT with restricted induction

PA version. Every consistent definable theory $T \supseteq \mathsf{PA}^-$ in $M \models \mathsf{PA}$ has a definable model in M that end extends M.

$\mathsf{I}\Sigma_1$ version (Hájek–Pudlák 1993). Every consistent Δ_1-definable theory $T \supseteq \mathsf{PA}^-$ in $M \models \mathsf{I}\Sigma_1$ has a $\Delta_0(\Sigma_1)$-definable model in M that end extends M.

$\mathsf{B}\Sigma_1 + \exp$ version (folklore). Every consistent Δ_1-definable theory $T \supseteq \mathsf{PA}^-$ in a countable $M \models \mathsf{B}\Sigma_1 + \exp$ has a ~~definable~~ model that end extends M.

Proposition (Paris–Kirby 1978)

If $M \subsetneq_{\mathrm{e}} K \models \mathsf{I}\Delta_0$, then $M \models \mathsf{B}\Sigma_1$.

$\Delta_0(\Sigma_1)$ is the closure of Σ_1 under $\neg$, $\wedge$, $\vee$, and bounded quantification.

Variants of Mc Aloon

Theorem (Enayat–W)

Let $\Theta(x)$ be a recursive set of $\mathscr{L}_{\mathrm{I}}$-formulas and $a \in M \models \mathrm{B}\Sigma_1 + \mathsf{exp}$. The following are equivalent provided ω is not Π_1-definable in M and M can be expanded to $(M, \mathscr{X}) \models \mathsf{WKL}_0^*$.

(a) There is an extension $K \supseteq M$ satisfying $\mathsf{PA} + \Theta(a)$.

(b) There is an end extension $K' \supsetneq_{\mathrm{e}} M$ satisfying $\mathsf{PA} + \Theta(a)$.

Theorem (Paris–Kirby 1978)

Let $\Theta(x)$ be a recursive set of $\mathscr{L}_{\mathrm{I}}$-formulas and $a \in M \models \mathrm{B}\Sigma_1$. The following are equivalent provided ω is not Π_1-definable in M and M is countable.

(a) There is an extension $K \succcurlyeq_{\Delta_0} M$ satisfying $\Theta(a)$.

(b) There is an end extension $K' \supseteq_{\mathrm{e}} M$ satisfying $\Theta(a)$.

A second-order version of the ACT

- $\mathscr{L}_{\mathrm{II}} = \{0, 1, +, \times, <, \in\}$ has a number sort and a set sort.
- Δ_n^0, Σ_n^0, $\mathrm{I}\Sigma_n^0$, $\mathrm{B}\Sigma_n^0$, ... are essentially Δ_n, Σ_n, $\mathrm{I}\Sigma_n$, $\mathrm{B}\Sigma_n$, ... with set variables added.
- RCA_0^* consists of $\mathrm{I}\Delta_0^0 + \exp$ and Δ_1^0-comprehension.
- $\mathrm{WKL}_0^* = \mathrm{RCA}_0^* + \mathrm{WKL}$, where WKL says

  every unbounded 0–1 tree contains an unbounded path.

Theorem (Simpson–Smith 1986)

Every countable $M \models \mathrm{B}\Sigma_1 + \exp$ expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

Theorem (Simpson)

WKL is equivalent to Gödel's Completeness Theorem over RCA_0^*.

ACT (WKL_0^* version)

Every consistent theory $T \supseteq \mathrm{PA}^-$ in $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$ has a model in $\mathscr{X}$ that end extends M.

Subsets coded in an end extension

Definition

Let $M \subseteq_e K \models I\Delta_0$. Then $c \in K$ is said to *code* $S \subseteq M$ if

$$S = \{i \in M : K \models \text{“}i\text{th prime divides } c\text{”}\}.$$

$\mathrm{Cod}(K/M)$ denotes the set of all $S \subseteq M$ coded in K.

Theorem (Scott 1962)

If $M \models I\Delta_0 + \exp$ and $K \models I\Delta_0$ properly end extending M, then $(M, \mathrm{Cod}(K/M)) \models \mathrm{WKL}_0^*$.

Theorem (Simpson–Smith 1986)

Every countable $M \models B\Sigma_1 + \exp$ expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

Proof

- Wilkie–Paris (1987) showed $I\Delta_0 + \exp \vdash \mathrm{CutFreeCon}(I\Delta_0)$.
- Apply the $B\Sigma_1 + \exp$ version of the ACT to $I\Delta_0$. □

Variations

Theorem (Simpson–Smith 1986)

Every countable $M \models \mathrm{B}\Sigma_1 + \exp$ expands to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

Theorem (Enayat–W)

Given countable $(M, \mathscr{X}), (M, \mathscr{X}') \models \mathrm{RCA}_0^*$ with $\mathscr{X} \cap \mathscr{X}' = \Delta_1\text{-Def}(M)$, we can find $(M, \mathscr{Y}) \models \mathrm{WKL}_0^*$ extending $(M, \mathscr{X})$ such that $\mathscr{Y} \cap \mathscr{X}' = \Delta_1\text{-Def}(M)$.

Theorem (Enayat–W)

For every $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$ and every $S \in \mathscr{X}$, there exists $Y \in \mathscr{X}$ such that if $\mathscr{Y} = \{(Y)_i : i \in M\}$, then

$$(M, \mathscr{Y}) \models \mathrm{WKL}_0^* \quad \text{and} \quad S \in \mathscr{Y}.$$

Here $(Y)_i = \{j \in M : \langle i, j \rangle \in Y\}$.

Π_1-definability of the standard cut

Theorem (Enayat–W)

Let $\Theta(x)$ be a recursive set of $\mathscr{L}_{\mathrm{I}}$-formulas and $a \in M \models \mathrm{B}\Sigma_1 + \mathrm{exp}$. The following are equivalent provided ω is not Π_1-definable in M and M can be expanded to $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

(a) There is an extension $K \supseteq M$ satisfying $\mathrm{PA} + \Theta(a)$.

(b) There is an end extension $K' \supsetneq_{\mathrm{e}} M$ satisfying $\mathrm{PA} + \Theta(a)$.

Proposition (Paris)

There is a countable $M \models \mathrm{B}\Sigma_1 + \mathrm{exp}$ with the following properties.

(1) There is an extension $K \supseteq M$ satisfying PA.

(2) There is no end extension $K' \supseteq_{\mathrm{e}} M$ satisfying PA.

Proof

Fix a countable $V \models \mathrm{ZFC}$ with $\nu \in \omega^V \setminus \omega$. Using the ACT in V, construct $M_\nu \subseteq_{\mathrm{e}} M_{\nu-1} \subseteq_{\mathrm{e}} \cdots \subseteq_{\mathrm{e}} M_1$ such that each $M_n \models \mathrm{I}\Sigma_n + \neg\,\mathrm{Con}(\mathrm{I}\Sigma_n)$. Set $M = \bigcup\{M_n : n \in \omega^V \setminus \omega\}$. $\square$

Conclusion

Summary

- The Arithmetized Completeness Theorem (ACT) says

  consistency implies satisfiability in an end extension.
- It is a powerful tool in making end extensions and models of the Weak König Lemma.
- ACT arguments apply to all countable models of $\mathrm{B}\Sigma_1 + \exp$.

Questions

(1) Is there a version of the ACT for Δ_0-definable theories in *uncountable* models of $\mathrm{B}\Sigma_1 + \exp$?

(2) Does arithmetization of some other theorem in mathematics have interesting model-theoretic consequences?