Combinatorial properties and strong colorings

(joint work with Liuzhen Wu)

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(d) Lindelöfness.

It's well-known that for regular spaces, Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact.

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Weaker version: is the square of hereditarily Lindelöf group normal or weakly paracompact?

Earlier results

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Theorem (Douwen, 1984)

There are two Lindelöf groups G and H such that $G \times H$ is not Lindelöf.

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Theorem (Todorcevic,1993)

Assume $Pr_0(\omega_1, \omega_1, 4, \omega)$. *There is a Lindelöf group whose square is not Lindelöf.*

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Theorem (Moore, 2006)

There is an L space.

A little strengthening - L group

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The first L group appeared quite early.

Theorem (Hajnal, Juhasz, 1973)

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Theorem

The group generated by Moore's L space is not Lindelöf.

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Note that for regular spaces, Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact. So none of these 4 properties is preserved by taking square.

Methodology

The *osc* map is constructed by Moore using the method of minimal walk which is introduced by Todorcevic.

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Definition

1. *A C-sequence is a sequence* $\langle C_\alpha : \alpha < \omega_1 \rangle$ *such that* $C_{\alpha+1} = \{\alpha\}$ *and* C_α *is a cofinal subset of* α *of order type* ω *for limit* α*'s.*

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2. $\rho_1 : [\omega_1]^2 \to \omega$*, defined recursively by* $\rho_1(\alpha, \beta) = \max\{|C_\beta \cap \alpha|, \rho_1(\alpha, \min(C_\beta \setminus \alpha))\}$ *with boundary value* $\rho_1(\alpha, \alpha) = 0$*.* $\rho_{1\beta} : \beta \to \omega$ *is defined by* $\rho_{1\beta}(\alpha) = \rho_1(\alpha, \beta)$ *for* $\alpha < \beta$*.*

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3. *For any C-sequence, the lower trace* $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$ *is recursively defined for any* $\alpha \leq \beta < \omega_1$ *as follows:*
   - $L(\alpha, \alpha) = 0$;
   - $L(\alpha, \beta) = (L(\alpha, \min(C_\beta \setminus \alpha)) \cup \{\max(C_\beta \cap \alpha)\}) \setminus \max(C_\beta \cap \alpha)$.

Definition

1. *For two functions* s, t *on a common finite set of ordinals* F*,* $Osc(s, t; F) = \{\alpha \in F \setminus \{\min F\} : s(\max F \cap \alpha) \leq t(\max F \cap \alpha)$ *and* $s(\alpha) > t(\alpha)\}$.

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It turns out that this form, together with these combinatorial properties has applications other than a solution to Arhangelskii's question.

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Theorem (Moore)

Let $\{\theta_\alpha : \alpha < \omega_1\}$ be a set of rationally independent reals and $\mathscr{A} \subset [\omega_1]^k$ be an uncountable family of pairwise disjoint sets, $B \in [\omega_1]^{\omega_1}$. Then for any sequence $U_i \subset (0,1)$ of open sets ($i < k$), there are $a \in \mathscr{A}$ and $\beta \in B \setminus a$ such that for any $i < k$, $frac(\theta_{a(i)} osc(a(i), \beta)) \in U_i$.

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Roughly speaking, $\{(frac(\theta_{a(0)} osc(a(0), \beta)), ..., frac(\theta_{a(k-1)} osc(a(k-1), \beta))) : a \in \mathscr{A}, \beta \in B \setminus a\}$ is dense in $(0,1)^k$ for any appropriate $\mathscr{A}, B$. And this is the key to get the L space property.

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Theorem (Combinatorial property 1)

For any uncountable families of pairwise disjoint sets $\mathscr{A} \subset [\omega_1]^k$ *and* $\mathscr{B} \subset [\omega_1]^l$, *there are* $\mathscr{A}' \in [\mathscr{A}]^{\omega_1}$, $\mathscr{B}' \in [\mathscr{B}]^{\omega_1}$ *and* $\langle c_{ij} : i < k, j < l \rangle \in \mathbb{Z}^{k \times l}$ *such that for any* $a \in \mathscr{A}'$, *for any* $b \in \mathscr{B}' \setminus a$, $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ *for any* $i < k, j < l$. *Moreover, we ca require* $\mathscr{A}' = \mathscr{B}'$ *if* $\mathscr{A} = \mathscr{B}$.

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This property allows us to refine $\mathscr{A}, \mathscr{B}$. As we are dealing with problems of the form: "for any uncountable $\mathscr{A}, \mathscr{B}$,...", combinatorial property 1 allows us dealing with the easier case: "for any uncountable $\mathscr{A}, \mathscr{B}$ with property mentioned above,...".

We also have a complement of combinatorial property 1.

Theorem (Combinatorial property 2)

For any $X \in [\omega_1]^{\omega_1}$, *for any* $k, l < \omega$, *for any* $\langle c_{ij} : i < k, j < l \rangle \in \mathbb{Z}^{k \times l}$ *such that* $c_{i0} = 0$ *for* $i < k$, *there are uncountable families* $\mathscr{A} \subset [X]^k$, $\mathscr{B} \subset [X]^l$ *that are pairwise disjoint and for any* $a \in \mathscr{A}, b \in \mathscr{B} \setminus a$, $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$ *for* $i < k, j < l$.

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Definition

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$grp(\mathscr{L})$ – the group generated by $\mathscr{L}$ – is what we need.

An L group with non-Lindelöf square

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Recall that for regular spaces, L $\Rightarrow$ hereditarily Lindelöf $\Rightarrow$ Lindelöf $\Rightarrow$ paracompact $\Rightarrow$ normal & weakly paracompact.

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So none of the properties mentioned above is preserved by taking square for topological groups.

Question

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Let $C_p(X)$ be Lindelöf. Is it then true that $C_p(X) \times C_p(X)$ is Lindelöf?

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Let X be a Banach space with weak topology w such that (X, w) is Lindelöf. Is it true that $(X, w)^2$ is Lindelöf?

Higher finite powers

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Assume MA_{ω_1}. For any regular space X, X^n is hereditarily Lindelöf for all finite n iff X^n is hereditarily separable for all finite n.

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The problem is that we didn't know whether there is an L space whose square is an L space.

Higher finite power and strong negative partition relation

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And previous mentioned Kunen's Theorem tell that this is the best we can do in ZFC.

On partition relations

Definition

(Strong coloring, Shelah) $Pr_0(\kappa, \kappa, \theta, \sigma)$ *asserts that there is a function* $c : [\kappa]^2 \to \theta$ *such that whenever we are given* $\gamma < \sigma$, *a family* $\mathscr{A} \subset [\kappa]^\gamma$ *of* κ *many pairwise disjoint sets and a function* $h : \gamma \times \gamma \to \theta$, *then there are* $a < b$ *in* $\mathscr{A}$ *such that* $c(a(i), b(j)) = h(i, j)$ *for any* $i, j < \gamma$.

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To make the notation simple, we will use the following "weaker" version.

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(Strong coloring, Shelah) $Pr_1(\omega_1, \omega_1, \theta, n)$ *asserts that there is a function* $c : [\omega_1]^2 \to \theta$ *such that whenever we are given* $m < n$, *a family* $\mathscr{A} \subset [\omega_1]^m$ *of pairwise disjoint sets and a* $\gamma < \theta$, *then there are* $a < b$ *in* $\mathscr{A}$ *such that* $c(a(i), b(j)) = \gamma$ *for any* $i, j < m$.

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What about the rest $n < \omega$?

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We will find appropriate $f : [0, 1) \to \omega$ and rationally independent $\{\theta_\alpha : \alpha < \omega_1\}$.

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Strong colorings

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By combinatorial property 1 of the *osc* map, we assume there is a $\langle c_{ij} : i, j < k \rangle \in \mathbb{Z}^{k \times k}$ such that for any $a < b$ in $\mathscr{A}$, for any $i, j < k$, $osc(a(i), b(j)) = osc(a(i), b(0)) + c_{ij}$.

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Thank you!